Generalized $\mathcal{I}p$-Closed sets in Ideal Topological Spaces

Abstract

In this study, we introduce a new class of generalized closed sets, $\mathcal{I}p$-closed sets, in ideal topological spaces. By using a few instances, we demonstrate $\mathcal{I}p$-closed sets and establish some fundamental properties of $\mathcal{I}p$-closed sets. We also investigate the relationship between $\mathcal{I}p$-closed sets and other classes of generalized closed sets in ideal topological spaces, such as $\mathcal{I}g$-closed sets, $\alpha\mathcal{I}g$-closed sets and $\mathcal{I}rg$-closed sets. Then, we focus on the topological implications of $\mathcal{I}p$-closed sets and investigate how they relate to the concepts of $\mathcal{I}p$-continuous map, $\mathcal{I}p$-irresolute map, and a strongly $\mathcal{I}p$-continuous map. First and foremost, we define the $\mathcal{I}p$-continuous map, investigate the behavior of $\mathcal{I}p$-continuous map with respect to $\mathcal{I}p$-closed sets, and derive several important properties of $\mathcal{I}p$-continuous map. Further, we studied their relationships with other classes of continuous maps in ideal topological spaces. Nevertheless, we defined the definitions of $\mathcal{I}p$-irresolute maps and strongly $\mathcal{I}p$-continuous maps in ideal topological spaces. We explored the connections with the notions of $\mathcal{I}p$-continuous map, $\mathcal{I}p$-irresolute map, and a strongly $\mathcal{I}p$-continuous maps. Our results provide new insights into the study of ideal topological spaces.

Keywords: Preopen set, Ideals, $\mathcal{I}p$-closed set, $\mathcal{I}p$-continuous map, $\mathcal{I}p$-irresolute map

1 Introduction

Today's mathematics incorporates topological concepts into nearly every discipline. It has grown to be an effective tool for mathematical research. In 1970, Levine[(7)] introduced the generalized closed sets (briefly g-closed sets) in a topological space. After that, several mathematicians turned their attention to various forms of topology by finding new types of generalized closed sets.

Ideals in topological spaces have been considered since 1930. The subject of ideals in topological spaces was studied by Kuratowski[(5)] and Vaidyanathaswamy[(17)] almost half-a century ago, which motivated the research in applying topological ideals to generalize the most basic properties in general topology. The notion of $\mathcal{I}g$-closed sets in ideal topological space was first introduced by Dontchev et al.[(1)] Navaneethakrishnan and Sivaraj[(10)] introduced the concept of $\mathcal{I}rg$-closed sets in ideal topological space. Futher, Maragathavalli, and Vinothini[(9)] introduced the concept of $\alpha\mathcal{I}g$-closed sets in ideal topological space.

In ideal topological spaces, the above-mentioned closed sets are defined as follows:
A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

1. $\mathcal{I}g$-closed set [(1)] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is open.
2. $\mathcal{I}rg$-closed set [(14)] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is regular open.
3. $\alpha\mathcal{I}g$-closed set [(9)] if $A^* \subseteq U$ whenever $A \subseteq U$ and U is α-open.

Moreover, the notion of continuous maps in ideal topological spaces was investigated by Jankovi and Hamlett[(3)]. After that many researchers introduced new types of continuous maps in ideal topological spaces namely $\mathcal{I}g$-continuous map, $\mathcal{I}rg$-continuous map, and $\alpha\mathcal{I}g$-continuous map.

In ideal topological spaces, the above-mentioned continuous maps are defined as follows:
A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called

1. $\mathcal{I}g$-continuous [(2)] if $f^{-1}(V)$ is $\mathcal{I}g$-closed in X for every closed set V of Y.
2. $\mathcal{I}rg$-continuous [(14)] if $f^{-1}(V)$ is $\mathcal{I}rg$-closed in X for every closed set V of Y.
3. $\alpha\mathcal{I}g$-continuous [(9)] if $f^{-1}(V)$ is $\alpha\mathcal{I}g$-closed in X for every closed set V of Y.

This paper is an elaborate study of a new type of generalized closed sets in ideal topological spaces called $\mathcal{I}p$-closed sets. The major properties of this new class of generalized closed sets will be studied, and its relation to other classes of generalized closed sets in ideal topological spaces will be investigated. We defined the Ip-continuous map with illustrated example. We investigated some properties of this new type of $\mathcal{I}p$-continuous map. Further, we defined the $\mathcal{I}p$-irresolute map and the strongly $\mathcal{I}p$-continuous map with some corresponding examples and investigated some significant properties

2 Materials and Methods

Definition 2.1. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies the following properties:

1. $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$.
2. $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$.

