Original Research Article

HERMITE POLYNOMIAL-BASED METHODS FOR OPTIMAL ORDER APPROXIMATION OF FIRST-ORDER ORDINARY DIFFERENTIAL EQUATIONS

Abstract

Thisstudy investigates the continuous linear multistep techniques utilized for solving first-order initial value problems in ordinary differential equations. Specifically, the study focuses on step k = 9, utilizing Hermite polynomials as basis functions. By applying collocation and interpolation methodologies, this study effectively constructs the Adams-Bashforth, Adams-Moulton, and optimal order methods. To demonstrate their efficacy and validity, the methods are thoroughly examined using various numerical instances. Notably, the optimal order method exhibits superior accuracy and efficiency compared to the traditional Adams-Bashforth and Adams-Moulton methods. The research results contribute novel and improved methodologies for solving initial value problems in differential equations, which have extensive applications across diverse mathematical and scientific domains.

Keywords: *Continuous linear multistep methods, First-order initial value problems, Ordinary differential equations, Hermite polynomials, Adams-Bashforth, Adams-Moulton, and optimal order method.*

1. Introduction

Linear multistep methods (LMM) are widely used in modeling dynamic systems in various scientific and engineering fields. However, these methods face challenges in determining an appropriate step size and dealing with stiffness where solutions change rapidly in some regions and slowly in others. To address these challenges, researchers have developed adaptive step size control and combined linear multistep methods with other numerical techniques, such as finite element and spectral methods.

In the study of numerical methods for solving initial value problems (IVPs) of ordinary differential equations (ODEs), several approaches have been developed to derive linear multistep methods in discrete form, such as interpolation, numerical integration, Taylor series expansion, and determination of the LMM order. Continuous collocation and interpolation techniques have gained popularity in the derivation of LMMs, including block and hybrid methods.

Several studies related to the primary objective of this research have been conducted. Alabi (2008) used Chebyshev polynomials in a multistep collocation technique to derive continuous solvers of IVPs. Mohammed (2011) derived a linear multistep method with continuous coefficients and used it to obtain multiple finite difference methods for solving first-order ODEs. Odekunle*et al.* (2012) developed a continuous linear multistep method using interpolation and collocation for the solution of first-order ODEs with a constant step size. Adesanya *et al.* (2012) proposed a method that combines the collocation of the differential system and interpolation of the approximate solution to generate a continuous LMM. Anake (2011) developed a new class of continuous implicit hybrid one-step methods using the collocation and interpolation techniques of the power series approximate solution to solve IVPs of general second-order ODEs.

Furthermore, Olaiya*et al.*(2019) proposed a new method for solving the Black-Scholes Partial Differential Equation by directly solving a system of second-order Ordinary Differential Equations using a two-step hybrid Block Method of Order seven developed using interpolation and collocation techniques. They investigated properties of the method such as zero stability, order, consistency, convergence, and the region of absolute stability. The new method was found to have better accuracy compared to existing methods in terms of error when applied to solve the Black-Scholes equation after converting it to the system of second-order ordinary differential equations.

Meanwhile, Aboiyar*et al.*(2015) developed continuous linear multistep methods for solving first-order IVPs of ODEs using Hermite polynomials as basis functions from step number k=3. They derived Adams-Bashforth, Adams-Moulton, and optimal order methods through collocation and interpolation techniques and evaluated their effectiveness and validity using two first-order IVPs of ODEs. Results showed that the optimal order method outperformed the standard Adams-Bashforth and Adams-Moulton methods in terms of accuracy and efficiency.

The primary objective of this research is to derive and evaluate a continuous linear multistep method using Hermite polynomials as basis functions of degree 8, an extension of the degree 4 scheme developed by Aboiyar*et al.*(2015). The study aims to increase accuracy and address ambiguity in solving ambiguous initial value problems using linear multistep methods. The primary objective of this research is to derive and evaluate continuous linear multistep methods using Hermite polynomial of degree 8 as basis function, an extension of the degree 4 scheme developed by Aboiyar*et al.* (2015). The study aims at increasing accuracy and addressing ambiguity in solving ambiguous initial value problems using linear multistep methods.The effectiveness and validity of the proposed method will be evaluated using numerical examples.The derivation of the linear multistep methods is introduced in Section 2. The numerical solutions and details of the results of the investigation into first-order differential equation problems are presented in Section 3. The findings of this research are finally summarized in Section 4.

2. Derivation of the Linear Multistep Methods

The characteristics feature of one step methods is that they need, for computing y_{k+1}, only the value from the previous approximation of the solution y_k.Methods that use,for computing y_{k+1}, more than one of the previous approximations $y_k, y_{k-1}, \ldots$ are called multi-step methods. A straightforward extension consists in constructing methods that use the computing y_{k+1} more than one of the previous approximations $y_k, y_{k-1}, \ldots$ such methods are called multi-step methods.

2.1 Definition: q-step method, linear q-step method.

A q-step method with $q \geq 1$ is a numerical method for approximately solving

$$y' = f(x, y(x)), y(x_0) = y_0, \qquad (2.1)$$

where y_{k+1} depends on y_{k+1-q} but not on y_i with $i < k + 1 - q$.

A q-step method is called linear, if it has the form

$$y_{k+1} = \sum_{j=0}^{q-1} a_j y_{k-j} + h \sum_{j=0}^{q-1} b_j f(x_{k-j}, y_{k-j}) + hb_{-1} f(x_{k+1}, y_{k+1}), k = q, q + 1, \ldots. \qquad (2.2)$$

with $q \geq 1, a_0, \ldots, a_{q-1}, b_{-1}, \ldots, b_{q-1} \in \mathbb{R}, a_{q-1} \neq 0$ and $b_{q-1} \neq 0$. For $q = 1$, the method is called a one step method. If $b_{-1} \neq 0$, then the linear q-step method is an implicit method, otherwise it is an explicit method.

In Awoyemi (1999) and Onumanyi, *et al.* (1993), some continuous LMM of the type in Equation (2.2) were developed using a collocation function of the form:

$$y(x) = \sum_{j=0}^{k} \alpha_j x^j \qquad (2.3)$$

Awoyemi*et al.* (2014) proposed a similar function of the type in Equation (2.3)

$$y(x) = \sum_{j=0}^{k} \alpha_j (x - x_k)^j \qquad (2.4)$$

to develop LMM for the solution of third-order IVPs. Adeniyi and Alabi (2006) used Chebyshev polynomial function of the form:

$$y(x) = \sum_{j=0}^{M} \alpha_j T_j \left(\frac{x - x_k}{h}\right) \qquad (2.5)$$

where $T_j(x)$ are some Chebyshev functions to develop continuous LMM.

In this research, we will apply the Probabilists' Hermite polynomial of the form (Koornwinder, *et al*., 2010), introduced by Aboiyar*et al.* (2015):

$$y(x) = \sum_{j=0}^{k} \alpha_j H_j(x - x_k) \quad (2.6)$$

where $H_j(x)$ are probabilists' Hermite polynomials generated by the formula:

$$H_n(x) = (-1)^n e^{\left(\frac{x^2}{2}\right)} \frac{d^n}{dx^n} e^{\left(-\frac{x^2}{2}\right)} = \left(x - \frac{d}{dx}\right)^n \quad (2.7)$$

and whose recursive relation is

$$H_{n+1}(x) = xH_n(x) - H_n(x) \quad (2.8)$$

to develop continuous LMMs for the solution of first-order IVPs of ODEs of the form:

$$y' = f(x, y(x)), y(x_0) = y_0 \quad (2.9)$$

The first six probabilists' Hermite polynomials are

$$H_0 = 1, H_1 = x, H_2 = x^2 - 1, H_3 = x^3 - 3x, H_4 = x^4 - 6x^2 + 3, H_5 = x^5 - 10x^3 + 15x.$$

We wish to approximate the exact solution $y(x)$ to the IVP in Equation (2.9) by a polynomial of degree n of the form:

$$y(x) = \sum_{j=0}^{n} \alpha_j H_j(x - x_k), x_k \leq x \leq x_{k+p} \quad (2.10)$$

which satisfies the equations

$$\left.\begin{aligned} y'(x) = f(x, y(x)), x_k \leq x \leq x_{k+p} \\ y(x_k) = y_k \end{aligned}\right\} \quad (2.11)$$

2.2 Derivation of Eight - step Adams-Bash forth method

To derive the eight-step Adams – Bash forth method using Hermite polynomials, we set n=8, in Equation (2.10) yielding

$$y(x) = a_0 + a_1(x - x_k) + a_2((x - x_k)^2 - 1) + a_3\left((x - x_k)^3 - 3(x - x_k)\right)$$
$$+ a_4((x - x_k)^4 - 6(x - x_k)^2 + 3)$$
$$+ a_5\left((x - x_k)^5 - 10(x - x_k)^3 + 15(x - x_k)\right)$$
$$+ a_6((x - x_k)^6 - 15(x - x_k)^4 + 45(x - x_k)^2 - 15)$$
$$+ a_7\left((x - x_k)^7 - 21(x - x_k)^5 + 105(x - x_k)^3 - 105(x - x_k)\right)$$
$$+ a_8((x - x_k)^8 - 28(x - x_k)^6 + 210(x - x_k)^4 - 420(x - x_k)^2 + 105); \quad (2.12)$$

Differentiating Eq. (2.12) once gives the equation:

$$\dot{y}(x) = a_1 + 2a_2(x - x_k) + 3a_3[(x - x_k)^2 - 1] + 4a_4[(x - x_k)^3 - 3(x - x_k)]$$
$$+ 5a_5[(x - x_k)^4 - 6(x - x_k)^2 + 3] + 6a_6[(x - x_k)^5 - 10(x - x_k)^3 + 15(x - x_k)]$$
$$+ 7a_7[(x - x_k)^6 - 15(x - x_k)^4 + 45(x - x_k)^2 - 15]$$
$$+ 8a_8[(x - x_k)^7 - 21(x - x_k)^5 + 105(x - x_k)^3 - 105(x - x_k)] \quad (2.13)$$

Interpolating Eq. (2.12) at $x = x_{k+7}$and collocating Eq. (2.13) at $x = x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}, x_{k+7}$ gives

$$y(x_{k+7}) = a_0 + a_1(x_{k+7} - x_k) + a_2((x_{k+7} - x_k)^2 - 1) + a_3\left((x_{k+7} - x_k)^3 - 3(x_{k+7} - x_k)\right) + a_4((x_{k+7} - x_k)^4 - 6(x_{k+7} - x_k)^2 + 3) + a_5\left((x_{k+7} - x_k)^5 - 10(x_{k+7} - x_k)^3 + 15(x_{k+7} - x_k)\right) + a_6((x_{k+7} - x_k)^6 - 15(x_{k+7} - x_k)^4 + 45(x_{k+7} - x_k)^2 - 15) + a_7\left((x_{k+7} - x_k)^7 - 21(x_{k+7} - x_k)^5 + 105(x_{k+7} - x_k)^3 - 105(x_{k+7} - x_k)\right) + a_8((x_{k+7} - x_k)^8 - 28(x_{k+7} - x_k)^6 + 210(x_{k+7} - x_k)^4 - 420(x_{k+7} - x_k)^2 + 105) \quad (2.14)$$

$$f_k = a_1 - 3a_3 + 15a_5 - 105a_7 = \dot{y}(x_k) \quad (2.15)$$

$$f_{k+1} = a_1 + 2a_2(x_{k+1} - x_k) + 3a_3[(x_{k+1} - x_k)^2 - 1] + 4a_4[(x_{k+1} - x_k)^3 - 3(x_{k+1} - x_k)] + 5a_5[(x_{k+1} - x_k)^4 - 6(x_{k+1} - x_k)^2 + 3] + 6a_6[(x_{k+1} - x_k)^5 - 10(x_{k+1} - x_k)^3 + 15(x_{k+1} - x_k)] + 7a_7[(x_{k+1} - x_k)^6 - 15(x_{k+1} - x_k)^4 + 45(x_{k+1} - x_k)^2 - 15] + 8a_8[(x_{k+1} - x_k)^7 - 21(x_{k+1} - x_k)^5 + 105(x_{k+1} - x_k)^3 - 105(x_{k+1} - x_k)] = \dot{y}(x_{k+1})$$
(2.16)

$$f_{k+2} = a_1 + 2a_2(x_{k+2} - x_k) + 3a_3[4(x_{k+2} - x_k)^2 - 1] + 8a_4[4(x_{k+2} - x_k)^3 - 3(x_{k+2} - x_k)] + 5a_5[16(x_{k+2} - x_k)^4 - 24(x_{k+2} - x_k)^2 + 3] + 12a_6[16(x_{k+2} - x_k)^5 - 40(x_{k+2} - x_k)^3 + 15(x_{k+2} - x_k)] + 7a_7[64(x_{k+2} - x_k)^6 - 240(x_{k+2} - x_k)^4 + 180(x_{k+2} - x_k)^2 - 15] + 16a_8[64(x_{k+2} - x_k)^7 - 336(x_{k+2} - x_k)^5 + 420(x_{k+2} - x_k)^3 - 105(x_{k+2} - x_k)] = \dot{y}(x_{k+2})$$
(2.17)

$$f_{k+3} = a_1 + 6a_2(x_{k+3} - x_k) + 3a_3[9(x_{k+3} - x_k)^2 - 1] + 36a_4[6(x_{k+3} - x_k)^3 - (x_{k+3} - x_k)] + 15a_5[27(x_{k+3} - x_k)^4 - 18(x_{k+3} - x_k)^2 + 1] + 54a_6[27(x_{k+3} - x_k)^5 - 30(x_{k+3} - x_k)^3 + 5(x_{k+3} - x_k)] + a_7[5103(x_{k+3} - x_k)^6 - 8505(x_{k+3} - x_k)^4 + 2835(x_{k+3} - x_k)^2 - 105] +$$