If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space.

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $P(X)$ is the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, called a local function [(5)] of A with respect to τ and $\mathcal{I}$ is defined as follows: for $A \subseteq X$, $A^*(\mathcal{I}, \tau)$ = $\{x \in X \mid U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x)$ = $\{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(\mathcal{I}, \tau)$ [(16)]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I}, \tau)$ and τ^* for $\tau^*(\mathcal{I}, \tau)$.

Definition 2.2. Let (X, τ) be a topological space and A be a subset of X. Then, A is called,

1. preopen set [(8)] if $A \subseteq int(cl(A))$.
2. α-open set [(11)] if $A \subseteq int(cl(int(A)))$.
3. regular open set [(15)] if $A = int(cl(A))$.

Lemma 2.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B subsets of X. Then the following properties hold:[(3)]*

1. $A \subseteq B = A^* \subseteq B^*$,
2. $A^* = cl(A^*) \subseteq cl(A)$,

3. $(A^*)^* \subseteq A^*$,
4. $(A \cup B)^* = A^* \cup B^*$,
5. $(A \cap B)^* \subseteq A^* \cap B^*$.

Using the above materials, we developed a new kind of generalized closed set in ideal topological spaces called $\mathcal{I}p$-closed sets. We consider other properties using these $\mathcal{I}p$-closed sets

3 Results and Discussion

Definition 3.1. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}p$-closed set if $A^* \subseteq U$ whenever $A \subseteq U$ and U is preopen.

Example 3.1.
1. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. Then, the preopen sets are X,$\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$. Therefore, the $\mathcal{I}p$-closed sets are X, $\emptyset$, $\{a\}$,$\{c\}$, $\{a, c\}$,$\{b, c\}$.*
2. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$.Then, the preopen sets are X,$\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, c\}$. Therefore, the $\mathcal{I}p$-closed sets are X, $\emptyset$, $\{b\}$,$\{c\}$,$\{b, c\}$.*
3. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{a, d\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. Then, the preopen sets are X,$\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, c\}$,$\{a, d\}$,$\{a, b, c\}$,$\{a, c, d\}$,$\{a, b, d\}$. Therefore, the $\mathcal{I}p$-closed sets are X, $\emptyset$, $\{b\}$,$\{c\}$,$\{d\}$,$\{b, c\}$,$\{b, d\}$,$\{c, d\}$,$\{b, c, d\}$.*

Remark 3.1. We see that every closed set is a $\mathcal{I}p$-closed set. This can be seen as follows: Let A be a closed set and let U be preopen. Then, whenever $A \subseteq U$, we have $A^* \subseteq U$. But in general, every $\mathcal{I}p$-closed set need not be a closed set. This can be seen in the above examples 1 and 3.

Definition 3.2. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be an $\mathcal{I}p$-open set if A^c is a $\mathcal{I}p$-closed set.

Proposition 3.1. *If A is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$ and $B \subseteq A$. Then, B is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$.*

Proof. Let A is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. Then $A^* \subseteq U$ whenever $A \subseteq U$ and U is preopen. Since $B^* \subseteq A^*$ if $B \subseteq A$. Then $B^* \subseteq U$ whenever $B \subseteq U$ and U is preopen. Hence B is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. □

Lemma 3.2. *Every $\mathcal{I}p$-closed set in an ideal topological space is a $\mathcal{I}g$-closed set.*

Proof. Let $A \subseteq U$ and U is open. Clearly every open set is a preopen set. Since A is a $\mathcal{I}p$-closed set, $A^* \subseteq U$, which implies that A is a $\mathcal{I}g$-closed set. □

But, in general, the converse of the above lemma need not be true. This can be seen in the following example:

Example 3.3. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then, the $\mathcal{I}g$-closed sets are X, $\emptyset$,$\{a\}$,$\{b\}$, $\{c\}$,$\{a, b\}$,$\{a, c\}$, $\{b, c\}$ and the $\mathcal{I}p$-closed sets are X, $\emptyset$,$\{a\}$,$\{b\}$,$\{a, b\}$, $\{b, c\}$. Since $\{c\}$ is a $\mathcal{I}g$-closed set but not a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$.*

Lemma 3.4. *Every $\mathcal{I}p$-closed set in an ideal topological space is a $\alpha\mathcal{I}g$-closed set.*

Proof. Let $A \subseteq U$ and U is α-open. Clearly every α-open set is a preopen set. Since A is a $\mathcal{I}p$-closed set, $A^* \subseteq U$, which implies that A is a $\alpha\mathcal{I}g$-closed set. □

But, in general, the converse of the above lemma need not be true. This can be seen in the following example:

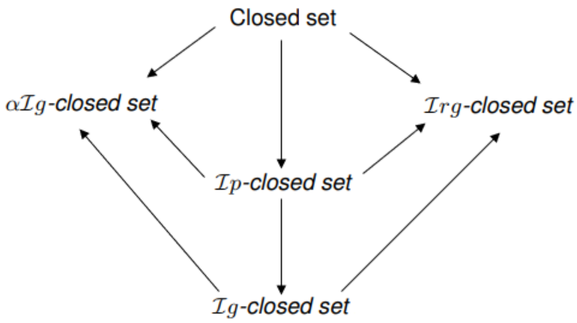
Example 3.5. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b\}, \{a, c\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}\}$. Then, the $\alpha\mathcal{I}g$-closed sets are X, $\emptyset$,$\{a\}$,$\{b\}$, $\{c\}$,$\{a, b\}$,$\{a, c\}$, $\{b, c\}$ and the $\mathcal{I}p$-closed sets are X, $\emptyset$,$\{b\}$,$\{a, c\}$. Since $\{a, b\}$ is a $\alpha\mathcal{I}g$-closed set but not a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$.*

Lemma 3.6. *Every $\mathcal{I}p$-closed set in an ideal topological space is a $\mathcal{I}rg$-closed set.*

Proof. Let $A \subseteq U$ and U is regular-open. Clearly every regular-open set is a preopen set. Since A is a $\mathcal{I}p$-closed set, $A^* \subseteq U$, which implies that A is a $\mathcal{I}rg$-closed set. □

But, in general, the converse of the above lemma need not be true. This can be seen in the following example:

Example 3.7. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then, the $\mathcal{I}rg$-closed sets are X, $\emptyset$,$\{c\}$,$\{a, b\}$,$\{a, c\}$, $\{b, c\}$ and the $\mathcal{I}p$-closed sets are X, $\emptyset$,$\{c\}$,$\{a, c\}$,$\{b, c\}$. Since $\{a, b\}$ is a $\mathcal{I}rg$-closed set but not a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$.*



The above diagram shows the connection between the newly found $\mathcal{I}p$-closed sets and some of the other existing closed sets in ideal topological spaces.

Proposition 3.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A and B be $\mathcal{I}p$-closed sets in $(X, \tau, \mathcal{I})$. Then,*

1. *$A \cup B$ is a $\mathcal{I}p$-closed set.*
2. *$A \cap B$ is a $\mathcal{I}p$-closed set.*

Proof.

1. Let U is preopen in X containing $A \cup B$. That is, $A \cup B \subseteq U$ where U is preopen. Then $A \subseteq U$ or $B \subseteq U$. Since A and B are $\mathcal{I}p$-closed, $A^* \subseteq U$ or $B^* \subseteq U$ and hence $A^* \cup B^* = (A \cup B)^* \subseteq U$. Thus, $A \cup B$ is a $\mathcal{I}p$-closed set.
2. Let A and B are $\mathcal{I}p$-closed sets in $(X, \tau, \mathcal{I})$. Then $A^* \subseteq U$ whenever $A \subseteq U$ and $B^* \subseteq U$ whenever $B \subseteq U$ where U is preopen. Since $A \cap B \subseteq A \subseteq U$ where U is preopen. Therefore, $A \cap B \subseteq U$. But $A^* \cap B^* \subseteq A^* \subseteq U$ where U is preopen. Therefore, $A^* \cap B^* \subseteq U$. Hence $(A \cap B)^* \subseteq A^* \cap B^* \subseteq U$ whenever $A \cap B \subseteq U$ and U is preopen. Thus, $A \cap B$ is a $\mathcal{I}p$-closed set.

□

Definition 3.3. Pre-closure of A is defined to be the intersection of all pre-closed sets containing A. It is denoted by $pcl(A)$.

Theorem 3.8. *If $(X, \tau, \mathcal{I})$ is any ideal topological space and $A \subseteq X$, then the following are equivalent:*

1. A is $\mathcal{I}p$-closed.
2. $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is preopen in X.
3. For all $x \in cl^*(A)$, $pcl(\{x\}) \cap A \neq \emptyset$.
4. $cl^*(A) - A$ contains no nonempty preclosed set.
5. $A^* - A$ contains no nonempty preclosed set.

Proof.

(1) ⇒ (2). If A is a $\mathcal{I}p$-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is preopen in X and so, $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is preopen in X.