$a_8[17496(x_{k+3} - x_k)^7 - 40824(x_{k+3} - x_k)^5 + 22680(x_{k+3} - x_k)^3 - 2520(x_{k+3} - x_k)] = \dot{y}(x_{k+3})$
(2.18)

$$f_{k+4} = a_1 + 8a_2(x_{k+4} - x_k) + 3a_3[16(x_{k+4} - x_k)^2 - 1] + a_4[256(x_{k+4} - x_k)^3 - 48(x_{k+4} - x_k)] + a_5[1280(x_{k+4} - x_k)^4 - 480(x_{k+4} - x_k)^2 + 15] + a_6[6144(x_{k+4} - x_k)^5 - 3840(x_{k+4} - x_k)^3 + 360(x_{k+4} - x_k)] + a_7[28672(x_{k+4} - x_k)^6 - 26880(x_{k+4} - x_k)^4 + 5040(x_{k+4} - x_k)^2 - 105] + a_8[1.31072 \times 10^5(x_{k+4} - x_k)^7 - 1.72032 \times 10^5(x_{k+4} - x_k)^5 + 53760(x_{k+4} - x_k)^3 - 3360(x_{k+4} - x_k)] = \dot{y}(x_{k+4}) \quad (2.19)$$

$$f_{k+5} = a_1 + 10a_2(x_{k+5} - x_k) + 3a_3[25(x_{k+5} - x_k)^2 - 1] + a_4[500(x_{k+5} - x_k)^3 - 60(x_{k+5} - x_k)] + a_5[3125(x_{k+5} - x_k)^4 - 750(x_{k+5} - x_k)^2 + 15] + a_6[18750(x_{k+5} - x_k)^5 - 7500(x_{k+5} - x_k)^3 + 450(x_{k+5} - x_k)] + a_7[1.09375 \times 10^5(x_{k+5} - x_k)^6 - 65625(x_{k+5} - x_k)^4 + 7875(x_{k+5} - x_k)^2 - 105] + a_8[6.25000 \times 10^5(x_{k+5} - x_k)^7 - 5.25000 \times 10^5(x_{k+5} - x_k)^5 + 1.05000 \times 10^5(x_{k+5} - x_k)^3 - 4200(x_{k+5} - x_k)] = \dot{y}(x_{k+5}) \quad (2.20)$$

$$f_{k+6} = a_1 + 12a_2(x_{k+6} - x_k) + 3a_3[36(x_{k+6} - x_k)^2 - 1] + a_4[864(x_{k+6} - x_k)^3 - 72(x_{k+6} - x_k)] + a_5[6480(x_{k+6} - x_k)^4 - 1080(x_{k+6} - x_k)^2 + 15] + a_6[46656(x_{k+6} - x_k)^5 - 12960(x_{k+6} - x_k)^3 + 540(x_{k+5} - x_k)] + a_7[3.26592 \times 10^5(x_{k+6} - x_k)^6 - 1.36080 \times 10^5(x_{k+6} - x_k)^4 + 11340(x_{k+5} - x_k)^2 - 105] + a_8[2.239488 \times 10^6(x_{k+6} - x_k)^7 - 1.306368 \times 10^6(x_{k+6} - x_k)^5 + 1.81440 \times 10^5(x_{k+6} - x_k)^3 - 5040(x_{k+6} - x_k)] = \dot{y}(x_{k+6}) \quad (2.21)$$

$$f_{k+7} = a_1 + 14a_2(x_{k+7} - x_k) + a_3[147(x_{k+7} - x_k)^2 - 3] + a_4[1372(x_{k+7} - x_k)^3 - 84(x_{k+7} - x_k)] + a_5[12005(x_{k+7} - x_k)^4 - 1470(x_{k+7} - x_k)^2 + 15] + a_6[1.00842 \times 10^5(x_{k+7} - x_k)^5 - 20580(x_{k+7} - x_k)^3 + 630(x_{k+7} - x_k)] + a_7[8.23543 \times 10^5(x_{k+7} - x_k)^6 - 2.52105 \times 10^5(x_{k+7} - x_k)^4 + 15435(x_{k+7} - x_k)^2 - 105] + a_8[6.588344 \times 10^6(x_{k+7} - x_k)^7 - 2.823576 \times 10^6(x_{k+7} - x_k)^5 + 2.88120 \times 10^5(x_{k+7} - x_k)^3 - 5880(x_{k+7} - x_k)] = \dot{y}(x_{k+7}) \quad (2.22)$$

In matrix form, we have:

$$\left[\begin{pmatrix} 1 & 7h & (49h^2 - 1) & (343h^3 - 21h) & (2401h^4 - 294h^2 + 3) & (16807h^5 - 3430h^3 + 105h) & (117649h^6 - 36015h^4 + 2205h^2 - 15) \\ 0 & 1 & 0 & -3 & 0 & 15 & 0 \\ 0 & 1 & 2h & (3h^2 - 3) & (4h^3 - 12h) & (5h^4 - 30h^2 + 15) & (6h^5 - 60h^3 + 90h) \\ 0 & 1 & 4h & (12h^2 - 3) & (32h^3 - 24h) & (80h^4 - 120h^2 + 15) & (192h^5 - 480h^3 + 180h) \\ 0 & 1 & 6h & (27h^2 - 3) & (108h^3 - 36h) & (405h^4 - 270h^2 + 15h) & (1458h^5 - 1620h^3 + 270h) \\ 0 & 1 & 8h & (48h^2 - 3) & (256h^3 - 48h) & (1280h^4 - 480h^2 + 15h) & (6144h^5 - 3840h^3 + 360h) \\ 0 & 1 & 10h & (75h^2 - 3) & (500h^3 - 60h) & (3125h^4 - 750h^2 + 15h) & (18750h^5 - 7500h^3 + 450h) \\ 0 & 1 & 12h & (108h^2 - 3) & (864h^3 - 72h) & (6480h^4 - 1080h^2 + 15h) & (46656h^5 - 12960h^3 + 540h) \\ 0 & 1 & 14h & (147h^2 - 3) & (1372h^3 - 84h) & (12005h^4 - 1470h^2 + 15h) & (100842h^5 - 20580h^3 + 630h) \end{pmatrix} + \begin{pmatrix} (823543h^7 - 352947h^5 + 36015h^3 - 735h) & (5764801h^8 - 3294172h^6 + 504210h^4 - 20580h^2 + 105) \\ -105 & 0 \\ (7h^6 - 105h^4 + 315h^2 - 105) & (8h^7 - 168h^5 + 840h^3 - 840h) \\ (448h^6 - 1680h^4 + 1260h^2 - 105) & (1024h^7 - 5376h^5 + 6720h^3 - 1680h) \\ (5103h^6 - 8505h^4 + 2835h^2 - 105) & (17496h^7 - 40824h^5 + 22680h^3 - 2520h) \\ (28672h^6 - 26880h^4 + 5040h^2 - 105) & (131072h^7 - 172032h^5 + 53760h^3 - 3360h) \\ (109375h^6 - 65625h^4 + 7875h^2 - 105) & (625000h^7 - 525000h^5 + 105000h^3 - 4200h) \\ (326592h^6 - 136080h^4 + 11340h^2 - 105) & (2239488h^7 - 1306368h^5 + 181440h^3 - 5040h) \\ (823543h^6 - 252105h^4 + 15435h^2 - 105) & (6588344h^7 - 2823576h^5 + 288120h^3 - 5880h) \end{pmatrix}\right] \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{k+7} \\ f_k \\ f_{k+1} \\ f_{k+2} \\ f_{k+3} \\ f_{k+4} \\ f_{k+5} \\ f_{k+6} \\ f_{k+7} \end{bmatrix} \quad (2.23)$$

Solving the system of equation (2.23) above by Gaussian elimination method, we have

$$a_0 = y_{k+7} - h\left(\left(\frac{5257}{17280}\right) f_k + \left(\frac{25039}{17280}\right) f_{k+1} + \left(\frac{343}{640}\right) f_{k+2} + \left(\frac{20923}{17280}\right) f_{k+3} + \left(\frac{20923}{17280}\right) f_{k+4} + \left(\frac{343}{640}\right) f_{k+5} + \left(\frac{25039}{17280}\right) f_{k+6} + \left(\frac{5257}{17280}\right) f_{k+7}\right) - \frac{1}{h}\left(\left(\frac{363}{280}\right) f_k - \left(\frac{7}{2}\right) f_{k+1} + \left(\frac{21}{4}\right) f_{k+2} - \left(\frac{35}{6}\right) f_{k+3} + \left(\frac{35}{8}\right) f_{k+4} - \left(\frac{21}{10}\right) f_{k+5} + \left(\frac{7}{12}\right) f_{k+6} - \left(\frac{1}{14}\right) f_{k+7}\right) - \frac{1}{h^3}\left(\left(\frac{967}{960}\right) f_k - \left(\frac{319}{60}\right) f_{k+1} + \left(\frac{3929}{320}\right) f_{k+2} - \left(\frac{389}{24}\right) f_{k+3} + \left(\frac{2545}{192}\right) f_{k+4} - \left(\frac{67}{10}\right) f_{k+5} + \left(\frac{1849}{960}\right) f_{k+6} - \left(\frac{29}{120}\right) f_{k+7}\right) - \frac{1}{h^5}\left(\left(\frac{23}{144}\right) f_k - \left(\frac{295}{288}\right) f_{k+1} + \left(\frac{45}{16}\right) f_{k+2} - \left(\frac{1235}{288}\right) f_{k+3} + \left(\frac{565}{144}\right) f_{k+4} - \left(\frac{69}{32}\right) f_{k+5} + \left(\frac{95}{144}\right) f_{k+6} - \left(\frac{25}{288}\right) f_{k+7}\right) - \frac{1}{h^7}\left(\left(\frac{7}{384}\right) f_{k+6} + \left(\frac{1}{384}\right) f_{k+7} + \left(\frac{7}{128}\right) f_{k+5} + \left(\frac{7}{384}\right) f_{k+1} + \left(\frac{35}{384}\right) f_{k+3} - \left(\frac{7}{128}\right) f_{k+2} - \left(\frac{1}{384}\right) f_k - \left(\frac{35}{384}\right) f_{k+4}\right) \tag{2.24}$$

$$a_1 = f_k + \frac{1}{h^2}\left(\left(\frac{469}{180}\right) f_k - \left(\frac{223}{20}\right) f_{k+1} + \left(\frac{879}{40}\right) f_{k+2} - \left(\frac{949}{36}\right) f_{k+3} + \left(\frac{41}{2}\right) f_{k+4} - \left(\frac{201}{20}\right) f_{k+5} + \left(\frac{1019}{360}\right) f_{k+6} - \left(\frac{7}{20}\right) f_{k+7}\right) + \frac{1}{h^4}\left(\left(\frac{7}{6}\right) f_k - \left(\frac{111}{16}\right) f_{k+1} + \left(\frac{71}{4}\right) f_{k+2} - \left(\frac{1219}{48}\right) f_{k+3} + 22 f_{k+4} - \left(\frac{185}{16}\right) f_{k+5} + \left(\frac{41}{12}\right) f_{k+6} - \left(\frac{7}{16}\right) f_{k+7}\right) + \frac{1}{h^6}\left(\left(\frac{1}{12}\right) f_k - \left(\frac{9}{16}\right) f_{k+1} + \left(\frac{13}{8}\right) f_{k+2} - \left(\frac{125}{48}\right) f_{k+3} + \left(\frac{5}{2}\right) f_{k+4} - \left(\frac{23}{16}\right) f_{k+5} + \left(\frac{11}{24}\right) f_{k+6} - \left(\frac{1}{16}\right) f_{k+7}\right) \tag{2.25}$$