(2) ⇒ (3). Suppose $x \in cl^*(A)$. If $pcl(\{x\}) \cap A = \emptyset$, then $A \subseteq X - pcl(\{x\})$. By (2), $cl^*(A) \subseteq X - pcl(\{x\})$. This is contradiction to our fact. Hence $pcl(\{x\}) \cap A \neq \emptyset$.

(3) ⇒ (4). Suppose $F \subseteq cl^*(A) - A$, F is preclosed and $x \in F$. Since $F \subseteq X - A$ and F is preclosed, $pcl(\{x\}) \cap A = \emptyset$. Since $x \in cl^*(A)$ by(3), $pcl(\{x\}) \cap A \neq \emptyset$, this is contradiction to our fact. Hence $cl^*(A) - A$ contains no nonempty preclosed set.

(4) ⇒ (5). Since $cl^*(A) - A = (A \cup A^*) - A = (A \cup A^*) \cap A^c = (A \cap A^c) \cup (A^* \cap A^c) = A^* \cap A^c = A^* - A$. Therefore, $A^* - A$ contains no nonempty preclosed set.

(5) ⇒ (6). Let $A \subseteq U$ and U is preopen. Therefore, $X - U \subseteq X - A$ and so, $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Since A^* is always closed, so, A^* is preclosed. So, $A^* \cap (X - U)$ is a preclosed set contained in $A^* - A$. Therefore, $A^* \cap (X - U) = \emptyset$, and hence $A^* \subseteq U$. Therefore A is a $\mathcal{I}p$-closed. □

Proposition 3.3. *If $(X, \tau, \mathcal{I})$ be an ideal topological space.*

1. *If $A \in \mathcal{I}$, then A is a $\mathcal{I}p$-closed set.*
2. *If $A \subseteq X$, then A^* is a $\mathcal{I}p$-closed set.*

Proof.

1. Let $A \in \mathcal{I}$ and let $A \subseteq U$ where U is preopen. Since $A^* = \emptyset$ for every $A \in \mathcal{I}$, then $cl^*(A) = (A \cup A^*) = A \subseteq U$. Therefore, by Theorem 3.8, A is a $\mathcal{I}p$-closed set.
2. Let $A \in (X, \tau, \mathcal{I})$ and let $A^* \subseteq U$ where U is preopen. Since $(A^*)^* \subseteq A^*$. Therefore, $(A^*)^* \subseteq U$ whenever $A^* \subseteq U$ and U is preopen. Hence, A^* is a $\mathcal{I}p$-closed.

□

Theorem 3.9. *If $(X, \tau, \mathcal{I})$ be an ideal topological space and let A and B are subsets of X such that $A \subseteq B \subseteq cl^*(A)$. If A is a $\mathcal{I}p$-closed set, then B is a $\mathcal{I}p$-closed set.*

Proof. If A is a $\mathcal{I}p$-closed set. Then, by Theorem 3.8, $cl^*(A) - A$ contains no nonempty preclosed set. Since $A \subseteq B \subseteq cl^*(A)$ implies, $cl^*(B) - B \subseteq cl^*(A) - A$ and so, $cl^*(B) - B$ contain no nonempty preclosed set. Again, by Theorem 3.8, B is a $\mathcal{I}p$-closed set. □

Theorem 3.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is a $\mathcal{I}p$-closed if and only if $A = F$-N where F is a $*$-closed and $N = A^*$-A .*

Proof. Suppose that, A is a $\mathcal{I}p$-closed then by Theorem 3.9, $N = A^* - A$ contain no nonempty preclosed set. Let $F = cl^*(A)$, then F is $*$-closed and $F - N = cl^*(A) - (A^* - A) = (A \cup A^*) \cap (A^* \cap A^c)^c = (A \cup A^*) \cap ((A^*)^c \cup A) = (A \cup A^*) \cap (A \cup (A^*)^c) = A \cup (A^* \cap (A^*)^c) = A$.

Conversely, suppose that $A = F - N$ where F is $*$-closed and N contain no nonempty preclosed set. Let U be preopen such that $A \subseteq U$. Then $F - N \subseteq U$ which implies $F \cap (X - U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$ then $A^* \subseteq F^*$ and so $A^* \cap (X - U) \subseteq F^* \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. By hypothesis, since $A^* \cap (X - U)$ is preclosed, $A^* \cap (X - U) = \emptyset$ and so, $A^* \subseteq U$. Hence A is $\mathcal{I}p$-closed. □

Theorem 3.11. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A is $\mathcal{I}p$-closed in X and B is $*$-closed in X, then $A \cap B$ is $\mathcal{I}p$-closed in $(X, \tau, \mathcal{I})$.*

Proof. Let U be a preopen set in X containing $A \cap B$. Then $A \subseteq U \cup (X - B)$. Since A is $\mathcal{I}p$-closed in X, therefore $A^* \subseteq U \cup (X - B)$ or $A^* \cap B \subseteq U$. Then $(A \cap B)^* \subseteq A^* \cap B^* \subseteq A^* \cap B \subseteq U$, since B is $*$-closed. Hence $A \cap B$ is $\mathcal{I}p$-closed in $(X, \tau, \mathcal{I})$. □

Definition 3.4. Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be two ideal topological spaces. A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called $\mathcal{I}p$-continuous map if $f^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$ for every closed set V in $(Y, \sigma, \mathcal{J})$.