$$a_2 = \frac{1}{h}\left(\left(\frac{1}{14}\right) f_{k+7} - \left(\frac{7}{12}\right) f_{k+6} + \left(\frac{21}{10}\right) f_{k+5} - \left(\frac{35}{8}\right) f_{k+4} + \left(\frac{35}{6}\right) f_{k+3} - \left(\frac{21}{4}\right) f_{k+2} + \left(\frac{7}{2}\right) f_{k+1} - \left(\frac{363}{280}\right) f_k\right) + \frac{1}{h^3}\left(\left(\frac{29}{60}\right) f_{k+7} - \left(\frac{1849}{480}\right) f_{k+6} + \left(\frac{67}{5}\right) f_{k+5} - \left(\frac{2545}{96}\right) f_{k+4} + \left(\frac{389}{12}\right) f_{k+3} - \left(\frac{3929}{160}\right) f_{k+2} + \left(\frac{319}{30}\right) f_{k+1} - \left(\frac{967}{480}\right) f_k\right) + \frac{1}{h^5}\left(\left(\frac{25}{96}\right) f_{k+7} - \left(\frac{95}{48}\right) f_{k+6} + \left(\frac{207}{32}\right) f_{k+5} - \left(\frac{565}{48}\right) f_{k+4} + \left(\frac{1235}{96}\right) f_{k+3} - \left(\frac{135}{16}\right) f_{k+2} + \left(\frac{295}{96}\right) f_{k+1} - \left(\frac{23}{48}\right) f_k\right) + \frac{1}{h^7}\left(\left(\frac{1}{96}\right) f_{k+7} - \left(\frac{7}{96}\right) f_{k+6} + \left(\frac{7}{32}\right) f_{k+5} - \left(\frac{35}{96}\right) f_{k+4} + \left(\frac{35}{96}\right) f_{k+3} - \left(\frac{7}{32}\right) f_{k+2} + \left(\frac{7}{96}\right) f_{k+1} - \left(\frac{1}{96}\right) f_k\right) \tag{2.26}$$

$$a_3 = \frac{1}{h^2}\left(\left(\frac{469}{540}\right) f_k - \left(\frac{223}{60}\right) f_{k+1} + \left(\frac{293}{40}\right) f_{k+2} - \left(\frac{949}{108}\right) f_{k+3} + \left(\frac{41}{6}\right) f_{k+4} - \left(\frac{67}{20}\right) f_{k+5} + \left(\frac{1019}{1080}\right) f_{k+6} - \left(\frac{7}{60}\right) f_{k+7}\right) + \frac{1}{h^4}\left(\left(\frac{7}{9}\right) f_k - \left(\frac{37}{8}\right) f_{k+1} + \left(\frac{71}{6}\right) f_{k+2} - \left(\frac{1219}{72}\right) f_{k+3} + \left(\frac{44}{3}\right) f_{k+4} - \left(\frac{185}{24}\right) f_{k+5} + \left(\frac{41}{18}\right) f_{k+6} - \left(\frac{7}{24}\right) f_{k+7}\right) + \frac{1}{h^6}\left(\left(\frac{1}{12}\right) f_k - \left(\frac{9}{16}\right) f_{k+1} + \left(\frac{13}{8}\right) f_{k+2} - \left(\frac{125}{48}\right) f_{k+3} + \left(\frac{5}{2}\right) f_{k+4} - \left(\frac{23}{16}\right) f_{k+5} + \left(\frac{11}{24}\right) f_{k+6} - \left(\frac{1}{16}\right) f_{k+7}\right) \tag{2.27}$$

$$a_4 = \frac{1}{h^3}\left(\left(\frac{29}{360}\right) f_{k+7} - \left(\frac{1849}{2880}\right) f_{k+6} + \left(\frac{67}{30}\right) f_{k+5} - \left(\frac{2545}{576}\right) f_{k+4} + \left(\frac{389}{72}\right) f_{k+3} - \left(\frac{3929}{960}\right) f_{k+2} + \left(\frac{319}{180}\right) f_{k+1} - \left(\frac{967}{2880}\right) f_k\right) + \frac{1}{h^5}\left(\left(\frac{25}{288}\right) f_{k+7} + \left(\frac{69}{32}\right) f_{k+5} - \left(\frac{95}{144}\right) f_{k+6} - \left(\frac{565}{144}\right) f_{k+4} + \left(\frac{1235}{288}\right) f_{k+3} - \right.$$

$$\left(\frac{45}{16}\right) f_{k+2} + \left(\frac{295}{288}\right) f_{k+1} - \left(\frac{23}{144}\right) f_k \Big) + \frac{1}{h^7}\Big(\left(\frac{1}{192}\right) f_{k+7} - \left(\frac{7}{192}\right) f_{k+6} + \left(\frac{7}{64}\right) f_{k+5} - \left(\frac{35}{192}\right) f_{k+4} + \left(\frac{35}{192}\right) f_{k+3} - \left(\frac{7}{64}\right) f_{k+2} + \left(\frac{7}{192}\right) f_{k+1} - \left(\frac{1}{192}\right) f_k \Big) \qquad (2.28)$$

$$a_5 = \frac{1}{h^4}\Big(\left(\frac{7}{90}\right) f_k - \left(\frac{37}{80}\right) f_{k+1} + \left(\frac{71}{60}\right) f_{k+2} - \left(\frac{1219}{720}\right) f_{k+3} + \left(\frac{22}{15}\right) f_{k+4} - \left(\frac{37}{48}\right) f_{k+5} + \left(\frac{41}{180}\right) f_{k+6} - \left(\frac{7}{240}\right) f_{k+7} \Big) + \frac{1}{h^6}\Big(\left(\frac{1}{60}\right) f_k - \left(\frac{9}{80}\right) f_{k+1} + \left(\frac{13}{40}\right) f_{k+2} - \left(\frac{25}{48}\right) f_{k+3} + \left(\frac{1}{2}\right) f_{k+4} - \left(\frac{23}{80}\right) f_{k+5} + \left(\frac{11}{120}\right) f_{k+6} - \left(\frac{1}{80}\right) f_{k+7} \Big) \qquad (2.29)$$

$$a_6 = \frac{1}{h^5}\Big(\left(\frac{5}{864}\right) f_{k+7} - \left(\frac{19}{432}\right) f_{k+6} + \left(\frac{23}{160}\right) f_{k+5} - \left(\frac{113}{432}\right) f_{k+4} + \left(\frac{247}{864}\right) f_{k+3} - \left(\frac{3}{16}\right) f_{k+2} + \left(\frac{59}{864}\right) f_{k+1} - \left(\frac{23}{2160}\right) f_k \Big) + \Big(\left(\frac{1}{1440}\right) f_{k+7} - \left(\frac{7}{1440}\right) f_{k+6} + \left(\frac{7}{480}\right) f_{k+5} - \left(\frac{7}{288}\right) f_{k+4} + \left(\frac{7}{288}\right) f_{k+3} - \left(\frac{7}{480}\right) f_{k+2} + \left(\frac{7}{1440}\right) f_{k+1} - \left(\frac{1}{1440}\right) f_k \Big) \qquad (2.30)$$

$$a_7 = \frac{1}{h^6}\Big(\left(\frac{1}{1260}\right) f_k - \left(\frac{3}{560}\right) f_{k+1} + \left(\frac{13}{840}\right) f_{k+2} - \left(\frac{25}{1008}\right) f_{k+3} + \left(\frac{1}{42}\right) f_{k+4} - \left(\frac{23}{1680}\right) f_{k+5} + \left(\frac{11}{2520}\right) f_{k+6} - \left(\frac{1}{1680}\right) f_{k+7} \Big) \qquad (2.31)$$

$$a_8 = \frac{1}{h^7}\Big(+ \left(\frac{1}{40320}\right) f_{k+7} - \left(\frac{1}{5760}\right) f_{k+6} + \left(\frac{1}{1920}\right) f_{k+5} - \left(\frac{1}{1152}\right) f_{k+4} + \left(\frac{1}{1152}\right) f_{k+3} - \left(\frac{1}{1920}\right) f_{k+2} + \left(\frac{1}{5760}\right) f_{k+1} - \left(\frac{1}{40320}\right) f_k \Big) \qquad (2.32)$$

Substituting for a_j = j =0,1,2,3,4,5,6,7,8 in equation (2.12) yields the continuous method

$$y(x) = y_{k+7} - h\Big(\left(\frac{5257}{17280}\right) f_{k+7} + \left(\frac{25039}{17280}\right) f_{k+6} + \left(\frac{343}{640}\right) f_{k+5} + \left(\frac{20923}{17280}\right) f_{k+4} + \left(\frac{20923}{17280}\right) f_{k+3} + \left(\frac{343}{640}\right) f_{k+2} + \left(\frac{25039}{17280}\right) f_{k+1} + \left(\frac{5257}{17280}\right) f_k \Big) + (x - x_k) f_k + \frac{(x-x_k)^2}{h}\Big(\left(\frac{1}{14}\right) f_{k+7} - \left(\frac{7}{12}\right) f_{k+6} + \left(\frac{21}{10}\right) f_{k+5} - \left(\frac{35}{8}\right) f_{k+4} + \left(\frac{35}{6}\right) f_{k+3} - \left(\frac{21}{4}\right) f_{k+2} + \left(\frac{7}{2}\right) f_{k+1} - \left(\frac{363}{280}\right) f_k \Big) - \frac{(x-x_k)^3}{h^2}\Big(\left(\frac{7}{60}\right) f_{k+7} - \left(\frac{1019}{1080}\right) f_{k+6} + \left(\frac{67}{20}\right) f_{k+5} - \left(\frac{41}{6}\right) f_{k+4} + \left(\frac{949}{108}\right) f_{k+3} - \left(\frac{293}{40}\right) f_{k+2} + \left(\frac{223}{60}\right) f_{k+1} - \left(\frac{469}{540}\right) f_k \Big) + \frac{(x-x_k)^4}{h^3}\Big(\left(\frac{29}{360}\right) f_{k+7} - \left(\frac{1849}{2880}\right) f_{k+6} + \left(\frac{67}{30}\right) f_{k+5} - \left(\frac{2545}{576}\right) f_{k+4} + \left(\frac{389}{72}\right) f_{k+3} - \left(\frac{3929}{960}\right) f_{k+2} + \left(\frac{319}{180}\right) f_{k+1} - \left(\frac{967}{2880}\right) f_k \Big) - \frac{(x-x_k)^5}{h^4}\Big(\left(\frac{7}{240}\right) f_{k+7} - \left(\frac{41}{180}\right) f_{k+6} + \left(\frac{37}{48}\right) f_{k+5} - \left(\frac{22}{15}\right) f_{k+4} + \left(\frac{1219}{720}\right) f_{k+3} - \left(\frac{71}{60}\right) f_{k+2} + \left(\frac{37}{80}\right) f_{k+1} - \left(\frac{7}{90}\right) f_k \Big) + \frac{(x-x_k)^6}{h^5}\Big(+ \left(\frac{5}{864}\right) f_{k+7} - \left(\frac{19}{432}\right) f_{k+6} + \left(\frac{23}{160}\right) f_{k+5} - \left(\frac{113}{432}\right) f_{k+4} + \left(\frac{247}{864}\right) f_{k+3} - \left(\frac{3}{16}\right) f_{k+2} + \left(\frac{59}{864}\right) f_{k+1} - \left(\frac{23}{2160}\right) f_k \Big) - \frac{(x-x_k)^7}{h^6}\Big(\left(\frac{1}{1680}\right) f_{k+7} - \left(\frac{11}{2520}\right) f_{k+6} + \left(\frac{23}{1680}\right) f_{k+5} - \left(\frac{1}{42}\right) f_{k+4} + \left(\frac{25}{1008}\right) f_{k+3} - \left(\frac{13}{840}\right) f_{k+2} +$$

$\left(\frac{3}{560}\right) f_{k+1} - \left(\frac{1}{1260}\right) f_k\Big) + \frac{(x-x_k)^8}{h^7}\Big(+ \left(\frac{1}{40320}\right) f_{k+7} - \left(\frac{1}{5760}\right) f_{k+6} + \left(\frac{1}{1920}\right) f_{k+5} - \left(\frac{1}{1152}\right) f_{k+4} +$
$\left(\frac{1}{1152}\right) f_{k+3} - \left(\frac{1}{1920}\right) f_{k+2} + \left(\frac{1}{5760}\right) f_{k+1} - \left(\frac{1}{40320}\right) f_k\Big) \qquad (2.32)$

Evaluating equation (2.32) at $x = y_{k+8}$, we have obtain the discrete form as:

$$y_{k+8} = y_{k+7} + h\Big(\left(\frac{16083}{4480}\right) f_{k+7} - \left(\frac{1152169}{120960}\right) f_{k+6} + \left(\frac{242653}{13440}\right) f_{k+5} - \left(\frac{296053}{13440}\right) f_{k+4} + \left(\frac{2102243}{120960}\right) f_{k+3} - \left(\frac{115747}{13440}\right) f_{k+2} + \left(\frac{32863}{13440}\right) f_{k+1} - \left(\frac{5257}{17280}\right) f_k\Big) \qquad (2.33)$$

The Eq. (2.33) is the eight-step Adams-Bash forth method.

2.3 Derivation of Eight-Step Adams-Moulton Method

To derive the eight-step Adams – Moulton method using Hermite polynomials, we set n=9, in Equation (2.10) yielding

$$\begin{aligned} y(x) = a_0 &+ a_1(x-x_k) + a_2((x-x_k)^2 - 1) + a_3((x-x_k)^3 - 3(x-x_k)) \\ &+ a_4((x-x_k)^4 - 6(x-x_k)^2 + 3) \\ &+ a_5((x-x_k)^5 - 10(x-x_k)^3 + 15(x-x_k)) \\ &+ a_6((x-x_k)^6 - 15(x-x_k)^4 + 45(x-x_k)^2 - 15) \\ &+ a_7((x-x_k)^7 - 21(x-x_k)^5 + 105(x-x_k)^3 - 105(x-x_k)) \\ &+ a_8((x-x_k)^8 - 28(x-x_k)^6 + 210(x-x_k)^4 - 420(x-x_k)^2 + 105) \\ &+ a_9((x-x_k)^9 - 36(x-x_k)^7 + 378(x-x_k)^5 - 1260(x-x_k)^2 \\ &+ 945(x-x_k)) \end{aligned} \qquad (2.34)$$

Differentiating Eq. (2.34) once gives the equation:

$$\begin{aligned} \dot{y}(x) = a_1 &+ 2a_2(x-x_k) + 3a_3[(x-x_k)^2 - 1] + 4a_4[(x-x_k)^3 - 3(x-x_k)] \\ &+ 5a_5[(x-x_k)^4 - 6(x-x_k)^2 + 3] + 6a_6[(x-x_k)^5 - 10(x-x_k)^3 + 15(x-x_k)] \\ &+ 7a_7[(x-x_k)^6 - 15(x-x_k)^4 + 45(x-x_k)^2 - 15] \\ &+ 8a_8[(x-x_k)^7 - 21(x-x_k)^5 + 105(x-x_k)^3 - 105(x-x_k)] \\ &+ a_9[9(x-x_k)^8 - 252(x-x_k)^6 + 1390(x-x_k)^4 - 2520(x-x_k) + 945] \end{aligned} \qquad (2.35)$$

Interpolating Eq. (2.34) at $x = x_{k+7}$and collocating Eq. (2.35) at
$x = x_k, x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}, x_{k+7}, x_{k+8}$ gives

$y(x_{k+7}) = a_0 + a_1(x_{k+7} - x_k) + a_2((x_{k+7} - x_k)^2 - 1) + a_3\left((x_{k+7} - x_k)^3 - 3(x_{k+7} - x_k)\right) +$
$a_4((x_{k+7} - x_k)^4 - 6(x_{k+7} - x_k)^2 + 3) + a_5\left((x_{k+7} - x_k)^5 - 10(x_{k+7} - x_k)^3 + 15(x_{k+7} -\right.$
$\left.x_k)\right) + a_6((x_{k+7} - x_k)^6 - 15(x_{k+7} - x_k)^4 + 45(x_{k+7} - x_k)^2 - 15) + a_7\left((x_{k+7} - x_k)^7 -\right.$
$\left.21(x_{k+7} - x_k)^5 + 105(x_{k+7} - x_k)^3 - 105(x_{k+7} - x_k)\right) + a_8((x_{k+7} - x_k)^8 - 28(x_{k+7} - x_k)^6 +$

$210(x_{k+7}-x_k)^4-420(x_{k+7}-x_k)^2+105)+a_9\left((x_{k+7}-x_k)^9-36(x_{k+7}-x_k)^7+378(x_{k+7}-x_k)^5-1260(x_{k+7}-x_k)^2+945(x_{k+7}-x_k)\right)$

(2.36)

$$f_k = a_1-3a_3+15a_5-105a_7+945=\dot{y}(x_k) \qquad (2.37)$$

$$f_{k+1} = a_1+2a_2(x_{k+1}-x_k)+3a_3[(x_{k+1}-x_k)^2-1]+4a_4[(x_{k+1}-x_k)^3-3(x_{k+1}-x_k)]+5a_5[(x_{k+1}-x_k)^4-6(x_{k+1}-x_k)^2+3]+6a_6[(x_{k+1}-x_k)^5-10(x_{k+1}-x_k)^3+15(x_{k+1}-x_k)]+7a_7[(x_{k+1}-x_k)^6-15(x_{k+1}-x_k)^4+45(x_{k+1}-x_k)^2-15]+8a_8[(x_{k+1}-x_k)^7-21(x_{k+1}-x_k)^5+105(x_{k+1}-x_k)^3-105(x_{k+1}-x_k)]+a_9[9(x_{k+1}-x_k)^8-252(x_{k+1}-x_k)^6+1890(x_{k+1}-x_k)^4-2520(x_{k+1}-x_k)+945]=\dot{y}(x_{k+1}) \qquad (2.38)$$

$$f_{k+2} = a_1+2a_2(x_{k+2}-x_k)+3a_3[(x_{k+2}-x_k)^2-1]+4a_4[(x_{k+2}-x_k)^3-3(x_{k+2}-x_k)]+5a_5[(x_{k+2}-x_k)^4-6(x_{k+2}-x_k)^2+3]+6a_6[(x_{k+2}-x_k)^5-10(x_{k+2}-x_k)^3+15(x_{k+2}-x_k)]+7a_7[(x_{k+2}-x_k)^6-15(x_{k+2}-x_k)^4+45(x_{k+2}-x_k)^2-15]+8a_8[(x_{k+2}-x_k)^7-21(x_{k+2}-x_k)^5+105(x_{k+2}-x_k)^3-105(x_{k+2}-x_k)]+a_9[9(x_{k+2}-x_k)^8-252(x_{k+2}-x_k)^6+1890(x_{k+2}-x_k)^4-2520(x_{k+2}-x_k)+945]=\dot{y}(x_{k+2}) \qquad (2.39)$$

$$f_{k+3} = a_1+2a_2(x_{k+3}-x_k)+3a_3[(x_{k+3}-x_k)^2-1]+4a_4[(x_{k+3}-x_k)^3-3(x_{k+3}-x_k)]+5a_5[(x_{k+3}-x_k)^4-6(x_{k+3}-x_k)^2+3]+6a_6[(x_{k+3}-x_k)^5-10(x_{k+3}-x_k)^3+15(x_{k+3}-x_k)]+7a_7[(x_{k+3}-x_k)^6-15(x_{k+3}-x_k)^4+45(x_{k+3}-x_k)^2-15]+8a_8[(x_{k+3}-x_k)^7-21(x_{k+3}-x_k)^5+105(x_{k+3}-x_k)^3-105(x_{k+3}-x_k)]+a_9[9(x_{k+3}-x_k)^8-252(x_{k+3}-x_k)^6+1890(x_{k+3}-x_k)^4-2520(x_{k+3}-x_k)+945]=\dot{y}(x_{k+3}) \qquad (2.40)$$

$$f_{k+4} = a_1+2a_2(x_{k+4}-x_k)+3a_3[(x_{k+4}-x_k)^2-1]+4a_4[(x_{k+4}-x_k)^3-3(x_{k+4}-x_k)]+5a_5[(x_{k+4}-x_k)^4-6(x_{k+4}-x_k)^2+3]+6a_6[(x_{k+4}-x_k)^5-10(x_{k+4}-x_k)^3+15(x_{k+4}-x_k)]+7a_7[(x_{k+4}-x_k)^6-15(x_{k+4}-x_k)^4+45(x_{k+4}-x_k)^2-15]+8a_8[(x_{k+4}-x_k)^7-21(x_{k+4}-x_k)^5+105(x_{k+4}-x_k)^3-105(x_{k+4}-x_k)]+a_9[9(x_{k+4}-x_k)^8-252(x_{k+4}-x_k)^6+1890(x_{k+4}-x_k)^4-2520(x_{k+4}-x_k)+945]=\dot{y}(x_{k+4}) \qquad (2.41)$$

$$f_{k+5} = a_1+2a_2(x_{k+5}-x_k)+3a_3[(x_{k+5}-x_k)^2-1]+4a_4[(x_{k+5}-x_k)^3-3(x_{k+5}-x_k)]+5a_5[(x_{k+5}-x_k)^4-6(x_{k+5}-x_k)^2+3]+6a_6[(x_{k+5}-x_k)^5-10(x_{k+5}-x_k)^3+15(x_{k+5}-x_k)]+7a_7[(x_{k+5}-x_k)^6-15(x_{k+5}-x_k)^4+45(x_{k+5}-x_k)^2-15]+8a_8[(x_{k+5}-x_k)^7-21(x_{k+5}-x_k)^5+105(x_{k+5}-x_k)^3-105(x_{k+5}-x_k)]+a_9[9(x_{k+5}-x_k)^8-252(x_{k+5}-x_k)^6+1890(x_{k+5}-x_k)^4-2520(x_{k+5}-x_k)+945]=\dot{y}(x_{k+5}) \qquad (2.42)$$

$$f_{k+6} = a_1+2a_2(x_{k+6}-x_k)+3a_3[(x_{k+6}-x_k)^2-1]+4a_4[(x_{k+6}-x_k)^3-3(x_{k+6}-x_k)]+5a_5[(x_{k+6}-x_k)^4-6(x_{k+6}-x_k)^2+3]+6a_6[(x_{k+6}-x_k)^5-10(x_{k+6}-x_k)^3+15(x_{k+6}-x_k)]+7a_7[(x_{k+6}-x_k)^6-15(x_{k+6}-x_k)^4+45(x_{k+6}-x_k)^2-15]+8a_8[(x_{k+6}-x_k)^7-21(x_{k+6}-x_k)^5+105(x_{k+6}-x_k)^3-105(x_{k+6}-x_k)]+a_9[9(x_{k+6}-x_k)^8-252(x_{k+6}-x_k)^6+1890(x_{k+6}-x_k)^4-2520(x_{k+6}-x_k)+945]=\dot{y}(x_{k+6}) \qquad (2.43)$$

$$f_{k+7} = a_1 + 2a_2(x_{k+7} - x_k) + 3a_3[(x_{k+7} - x_k)^2 - 1] + 4a_4[(x_{k+7} - x_k)^3 - 3(x_{k+7} - x_k)] + 5a_5[(x_{k+7} - x_k)^4 - 6(x_{k+7} - x_k)^2 + 3] + 6a_6[(x_{k+7} - x_k)^5 - 10(x_{k+7} - x_k)^3 + 15(x_{k+7} - x_k)] + 7a_7[(x_{k+7} - x_k)^6 - 15(x_{k+7} - x_k)^4 + 45(x_{k+7} - x_k)^2 - 15] + 8a_8[(x_{k+7} - x_k)^7 - 21(x_{k+7} - x_k)^5 + 105(x_{k+7} - x_k)^3 - 105(x_{k+7} - x_k)] + a_9[9(x_{k+7} - x_k)^8 - 252(x_{k+7} - x_k)^6 + 1890(x_{k+7} - x_k)^4 - 2520(x_{k+7} - x_k) + 945] = \dot{y}(x_{k+7}) \quad (2.44)$$

$$f_{k+8} = a_1 + 2a_2(x_{k+8} - x_k) + 3a_3[(x_{k+8} - x_k)^2 - 1] + 4a_4[(x_{k+8} - x_k)^3 - 3(x_{k+8} - x_k)] + 5a_5[(x_{k+8} - x_k)^4 - 6(x_{k+8} - x_k)^2 + 3] + 6a_6[(x_{k+8} - x_k)^5 - 10(x_{k+8} - x_k)^3 + 15(x_{k+8} - x_k)] + 7a_7[(x_{k+8} - x_k)^6 - 15(x_{k+8} - x_k)^4 + 45(x_{k+8} - x_k)^2 - 15] + 8a_8[(x_{k+8} - x_k)^7 - 21(x_{k+8} - x_k)^5 + 105(x_{k+8} - x_k)^3 - 105(x_{k+8} - x_k)] + a_9[9(x_{k+8} - x_k)^8 - 252(x_{k+8} - x_k)^6 + 1890(x_{k+8} - x_k)^4 - 2520(x_{k+8} - x_k) + 945] = \dot{y}(x_{k+8}) \quad (2.45)$$

In matrix form, we have:

$$\left[\begin{pmatrix} 1 & 7h & (49h^2-1) & (343h^3-21h) & (2401h^4-294h^2+3) & (16807h^5-3430h^3+105h) & (117649h^6-36015h^4+2205h^2-15) \\ 0 & 1 & 0 & -3 & 0 & 15 & 0 \\ 0 & 1 & 2h & (3h^2-3) & (4h^3-12h) & (5h^4-30h^2+15) & (6h^5-60h^3+90h) \\ 0 & 1 & 4h & (12h^2-3) & (32h^3-24h) & (80h^4-120h^2+15) & (192h^5-480h^3+180h) \\ 0 & 1 & 6h & (27h^2-3) & (108h^3-36h) & (405h^4-270h^2+15h) & (145h^5-1620h^3+270h) \\ 0 & 1 & 8h & (48h^2-3) & (256h^3-48h) & (1280h^4-480h^2+15h) & (6144h^5-3840h^3+360h) \\ 0 & 1 & 10h & (75h^2-3) & (500h^3-60h) & (3125h^4-750h^2+15h) & (18750h^5-7500h^3+450h) \\ 0 & 1 & 12h & (108h^2-3) & (864h^3-72h) & (6480h^4-1080h^2+15h) & (46656h^5-12960h^3+540h) \\ 0 & 1 & 14h & (147h^2-3) & (1372h^3-84h) & (12005h^4-1470h^2+15h) & (100842h^5-20580h^3+630h) \\ 0 & 1 & 16h & (192h^2-3) & (2048h^3-96h) & (20480h^4-1920h^2+15h) & (196608h^5-30720h^3+720h) \end{pmatrix} + \begin{pmatrix} (823543h^7-352947h^5+36015h^3-735h) & (5764801h^8-3294172h^6+504210h^4-20580h^2+105) & (40353607h^9-29647548h^7+6353046h^5-61740h^2+6615h) \\ -105 & 0 & 945 \\ (7h^6-105h^4+315h^2-105) & (8h^7-168h^5+840h^3-840h) & (9h^8-252h^6+1890h^4-2520h+945) \\ (448h^6-1680h^4+1260h^2-105) & (1024h^7-5376h^5+6720h^3-1680h) & (2304h^8-16128h^6+30240h^4-5040h+945) \\ (5103h^6-8505h^4+2835h^2-105) & (17496h^7-40824h^5+22680h^3-2520h) & (59049h^8-183708h^6+112590h^4-7560h+945) \\ (28672h^6-26880h^4+5040h^2-105) & (131072h^7-172032h^5+53760h^3-3360h) & (589824h^8-1032192h^6+355840h^4-10080h+945) \\ (109375h^6-65625h^4+7875h^2-105) & (625000h^7-525000h^5+105000h^3-4200h) & (3515625h^8-3937500h^6+868750h^4-12600h+945) \\ (326592h^6-136080h^4+11340h^2-105) & (2239488h^7-1306368h^5+181440h^3-5040h) & (15116544h^8-11757312h^6+1801440h^4-15120h+945) \\ (823543h^6-252105h^4+15435h^2-105) & (6588344h^7-2823576h^5+288120h^3-5880h) & (51883209h^8-29647548h^6+3337390h^4-17640h+945) \\ (1835008h^6-430080h^4+20160h^2-105) & (16777216h^7-5505024h^5+4.30080h^3-6720h) & (150994944h^8-66060288h^6+5693440h^4-20160h+945) \end{pmatrix}\right]$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} y_{k+7} \\ f_k \\ f_{k+1} \\ f_{k+2} \\ f_{k+3} \\ f_{k+4} \\ f_{k+5} \\ f_{k+6} \\ f_{k+7} \\ f_{k+8} \end{bmatrix} \quad (2.46)$$

Solving the system of equation (2.46) above by Gaussian elimination method

$$a_0 = y_{k+7} - h\left(\left(\frac{149527}{518400}\right)f_k + \left(\frac{408317}{259200}\right)f_{k+1} + \left(\frac{542969}{259200}\right)f_{k+3} - \left(\frac{368039}{259200}\right)f_{k+5} - \left(\frac{343}{3240}\right)f_{k+4} - \left(\frac{24353}{259200}\right)f_{k+2} - \left(\frac{111587}{259200}\right)f_{k+7} - \left(\frac{261023}{259200}\right)f_{k+6} + \left(\frac{8183}{518400}\right)f_{k+8}\right) - \frac{1}{h}\left(\left(\frac{761}{560}\right)f_k - 4f_{k+1} + 7f_{k+2} - \left(\frac{28}{3}\right)f_{k+3} + \left(\frac{35}{4}\right)f_{k+4} - \left(\frac{28}{5}\right)f_{k+5} + \left(\frac{7}{3}\right)f_{k+6} - \left(\frac{4}{7}\right)f_{k+7} + \left(\frac{1}{16}\right)f_{k+8}\right) - \frac{1}{h^3}\left(\left(\frac{801}{640}\right)f_k - \left(\frac{349}{48}\right)f_{k+1} + \left(\frac{18353}{960}\right)f_{k+2} - \left(\frac{2391}{80}\right)f_{k+3} + \left(\frac{1457}{48}\right)f_{k+4} - \left(\frac{4891}{240}\right)f_{k+5} + \left(\frac{561}{64}\right)f_{k+6} - \right.$$

$$\left(\frac{527}{240}\right)f_{k+7}+\left(\frac{469}{1920}\right)f_{k+8}\bigg)-\frac{1}{h^5}\bigg(\left(\frac{9}{32}\right)f_k-\left(\frac{575}{288}\right)f_{k+1}+\left(\frac{895}{144}\right)f_{k+2}-\left(\frac{355}{32}\right)f_{k+3}+\left(\frac{895}{72}\right)f_{k+4}-\left(\frac{2581}{288}\right)f_{k+5}+\left(\frac{65}{16}\right)f_{k+6}-\left(\frac{305}{288}\right)f_{k+7}+\left(\frac{35}{288}\right)f_{k+8}\bigg)-\frac{1}{h^7}\bigg(\left(\frac{3}{256}\right)f_k-\left(\frac{35}{384}\right)f_{k+1}+\left(\frac{119}{384}\right)f_{k+2}-\left(\frac{77}{128}\right)f_{k+3}+\left(\frac{35}{48}\right)f_{k+4}-\left(\frac{217}{384}\right)f_{k+5}+\left(\frac{35}{128}\right)f_{k+6}-\left(\frac{29}{384}\right)f_{k+7}+\left(\frac{7}{768}\right)f_{k+8}\bigg)+\frac{1}{h^8}\bigg(\left(\frac{1}{288}\right)f_k-\left(\frac{1}{36}\right)f_{k+1}+\left(\frac{7}{72}\right)f_{k+2}-\left(\frac{7}{36}\right)f_{k+3}+\left(\frac{35}{144}\right)f_{k+4}-\left(\frac{7}{36}\right)f_{k+5}+\left(\frac{7}{72}\right)f_{k+6}-\left(\frac{1}{36}\right)f_{k+7}+\left(\frac{1}{288}\right)f_{k+8}\bigg) \tag{2.47}$$

$$a_1=f_k+\frac{1}{h^2}\bigg(\left(\frac{363}{1120}\right)f_{k+8}-\left(\frac{103}{35}\right)f_{k+7}+\left(\frac{2143}{180}\right)f_{k+6}-\left(\frac{141}{5}\right)f_{k+5}+\left(\frac{691}{16}\right)f_{k+4}-\left(\frac{2003}{45}\right)f_{k+3}+\left(\frac{621}{20}\right)f_{k+2}-\left(\frac{481}{35}\right)f_{k+1}+\left(\frac{29531}{10080}\right)f_k\bigg)+\frac{1}{h^4}\bigg(\left(\frac{967}{1920}\right)f_{k+8}-\left(\frac{67}{15}\right)f_{k+7}+\left(\frac{2803}{160}\right)f_{k+6}-\left(\frac{1193}{30}\right)f_{k+5}+\left(\frac{10993}{192}\right)f_{k+4}-\left(\frac{268}{5}\right)f_{k+3}+\left(\frac{15289}{480}\right)f_{k+2}-\left(\frac{329}{30}\right)f_{k+1}+\left(\frac{1069}{640}\right)f_k\bigg)+\frac{1}{h^6}\bigg(\left(\frac{23}{192}\right)f_{k+8}-\left(\frac{49}{48}\right)f_{k+7}+\left(\frac{61}{16}\right)f_{k+6}-\left(\frac{391}{48}\right)f_{k+5}+\left(\frac{1045}{96}\right)f_{k+4}-\left(\frac{149}{16}\right)f_{k+3}+\left(\frac{239}{48}\right)f_{k+2}-\left(\frac{73}{48}\right)f_{k+1}+\left(\frac{13}{64}\right)f_k\bigg)-\frac{1}{h^8}\bigg(\left(\frac{89}{24192}\right)f_{k+8}-\left(\frac{89}{3024}\right)f_{k+7}+\left(\frac{89}{864}\right)f_{k+6}-\left(\frac{89}{432}\right)f_{k+5}+\left(\frac{445}{1728}\right)f_{k+4}-\left(\frac{89}{432}\right)f_{k+3}+\left(\frac{89}{864}\right)f_{k+2}-\left(\frac{89}{3024}\right)f_{k+1}+\left(\frac{89}{24192}\right)f_k\bigg) \tag{2.48}$$

$$a_2=-\frac{1}{h}\bigg(\left(\frac{1}{16}\right)f_{k+8}-\left(\frac{4}{7}\right)f_{k+7}+\left(\frac{7}{3}\right)f_{k+6}-\left(\frac{28}{5}\right)f_{k+5}+\left(\frac{35}{4}\right)f_{k+4}-\left(\frac{28}{3}\right)f_{k+3}+7f_{k+2}-4f_{k+1}+\left(\frac{761}{560}\right)f_k\bigg)-\frac{1}{h^3}\bigg(\left(\frac{469}{960}\right)f_{k+8}-\left(\frac{527}{120}\right)f_{k+7}+\left(\frac{561}{32}\right)f_{k+6}-\left(\frac{4891}{120}\right)f_{k+5}+\left(\frac{1457}{24}\right)f_{k+4}-\left(\frac{2391}{40}\right)f_{k+3}+\left(\frac{18353}{480}\right)f_{k+2}-\left(\frac{349}{24}\right)f_{k+1}+\left(\frac{801}{320}\right)f_k\bigg)-\frac{1}{h^5}\bigg(\left(\frac{35}{96}\right)f_{k+8}-\left(\frac{305}{96}\right)f_{k+7}+\left(\frac{195}{16}\right)f_{k+6}-\left(\frac{2581}{96}\right)f_{k+5}+\left(\frac{895}{24}\right)f_{k+4}-\left(\frac{1065}{32}\right)f_{k+3}+\left(\frac{895}{48}\right)f_{k+2}-\left(\frac{575}{96}\right)f_{k+1}+\left(\frac{27}{32}\right)f_k\bigg)-\frac{1}{h^7}\bigg(\left(\frac{7}{192}\right)f_{k+8}-\left(\frac{29}{96}\right)f_{k+7}+\left(\frac{35}{32}\right)f_{k+6}-\left(\frac{217}{96}\right)f_{k+5}+\left(\frac{35}{12}\right)f_{k+4}-\left(\frac{77}{32}\right)f_{k+3}+\left(\frac{119}{96}\right)f_{k+2}-\left(\frac{35}{96}\right)f_{k+1}+\left(\frac{3}{64}\right)f_k\bigg)+\frac{1}{h^8}\bigg(\left(\frac{1}{288}\right)f_{k+8}-\left(\frac{1}{36}\right)f_{k+7}+\left(\frac{7}{72}\right)f_{k+6}-\left(\frac{7}{36}\right)f_{k+5}+\left(\frac{35}{144}\right)f_{k+4}-\left(\frac{7}{36}\right)f_{k+3}+\left(\frac{7}{72}\right)f_{k+2}-\left(\frac{1}{36}\right)f_{k+1}+\left(\frac{1}{288}\right)f_k\bigg) \tag{2.49}$$

$$a_3=\frac{1}{h^2}\bigg(\left(\frac{121}{1120}\right)f_{k+8}-\left(\frac{103}{105}\right)f_{k+7}+\left(\frac{2143}{540}\right)f_{k+6}-\left(\frac{47}{5}\right)f_{k+5}+\left(\frac{691}{48}\right)f_{k+4}-\left(\frac{2003}{135}\right)f_{k+3}+\left(\frac{207}{20}\right)f_{k+2}-\left(\frac{481}{105}\right)f_{k+1}+\left(\frac{29531}{30240}\right)f_k\bigg)+\frac{1}{h^4}\bigg(\left(\frac{967}{2880}\right)f_{k+8}-\left(\frac{134}{45}\right)f_{k+7}+\left(\frac{2803}{240}\right)f_{k+6}-\left(\frac{1193}{45}\right)f_{k+5}+\left(\frac{10993}{288}\right)f_{k+4}-\left(\frac{536}{15}\right)f_{k+3}+\left(\frac{15289}{720}\right)f_{k+2}-\left(\frac{329}{45}\right)f_{k+1}+\left(\frac{1069}{960}\right)f_k\bigg)+\frac{1}{h^6}\bigg(\left(\frac{23}{192}\right)f_{k+8}-\left(\frac{49}{48}\right)f_{k+7}+\left(\frac{61}{16}\right)f_{k+6}-\left(\frac{391}{48}\right)f_{k+5}+\left(\frac{1045}{96}\right)f_{k+4}-\left(\frac{149}{16}\right)f_{k+3}+\left(\frac{239}{48}\right)f_{k+2}-$$

$$\left(\frac{73}{48}\right) f_{k+1} + \left(\frac{13}{64}\right) f_k\Bigg) + \frac{1}{h^8}\Bigg(\left(\frac{25}{9072}\right) f_{k+8} - \left(\frac{25}{1134}\right) f_{k+7} + \left(\frac{25}{324}\right) f_{k+6} - \left(\frac{25}{162}\right) f_{k+5} + \left(\frac{125}{648}\right) f_{k+4} - \left(\frac{25}{162}\right) f_{k+3} + \left(\frac{25}{324}\right) f_{k+2} - \left(\frac{25}{1134}\right) f_{k+1} + \left(\frac{25}{9072}\right) f_k\Bigg) \qquad (2.50)$$

$$a_4 = -\frac{1}{h^3}\Bigg(\left(\frac{469}{5760}\right) f_{k+8} - \left(\frac{527}{720}\right) f_{k+7} + \left(\frac{187}{64}\right) f_{k+6} - \left(\frac{4891}{720}\right) f_{k+5} + \left(\frac{1457}{144}\right) f_{k+4} - \left(\frac{797}{80}\right) f_{k+3} + \left(\frac{18353}{2880}\right) f_{k+2} - \left(\frac{349}{144}\right) f_{k+1} + \left(\frac{267}{640}\right) f_k\Bigg) - \frac{1}{h^5}\Bigg(\left(\frac{35}{288}\right) f_{k+8} - \left(\frac{305}{288}\right) f_{k+7} + \left(\frac{65}{16}\right) f_{k+6} - \left(\frac{2581}{288}\right) f_{k+5} + \left(\frac{895}{72}\right) f_{k+4} - \left(\frac{355}{32}\right) f_{k+3} + \left(\frac{895}{144}\right) f_{k+2} - \left(\frac{575}{288}\right) f_{k+1} + \left(\frac{9}{32}\right) f_k\Bigg) - \frac{1}{h^7}\Bigg(\left(\frac{7}{384}\right) f_{k+8} - \left(\frac{29}{192}\right) f_{k+7} + \left(\frac{35}{64}\right) f_{k+6} - \left(\frac{217}{192}\right) f_{k+5} + \left(\frac{35}{24}\right) f_{k+4} - \left(\frac{77}{64}\right) f_{k+3} + \left(\frac{119}{192}\right) f_{k+2} - \left(\frac{35}{192}\right) f_{k+1} + \left(\frac{3}{128}\right) f_k\Bigg) \qquad (2.51)$$

$$a_5 = \frac{1}{h^4}\Bigg(\left(\frac{967}{28800}\right) f_{k+8} - \left(\frac{67}{225}\right) f_{k+7} + \left(\frac{2803}{2400}\right) f_{k+6} - \left(\frac{1193}{450}\right) f_{k+5} + \left(\frac{10993}{2880}\right) f_{k+4} - \left(\frac{268}{75}\right) f_{k+3} + \left(\frac{15289}{7200}\right) f_{k+2} - \left(\frac{329}{450}\right) f_{k+1} + \left(\frac{1069}{9600}\right) f_k\Bigg) + \frac{1}{h^6}\Bigg(\left(\frac{23}{960}\right) f_{k+8} - \left(\frac{49}{240}\right) f_{k+7} + \left(\frac{61}{80}\right) f_{k+6} - \left(\frac{391}{240}\right) f_{k+5} + \left(\frac{209}{96}\right) f_{k+4} - \left(\frac{149}{80}\right) f_{k+3} + \left(\frac{239}{240}\right) f_{k+2} - \left(\frac{73}{240}\right) f_{k+1} + \left(\frac{13}{320}\right) f_k\Bigg) + \frac{1}{h^8}\Bigg(\left(\frac{239}{181440}\right) f_{k+8} - \left(\frac{239}{22680}\right) f_{k+7} + \left(\frac{239}{6480}\right) f_{k+6} - \left(\frac{239}{3240}\right) f_{k+5} + \left(\frac{239}{2592}\right) f_{k+4} - \left(\frac{239}{3240}\right) f_{k+3} + \left(\frac{239}{6480}\right) f_{k+2} - \left(\frac{239}{22680}\right) f_{k+1} + \left(\frac{239}{181440}\right) f_k\Bigg) \qquad (2.52)$$

$$a_6 = -\frac{1}{h^5}\Bigg(\left(\frac{3}{160}\right) f_k - \left(\frac{115}{864}\right) f_{k+1} + \left(\frac{179}{432}\right) f_{k+2} - \left(\frac{71}{96}\right) f_{k+3} + \left(\frac{179}{216}\right) f_{k+4} - \left(\frac{2581}{4320}\right) f_{k+5} + \left(\frac{13}{48}\right) f_{k+6} - \left(\frac{61}{864}\right) f_{k+7} + \left(\frac{7}{864}\right) f_{k+8}\Bigg) - \frac{1}{h^7}\Bigg(\left(\frac{1}{320}\right) f_k - \left(\frac{7}{288}\right) f_{k+1} + \left(\frac{119}{1440}\right) f_{k+2} - \left(\frac{77}{480}\right) f_{k+3} + \left(\frac{7}{36}\right) f_{k+4} - \left(\frac{217}{1440}\right) f_{k+5} + \left(\frac{7}{96}\right) f_{k+6} - \left(\frac{29}{1440}\right) f_{k+7} + \left(\frac{7}{2880}\right) f_{k+8}\Bigg) \qquad (2.53)$$

$$a_7 = \frac{1}{h^6}\Bigg(\left(\frac{13}{6720}\right) f_k - \left(\frac{73}{5040}\right) f_{k+1} + \left(\frac{239}{5040}\right) f_{k+2} - \left(\frac{149}{1680}\right) f_{k+3} \left(\frac{209}{2016}\right) f_{k+4} - \left(\frac{391}{5040}\right) f_{k+5} + \left(\frac{61}{1680}\right) f_{k+6} - \left(\frac{7}{720}\right) f_{k+7} + \left(\frac{23}{20160}\right) f_{k+8}\Bigg) + \frac{1}{h^8}\Bigg(+ \left(\frac{1}{10080}\right) f_k - \left(\frac{1}{1260}\right) f_{k+1} + \left(\frac{1}{360}\right) f_{k+2} - \left(\frac{1}{180}\right) f_{k+3} + \left(\frac{1}{144}\right) f_{k+4} - \left(\frac{1}{180}\right) f_{k+5} + \left(\frac{1}{360}\right) f_{k+6} - \left(\frac{1}{1260}\right) f_{k+7} + \left(\frac{1}{10080}\right) f_{k+8}\Bigg) \qquad (2.54)$$

$$a_8 = -\frac{1}{h^7}\Bigg(\left(\frac{1}{8960}\right) f_k - \left(\frac{1}{1152}\right) f_{k+1} + \left(\frac{17}{5760}\right) f_{k+2} - \left(\frac{11}{1920}\right) f_{k+3} + \left(\frac{1}{144}\right) f_{k+4} - \left(\frac{31}{5760}\right) f_{k+5} + \left(\frac{1}{384}\right) f_{k+6} - \left(\frac{29}{40320}\right) f_{k+7} + \left(\frac{1}{11520}\right) f_{k+8}\Bigg) \qquad (2.55)$$

$$a_9 = \frac{1}{h^8}\left(\left(\frac{1}{362880}\right) f_k - \left(\frac{1}{45360}\right) f_{k+1} + \left(\frac{1}{12960}\right) f_{k+2} - \left(\frac{1}{6480}\right) f_{k+3} + \left(\frac{1}{5184}\right) f_{k+4} - \left(\frac{1}{6480}\right) f_{k+5} + \left(\frac{1}{12960}\right) f_{k+6} - \left(\frac{1}{45360}\right) f_{k+7} + \left(\frac{1}{362880}\right) f_{k+8}\right) \quad (2.56)$$

Substituting for a_j = j =0,1,2,3,4,5,6,7,8,9 in equation (2.34) yields the continuous method

$$y(x) = y_{k+7} - h\left(-\left(\frac{24353}{259200}\right) f_{k+2} - \left(\frac{542969}{259200}\right) f_{k+3} - \left(\frac{343}{3240}\right) f_{k+4} - \left(\frac{368039}{259200}\right) f_{k+5} - \left(\frac{261023}{259200}\right) f_{k+6} - \left(\frac{111587}{259200}\right) f_{k+7} + \left(\frac{8183}{518400}\right) f_{k+8} - \left(\frac{149527}{518400}\right) f_k - \left(\frac{408317}{259200}\right) f_{k+1}\right) + (x - x_k) f_k + \frac{(x-x_k)^2}{h}\left(\left(\frac{28}{3}\right) f_{k+3} + \left(\frac{28}{5}\right) f_{k+5} - \left(\frac{1}{16}\right) f_{k+8} - \left(\frac{35}{4}\right) f_{k+4} - \left(\frac{7}{3}\right) f_{k+6} - 7 f_{k+2} + 4 f_{k+1} - \left(\frac{761}{560}\right) f_k + \left(\frac{4}{7}\right) f_{k+7}\right) - \frac{(x-x_k)^3}{h^2}\left(-\left(\frac{47}{5}\right) f_{k+5} - \left(\frac{103}{105}\right) f_{k+7} + \left(\frac{2143}{540}\right) f_{k+6} + \left(\frac{29531}{30240}\right) f_k - \left(\frac{2003}{135}\right) f_{k+3} + \left(\frac{121}{1120}\right) f_{k+8} - \left(\frac{481}{105}\right) f_{k+1} + \left(\frac{207}{20}\right) f_{k+2} + \left(\frac{691}{48}\right) f_{k+4}\right) + \frac{(x-x_k)^4}{h^3}\left(-\left(\frac{187}{64}\right) f_{k+6} - \left(\frac{18353}{2880}\right) f_{k+2} - \left(\frac{267}{640}\right) f_k + \left(\frac{4891}{720}\right) f_{k+5} + \left(\frac{527}{720}\right) f_{k+7} + \left(\frac{349}{144}\right) f_{k+1} - \left(\frac{1457}{144}\right) f_{k+4} + \left(\frac{797}{80}\right) f_{k+3} - \left(\frac{469}{5760}\right) f_{k+8}\right) - \frac{(x-x_k)^5}{h^4}\left(\left(\frac{15289}{7200}\right) f_{k+2} + \left(\frac{1069}{9600}\right) f_k + \left(\frac{967}{28800}\right) f_{k+8} - \left(\frac{67}{225}\right) f_{k+7} - \left(\frac{1193}{450}\right) f_{k+5} + \left(\frac{10993}{2880}\right) f_{k+4} + \left(\frac{2803}{2400}\right) f_{k+6} - \left(\frac{329}{450}\right) f_{k+1} - \left(\frac{268}{75}\right) f_{k+3}\right) + \frac{(x-x_k)^6}{h^5}\left(\left(\frac{71}{96}\right) f_{k+3} - \left(\frac{179}{216}\right) f_{k+4} + \left(\frac{2581}{4320}\right) f_{k+5} - \left(\frac{13}{48}\right) f_{k+6} + \left(\frac{61}{864}\right) f_{k+7} - \left(\frac{7}{864}\right) f_{k+8} - \left(\frac{3}{160}\right) f_k + \left(\frac{115}{864}\right) f_{k+1} - \left(\frac{179}{432}\right) f_{k+2}\right) - \frac{(x-x_k)^7}{h^6}\left(\left(\frac{209}{2016}\right) f_{k+4} + \left(\frac{23}{20160}\right) f_{k+8} - \left(\frac{391}{5040}\right) f_{k+5} + \left(\frac{61}{1680}\right) f_{k+6} - \left(\frac{7}{720}\right) f_{k+7} - \left(\frac{149}{1680}\right) f_{k+3} + \left(\frac{13}{6720}\right) f_k - \left(\frac{73}{5040}\right) f_{k+1} + \left(\frac{239}{5040}\right) f_{k+2}\right) + \frac{(x-x_k)^8}{h^7}\left(-\left(\frac{1}{8960}\right) f_k + \left(\frac{1}{1152}\right) f_{k+1} - \left(\frac{17}{5760}\right) f_{k+2} + \left(\frac{11}{1920}\right) f_{k+3} + \left(\frac{29}{40320}\right) f_{k+7} - \left(\frac{1}{384}\right) f_{k+6} - \left(\frac{1}{11520}\right) f_{k+8} - \left(\frac{1}{144}\right) f_{k+4} + \left(\frac{31}{5760}\right) f_{k+5}\right) + \frac{(x-x_k)^9}{h^8}\left(\left(\frac{1}{362880}\right) f_k - \left(\frac{1}{45360}\right) f_{k+1} + \left(\frac{1}{12960}\right) f_{k+2} - \left(\frac{1}{6480}\right) f_{k+3} + \left(\frac{1}{362880}\right) f_{k+8} - \left(\frac{1}{45360}\right) f_{k+7} + \left(\frac{1}{5184}\right) f_{k+4} - \left(\frac{1}{6480}\right) f_{k+5} + \left(\frac{1}{12960}\right) f_{k+6}\right) \quad (2.57)$$

Evaluating equation (2.57) at $x = y_{k+8}$, we have obtain the discrete form as:

$$y_{k+8} = y_{k+7} + h\left(-\left(\frac{33953}{3628800}\right) f_k + \left(\frac{156437}{1814400}\right) f_{k+1} - \left(\frac{645607}{1814400}\right) f_{k+2} + \left(\frac{1573169}{1814400}\right) f_{k+3} - \left(\frac{31457}{22680}\right) f_{k+4} + \left(\frac{2797679}{1814400}\right) f_{k+5} - \left(\frac{2302297}{1814400}\right) f_{k+6} + \left(\frac{2233547}{1814400}\right) f_{k+7} + \left(\frac{1070017}{3628800}\right) f_{k+8}\right) \quad (2.58)$$

The Eq. (2.58) is the eight-step Adams-Moulton method.

2.4 Eight-step optimal order method

The optimal scheme is an implicit multi step method similar to Adams-Moulton method. To derive the eight-step optimal method, we shall consider the system of equation in Eq. [2.34 - 2.45] except for $y(x_{k+7})$. Interpolating equation (2.34) at $x = x_{k+6}$, we have

$$y(x_{k+6}) = a_0 + a_1(x_{k+6} - x_k) + a_2((x_{k+6} - x_k)^2 - 1) + a_3((x_{k+6} - x_k)^3 - 3(x_{k+6} - x_k)) + a_4((x_{k+6} - x_k)^4 - 6(x_{k+6} - x_k)^2 + 3) + a_5((x_{k+6} - x_k)^5 - 10(x_{k+6} - x_k)^3 + 15(x_{k+6} - x_k)) + a_6((x_{k+6} - x_k)^6 - 15(x_{k+6} - x_k)^4 + 45(x_{k+6} - x_k)^2 - 15) + a_7((x_{k+6} - x_k)^7 - 21(x_{k+6} - x_k)^5 + 105(x_{k+6} - x_k)^3 - 105(x_{k+6} - x_k)) + a_8((x_{k+6} - x_k)^8 - 28(x_{k+6} - x_k)^6 + 210(x_{k+6} - x_k)^4 - 420(x_{k+6} - x_k)^2 + 105) + a_9((x_{k+6} - x_k)^9 - 36(x_{k+6} - x_k)^7 + 378(x_{k+6} - x_k)^5 - 1260(x_{k+6} - x_k)^2 + 945(x_{k+6} - x_k)) \quad (2.59)$$

The corresponding matrix of the equation is

$$\left[\begin{pmatrix} 1 & 6h & (36h^2-1) & (216h^3-18h) & (1296h^4-216h^2+3) & (7776h^5-2160h^3+90h) & (46656h^6-19440h^4+1620h^2-15) \\ 0 & 1 & 0 & -3 & 0 & 15 & 0 \\ 0 & 1 & 2h & (3h^2-3) & (4h^3-12h) & (5h^4-30h^2+15) & (6h^5-60h^3+90h) \\ 0 & 1 & 4h & (12h^2-3) & (32h^3-24h) & (80h^4-120h^2+15) & (192h^5-480h^3+180h) \\ 0 & 1 & 6h & (27h^2-3) & (108h^3-36h) & (405h^4-270h^2+15h) & (145h^5-1620h^3+270h) \\ 0 & 1 & 8h & (48h^2-3) & (256h^3-48h) & (1280h^4-480h^2+15h) & (6144h^5-3840h^3+360h) \\ 0 & 1 & 10h & (75h^2-3) & (500h^3-60h) & (3125h^4-750h^2+15h) & (18750h^5-7500h^3+450h) \\ 0 & 1 & 12h & (108h^2-3) & (864h^3-72h) & (6480h^4-1080h^2+15h) & (46656h^5-12960h^3+540h) \\ 0 & 1 & 14h & (147h^2-3) & (1372h^3-84h) & (12005h^4-1470h^2+15h) & (100842h^5-20580h^3+630h) \\ 0 & 1 & 16h & (192h^2-3) & (2048h^3-96h) & (20480h^4-1920h^2+15h) & (196608h^5-30720h^3+720h) \end{pmatrix} + \right.$$

$$\left.\begin{pmatrix} (279936h^7-163296h^5+22680h^3-630h) & (1679616h^8-1306368h^6+272160h^4-15120h^2+105) & (10077696h^9-10077696h^7+2161728h^5-45360h^2+5670h) \\ -105 & 0 & 945 \\ (7h^6-105h^4+315h^2-105) & (8h^7-168h^5+840h^3-840h) & (9h^8-252h^6+1390h^4-2520h+945) \\ (448h^6-1680h^4+1260h^2-105) & (1024h^7-5376h^5+6720h^3-1680h) & (2304h^8-16128h^6+22240h^4-5040h+945) \\ (5103h^6-8505h^4+2835h^2-105) & (17496h^7-40824h^5+22680h^3-2520h) & (59049h^8-183708h^6+112590h^4-7560h+945) \\ (28672h^6-26880h^4+5040h^2-105) & (131072h^7-172032h^5+53760h^3-3360h) & (589824h^8-1032192h^6+355840h^4-10080h+945) \\ (109375h^6-65625h^4+7875h^2-105) & (625000h^7-525000h^5+105000h^3-4200h) & (3515625h^8-3937500h^6+868750h^4-12600h+945) \\ (326592h^6-136080h^4+11340h^2-105) & (2239488h^7-1306368h^5+181440h^3-5040h) & (15116544h^8-11757312h^6+1801440h^4-15120h+945) \\ (823543h^6-252105h^4+15435h^2-105) & (6588344h^7-2823576h^5+288120h^3-5880h) & (51883209h^8-29647548h^6+3337390h^4-17640h+945) \\ (1835008h^6-430080h^4+20160h^2-105) & (16777216h^7-5505024h^5+4.30080h^3-6720h) & (150994944h^8-66060288h^6+5693440h^4-20160h+945) \end{pmatrix}\right]$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \\ a_9 \end{bmatrix} = \begin{bmatrix} y_{k+6} \\ f_k \\ f_{k+1} \\ f_{k+2} \\ f_{k+3} \\ f_{k+4} \\ f_{k+5} \\ f_{k+6} \\ f_{k+7} \\ f_{k+8} \end{bmatrix} \quad (2.60)$$

Solving the system of equation by Gaussian's elimination methods gives the same result in equation (2.46 – 2.59) above except for a_0. Now, we have a_0 to be

$$a_0 = y_{k+6} - h\left(-\left(\frac{333}{175}\right)f_{k+5} - \left(\frac{79}{700}\right)f_{k+6} - \left(\frac{9}{175}\right)f_{k+7} + \left(\frac{9}{1400}\right)f_{k+8} - \left(\frac{401}{1400}\right)f_k - \left(\frac{279}{175}\right)f_{k+1} - \left(\frac{9}{700}\right)f_{k+2} - \left(\frac{403}{175}\right)f_{k+3} + \left(\frac{9}{35}\right)f_{k+4}\right) - \frac{1}{h}\left(\left(\frac{4}{7}\right)f_{k+7} + 4f_{k+1} - 7f_{k+2} + \left(\frac{28}{3}\right)f_{k+3} - \left(\frac{761}{560}\right)f_k + \left(\frac{28}{5}\right)f_{k+5} - \left(\frac{7}{3}\right)f_{k+6} - \left(\frac{1}{16}\right)f_{k+8} - \left(\frac{35}{4}\right)f_{k+4}\right) - \frac{1}{h^3}\left(-\left(\frac{561}{64}\right)f_{k+6} + \left(\frac{527}{240}\right)f_{k+7} + \left(\frac{2391}{80}\right)f_{k+3} + \left(\frac{4891}{240}\right)f_{k+5} - \left(\frac{801}{640}\right)f_k - \left(\frac{469}{1920}\right)f_{k+8} + \left(\frac{349}{48}\right)f_{k+1} - \left(\frac{1457}{48}\right)f_{k+4} - \right.$$

$$\left(\frac{18353}{960}\right) f_{k+2}\Big) - \frac{1}{h^5}\Big(-\left(\frac{895}{144}\right) f_{k+2} + \left(\frac{2581}{288}\right) f_{k+5} - \left(\frac{35}{288}\right) f_{k+8} + \left(\frac{575}{288}\right) f_{k+1} + \left(\frac{355}{32}\right) f_{k+3} - \left(\frac{65}{16}\right) f_{k+6} + \left(\frac{305}{288}\right) f_{k+7} - \left(\frac{9}{32}\right) f_k - \left(\frac{895}{72}\right) f_{k+4}\Big) - \frac{1}{h^7}\Big(-\left(\frac{35}{128}\right) f_{k+6} + \left(\frac{29}{384}\right) f_{k+7} - \left(\frac{119}{384}\right) f_{k+2} + \left(\frac{217}{384}\right) f_{k+5} - \left(\frac{7}{768}\right) f_{k+8} + \left(\frac{35}{384}\right) f_{k+1} + \left(\frac{77}{128}\right) f_{k+3} - \left(\frac{35}{48}\right) f_{k+4} - \left(\frac{3}{256}\right) f_k\Big) + \frac{1}{h^8}\Big(-\left(\frac{7}{36}\right) f_{k+3} + \left(\frac{7}{72}\right) f_{k+6} + \left(\frac{1}{288}\right) f_{k+8} + \left(\frac{7}{72}\right) f_{k+2} + \left(\frac{35}{144}\right) f_{k+4} - \left(\frac{1}{36}\right) f_{k+7} - \left(\frac{1}{36}\right) f_{k+1} + \left(\frac{1}{288}\right) f_k - \left(\frac{7}{36}\right) f_{k+5}\Big) \qquad (2.61)$$

Substituting for $a_j, j = 0,1,2,3,4,5,6,7,8,9$ in equation (2.61) yields the continuous eight-step optimal order method:

$$y(x) = y_{k+6} - h\Big(\left(\frac{9}{1400}\right) f_{k+8} - \left(\frac{401}{1400}\right) f_k - \left(\frac{279}{175}\right) f_{k+1} - \left(\frac{9}{700}\right) f_{k+2} - \left(\frac{403}{175}\right) f_{k+3} + \left(\frac{9}{35}\right) f_{k+4} - \left(\frac{333}{175}\right) f_{k+5} - \left(\frac{79}{700}\right) f_{k+6} - \left(\frac{9}{175}\right) f_{k+7}\Big) + (x - x_k) f_k + \frac{(x-x_k)^2}{h}\Big(-7 f_{k+2} + \left(\frac{28}{3}\right) f_{k+3} - \left(\frac{761}{560}\right) f_k + 4 f_{k+1} + \left(\frac{4}{7}\right) f_{k+7} - \left(\frac{35}{4}\right) f_{k+4} + \left(\frac{28}{5}\right) f_{k+5} - \left(\frac{7}{3}\right) f_{k+6} - \left(\frac{1}{16}\right) f_{k+8}\Big) - \frac{(x-x_k)^3}{h^2}\Big(-\left(\frac{2003}{135}\right) f_{k+3} + \left(\frac{207}{20}\right) f_{k+2} + \left(\frac{29531}{30240}\right) f_k - \left(\frac{481}{105}\right) f_{k+1} + \left(\frac{2143}{540}\right) f_{k+6} - \left(\frac{103}{105}\right) f_{k+7} + \left(\frac{691}{48}\right) f_{k+4} - \left(\frac{47}{5}\right) f_{k+5} + \left(\frac{121}{1120}\right) f_{k+8}\Big) + \frac{(x-x_k)^4}{h^3}\Big(-\left(\frac{187}{64}\right) f_{k+6} + \left(\frac{797}{80}\right) f_{k+3} - \left(\frac{1457}{144}\right) f_{k+4} + \left(\frac{349}{144}\right) f_{k+1} - \left(\frac{267}{640}\right) f_k + \left(\frac{527}{720}\right) f_{k+7} - \left(\frac{469}{5760}\right) f_{k+8} + \left(\frac{4891}{720}\right) f_{k+5} - \left(\frac{18353}{2880}\right) f_{k+2}\Big) - \frac{(x-x_k)^5}{h^4}\Big(-\left(\frac{67}{225}\right) f_{k+7} + \left(\frac{1069}{9600}\right) f_k - \left(\frac{329}{450}\right) f_{k+1} + \left(\frac{15289}{7200}\right) f_{k+2} + \left(\frac{2803}{2400}\right) f_{k+6} - \left(\frac{268}{75}\right) f_{k+3} + \left(\frac{10993}{2880}\right) f_{k+4} - \left(\frac{1193}{450}\right) f_{k+5} + \left(\frac{967}{28800}\right) f_{k+8}\Big) + \frac{(x-x_k)^6}{h^5}\Big(\left(\frac{71}{96}\right) f_{k+3} - \left(\frac{179}{432}\right) f_{k+2} - \left(\frac{3}{160}\right) f_k + \left(\frac{115}{864}\right) f_{k+1} - \left(\frac{13}{48}\right) f_{k+6} + \left(\frac{61}{864}\right) f_{k+7} - \left(\frac{179}{216}\right) f_{k+4} + \left(\frac{2581}{4320}\right) f_{k+5} - \left(\frac{7}{864}\right) f_{k+8}\Big) - \frac{(x-x_k)^7}{h^6}\Big(\left(\frac{61}{1680}\right) f_{k+6} - \left(\frac{149}{1680}\right) f_{k+3} + \left(\frac{209}{2016}\right) f_{k+4} - \left(\frac{73}{5040}\right) f_{k+1} + \left(\frac{13}{6720}\right) f_k - \left(\frac{7}{720}\right) f_{k+7} + \left(\frac{23}{20160}\right) f_{k+8} - \left(\frac{391}{5040}\right) f_{k+5} + \left(\frac{239}{5040}\right) f_{k+2}\Big) + \frac{(x-x_k)^8}{h^7}\Big(-\left(\frac{17}{5760}\right) f_{k+2} - \left(\frac{1}{8960}\right) f_k + \left(\frac{1}{1152}\right) f_{k+1} - \left(\frac{1}{11520}\right) f_{k+8} + \left(\frac{29}{40320}\right) f_{k+7} + \left(\frac{11}{1920}\right) f_{k+3} + \left(\frac{31}{5760}\right) f_{k+5} - \left(\frac{1}{384}\right) f_{k+6} - \left(\frac{1}{144}\right) f_{k+4}\Big) + \frac{(x-x_k)^9}{h^8}\Big(-\left(\frac{1}{6480}\right) f_{k+3} + \left(\frac{1}{362880}\right) f_{k+8} - \left(\frac{1}{45360}\right) f_{k+1} + \left(\frac{1}{12960}\right) f_{k+2} - \left(\frac{1}{45360}\right) f_{k+7} + \left(\frac{1}{5184}\right) f_{k+4} - \left(\frac{1}{6480}\right) f_{k+5} + \left(\frac{1}{12960}\right) f_{k+6} + \left(\frac{1}{362880}\right) f_k\Big) \qquad (2.62)$$

Evaluating equation (2.62) at $x = x_{k+8}$, we obtain the discrete form as

$$y_{k+8} = y_{k+6} + h\Big(-\left(\frac{119}{16200}\right) f_k + \left(\frac{953}{14175}\right) f_{k+1} - \left(\frac{15577}{56700}\right) f_{k+2} + \left(\frac{9341}{14175}\right) f_{k+3} - \left(\frac{2903}{2835}\right) f_{k+4} + \left(\frac{15011}{14175}\right) f_{k+5} - \left(\frac{21247}{56700}\right) f_{k+6} + \left(\frac{22823}{14175}\right) f_{k+7} + \left(\frac{32377}{113400}\right) f_{k+8}\Big) \qquad (2.63)$$

Thus, the equation (2.63) is the eight-step optimal order method.

3. Numerical Solutions: An application of the Linear Multistep Scheme

In this study, we developed continuous multistep collocation methods for the solution of first-order IVPs of ODEs using the probabilists' Hermite polynomials as the basis function. The corresponding discrete schemes were also obtained.

In this section, we apply the derived eight-step methods of Adams-Bashforth, Adams-Moulton and the proposed optimal order for solving the non-stiff IVPs of ODEs that were the continuous interpolant derived and collocated at grid and off-grid points. Errors associated with the methods will also be obtained. The fourth order Runge-Kutta method is used to obtain the starting values, and the fourth order Adams-Bashforth method is used as a predictor to the implicit schemes. The result and errors obtained are tabulated for clarity. The fourth order Runge-Kutta method is used to evaluate the starting values $y_n, n = 0, 1, \dots, 5$, since the Runge-Kutta methods constitute the most efficient method for generating starting values for linear multistep methods.

Test Problem 1: Consider the IVP,

$$\frac{dy}{dx} = -y; \ y(0) = 1$$

Exact Solution:

$$y(x) = e^{-x}$$

Table 1: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 1, with stepsize h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	0.904837418035960	0.904837500000000	0.904837500000000	0.904837500000000
0.2	0.818730753077982	0.818730901406250	0.818730901406250	0.818730901406250
0.3	0.740818220681718	0.740818422001178	0.740818422001178	0.740818422001178
0.4	0.670320046035639	0.670320288917491	0.670320288917491	0.670320288917491
0.5	0.606530659712633	0.606530934423380	0.606530934423380	0.606530934423380
0.6	0.548811636094026	0.548811934376315	0.548811934376315	0.548811934376315
0.7	0.496585303791409	0.496585618671229	0.496585618671229	0.496585618671229
0.8	0.449328964117222	0.449329247126416	0.449329248184475	0.449329200835643
0.9	0.406569659740599	0.406569925334822	0.406569917974354	0.406569925854396
1	0.367879441171442	0.367879656723068	0.367879673352084	0.367879622283663

Table 2: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 1, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	8.20E-08	8.20E-08	8.20E-08
0.2	1.48E-07	1.48E-07	1.48E-07
0.3	2.01E-07	2.01E-07	2.01E-07
0.4	2.43E-07	2.43E-07	2.43E-07
0.5	2.75E-07	2.75E-07	2.75E-07
0.6	2.98E-07	2.98E-07	2.98E-07
0.7	3.15E-07	3.15E-07	3.15E-07
0.8	2.83E-07	2.84E-07	2.37E-07
0.9	2.66E-07	2.58E-07	2.66E-07
1	2.16E-07	2.32E-07	1.81E-07

Test Problem 2: Consider the IVP,

$$\frac{dy}{dx} = 1 - x + 4y; \quad y(0) = 1$$

Exact Solution:

$$y(x) = \frac{1}{16}(4x + 19e^{4x} - 3)$$

Table 3: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 2, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	1.609041828449010	1.608933333333330	1.608933333333330	1.608933333333330
0.2	2.505329852584810	2.505006151111110	2.505006151111110	2.505006151111110
0.3	3.830138845749650	3.829414509150810	3.829414509150810	3.829414509150810
0.4	5.794226003969200	5.792785270450580	5.792785270450580	5.792785270450580
0.5	8.712004117480150	8.709317547440140	8.709317547440140	8.709317547440140
0.6	13.052521952011900	13.047712629434700	13.047712629434700	13.047712629434700
0.7	19.515518040677800	19.507147853082100	19.507147853082100	19.507147853082100
0.8	29.144879609067300	29.131606335987000	29.132202731569100	29.133183299790600
0.9	43.497903401867600	43.477870180375300	43.478995629506700	43.479729037231900
1	64.897803164358800	64.866065719536600	64.869044474654400	64.870397343912000

Table 4: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 2, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method

0.1	1.08E-04	1.08E-04	1.08E-04
0.2	3.24E-04	3.24E-04	3.24E-04
0.3	7.24E-04	7.24E-04	7.24E-04
0.4	1.44E-03	1.44E-03	1.44E-03
0.5	2.69E-03	2.69E-03	2.69E-03
0.6	4.81E-03	4.81E-03	4.81E-03
0.7	8.37E-03	8.37E-03	8.37E-03
0.8	1.33E-02	1.27E-02	1.17E-02
0.9	2.00E-02	1.89E-02	1.82E-02
1	3.17E-02	2.88E-02	2.74E-02

Test Problem 3: Consider the IVP,

$$\frac{dy}{dx} = 5y + \frac{e^{-2x}}{y^2}, \quad y(0) = 2$$

Exact Solution:

$$y(x) = \sqrt[3]{\frac{(139e^{15x} - 3e^{-2x})}{17}}$$

Table 5: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 3, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	3.317141203207850	3.316705869531850	3.316705869531850	3.316705869531850
0.2	5.474932869673380	5.473315121398090	5.473315121398090	5.473315121398090
0.3	9.028410539856130	9.024203136693940	9.024203136693940	9.024203136693940
0.4	14.885866168811800	14.876372621469500	14.876372621469500	14.876372621469500
0.5	24.542804917158000	24.522932513230200	24.522932513230200	24.522932513230200
0.6	40.464292920687700	40.424570380692300	40.424570380692300	40.424570380692300
0.7	66.714355023308700	66.637392445034900	66.637392445034900	66.637392445034900
0.8	109.993380579720000	109.852968811974000	109.863396523534000	109.872374058953000
0.9	181.348427520768000	181.101308915549000	181.130798254438000	181.139931713824000
1	298.993010259922000	298.552128950694000	298.620412667443000	298.638430344442000

Table 6: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 3, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	4.35E-04	4.35E-04	4.35E-04
0.2	1.62E-03	1.62E-03	1.62E-03
0.3	4.21E-03	4.21E-03	4.21E-03
0.4	9.49E-03	9.49E-03	9.49E-03
0.5	1.99E-02	1.99E-02	1.99E-02

0.6	3.97E-02	3.97E-02	3.97E-02
0.7	7.70E-02	7.70E-02	7.70E-02
0.8	1.40E-01	1.30E-01	1.21E-01
0.9	2.47E-01	2.18E-01	2.08E-01
1	4.41E-01	3.73E-01	3.55E-01

Test Problem 4: Consider the IVP,

$$\frac{dy}{dx} - y = -\frac{1}{2}e^{\frac{x}{2}}\sin(5x) + 5e^{\frac{x}{2}}\cos(5x),\quad y(0) = 0$$

Exact Solution:

$$y(x) = e^{\frac{x}{2}}\sin(5x)$$

Table 7: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 4, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	0.504006211599106	0.504014759887403	0.504014759887403	0.504014759887403
0.2	0.929969260814162	0.929983373647291	0.929983373647291	0.929983373647291
0.3	1.158923832386380	1.158938590752240	1.158938590752240	1.158938590752240
0.4	1.110618385112830	1.110627991859330	1.110627991859330	1.110627991859330
0.5	0.768453444209089	0.768452618751726	0.768452618751726	0.768452618751726
0.6	0.190492085804808	0.190477469066620	0.190477469066620	0.190477469066620
0.7	-0.497785095005140	-0.497813889877322	-0.497813889877322	-0.497813889877322
0.8	-1.129016653736910	-1.129352840215060	-1.129066324157410	-1.129052476475610
0.9	-1.533072395217810	-1.533494108291160	-1.533148367354660	-1.533126945181480
1	-1.580998848627810	-1.580895807556460	-1.581085845119980	-1.581047653893670

Table 8: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 4, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	8.55E-06	8.55E-06	8.55E-06
0.2	1.41E-05	1.41E-05	1.41E-05
0.3	1.48E-05	1.48E-05	1.48E-05
0.4	9.61E-06	9.61E-06	9.61E-06
0.5	8.25E-07	8.25E-07	8.25E-07
0.6	1.46E-05	1.46E-05	1.46E-05
0.7	2.88E-05	2.88E-05	2.88E-05
0.8	3.36E-04	4.97E-05	3.58E-05
0.9	4.22E-04	7.60E-05	5.45E-05
1	1.03E-04	8.70E-05	4.88E-05

Test Problem 5: Consider the IVP,

$$\frac{dy}{dx} - \frac{y\ln y}{x+1} = (x+1)y, \quad y(0) = 1$$

Exact Solution:

$$y(x) = e^{x(x+1)}$$

Table 9: Comparison of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 5, with step-size h=0.1

x-value	Exact Solution	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	1.116278070458870	1.116276566958480	1.116276566958480	1.116276566958480
0.2	1.271249150321400	1.271244993844780	1.271244993844780	1.271244993844780
0.3	1.476980793882640	1.476971897098690	1.476971897098690	1.476971897098690
0.4	1.750672500296100	1.750655088565740	1.750655088565740	1.750655088565740
0.5	2.117000016612680	2.116967258713190	2.116967258713190	2.116967258713190
0.6	2.611696473423120	2.611635968023550	2.611635968023550	2.611635968023550
0.7	3.287081207383120	3.286970339191010	3.286970339191010	3.286970339191010
0.8	4.220695816996550	4.220471385469230	4.220537241735280	4.220563216666740
0.9	5.528961477624000	5.528507197783100	5.528723529870500	5.528742731240950
1	7.389056098930650	7.388193146884370	7.388665357756570	7.388716640390750

Table 10: Comparison of Absolute Error of 8-order Adams-Bashforth, Adams-Moulton, Optimal Method for Test Problem 5, with step-size h=0.1

x-value	8-order Adams-Bashforth	8-order Adams-Moulton	8- order Optimal Method
0.1	1.50E-06	1.50E-06	1.50E-06
0.2	4.16E-06	4.16E-06	4.16E-06
0.3	8.90E-06	8.90E-06	8.90E-06
0.4	1.74E-05	1.74E-05	1.74E-05
0.5	3.28E-05	3.28E-05	3.28E-05
0.6	6.05E-05	6.05E-05	6.05E-05
0.7	1.11E-04	1.11E-04	1.11E-04
0.8	2.24E-04	1.59E-04	1.33E-04
0.9	4.54E-04	2.38E-04	2.19E-04
1	8.63E-04	3.91E-04	3.39E-04

Conclusion

The continuous and discrete LMMs are derived through the technique of collocation and interpolation using the probabilists' Hermite polynomials as basis functions. The result has shown that continuous and discrete LMMs can be derived using any polynomial function and approach. From the results obtained, the Adams-Moulton method produced better results than the Adams-Bashforthmethod but the proposed optimal order method is the most accurate.

The linear multistep scheme is a very significant and record breaking scheme as it help to solve numerical solutions for ordinary differential equations of all types. The approach and basis functions applied in deriving the LMMs in this research work are different from those of some other researchers, though, the obtained LMMs (continuous and discrete) are same. Also, the proposed optimal order scheme has shown superiority over the standard existing methods of Adams-Bashforth and Adams-Moulton of the same step number, in terms of accuracy. It builds its scheme from the Runge-Kutta methods and has seen great impact in easily reaching convergence in evaluating IVPs. More so, from the numerous solutions of differential equations deduced by numerical method it is safe to conclude that the method is accurate as it produced results which are comparable with those produced by other similar methods.

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