Example 3.12. *Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{c\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset\}$, $\sigma = \{Y, \emptyset, \{c\}\}$ and $\mathcal{J} = \{\emptyset, \{c\}\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = b$, $f(b) = a$ and $f(c) = c$. Then, f is a $\mathcal{I}p$-continuous map.*

Lemma 3.13. *Every continuous map in an ideal topological space is a $\mathcal{I}p$-continuous map.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a continuous map between the ideal topological spaces and let V be a closed set in Y. Since f is continuous, $f^{-1}(V)$ is closed in X and then $\mathcal{I}p$-closed in X. Therefore, f is a $\mathcal{I}p$-continuous map. □

Converse of the above lemma need not be true in general. This can be seen in the following example:

Example 3.14. *let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$, $\sigma = \{Y, \emptyset, \{b, c\}\}$, and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Since for a closed set $\{a\}$ in Y, $f^{-1}(\{a\}) = \{b\}$ is not closed in X. Hence, f is a $\mathcal{I}p$-continuous map but not a continuous map.*

Lemma 3.15. *Every $\mathcal{I}p$-continuous map in an ideal topological space is a $\mathcal{I}g$-continuous map.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $\mathcal{I}p$-continuous map between the ideal topological spaces and let V be a closed set in Y. Since f is $\mathcal{I}p$-continuous, $f^{-1}(V)$ is $\mathcal{I}p$-closed in X then by Lemma 3.2, $f^{-1}(V)$ is $\mathcal{I}g$-closed in X. Therefore, f is a $\mathcal{I}g$-continuous map. □

Converse of the above lemma need not be true in general. This can be seen in the following example:

Example 3.16. *let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b, c\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$, $\sigma = \{Y, \emptyset, \{a\}, \{b, c\}\}$, and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then the $\mathcal{I}g$-closed sets are X,$\emptyset$,$\{a\}$,$\{a, b\}$,$\{a, c\}$ and the $\mathcal{I}p$-closed sets are X,$\emptyset$,$\{a\}$. Since for a closed set {b,c} in Y, $f^{-1}(\{b, c\}) = \{a, c\}$ is not a $\mathcal{I}p$-closed set in X. Hence, f is a $\mathcal{I}g$-continuous map but not a $\mathcal{I}p$-continuous map.*

Lemma 3.17. *Every $\mathcal{I}p$-continuous map in an ideal topological space is a $\alpha\mathcal{I}g$-continuous map.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $\mathcal{I}p$-continuous map between the ideal topological spaces and let V be a closed set in Y. Since f is $\mathcal{I}p$-continuous, $f^{-1}(V)$ is $\mathcal{I}p$-closed in X then by Lemma 3.4, $f^{-1}(V)$ is $\alpha\mathcal{I}g$-closed in X. Therefore, f is a $\alpha\mathcal{I}g$-continuous map. □

Converse of the above lemma need not be true in general. This can be seen in the following example:

Example 3.18. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b\}, \{a, c\}\}$, $\mathcal{I} = \{\emptyset, \{b\}\}$, $\sigma = \{Y, \emptyset, \{a\}\}$, and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = a$, $f(b) = c$, $f(c) = b$. Then the $\alpha\mathcal{I}g$-closed sets are X, $\emptyset$,$\{a\}$,$\{b\}$, $\{c\}$,$\{a, b\}$,$\{a, c\}$, $\{b, c\}$ and the $\mathcal{I}p$-closed sets are X, $\emptyset$,$\{b\}$,$\{a, c\}$. Since for a closed set {b,c} in Y, $f^{-1}(\{b, c\}) = \{b, c\}$ is not a $\mathcal{I}p$-closed set in X. Hence, f is a $\alpha\mathcal{I}g$-continuous map but not a $\mathcal{I}p$-continuous map.*

Lemma 3.19. *Every $\mathcal{I}p$-continuous map in an ideal topological space is a $\mathcal{I}rg$-continuous map.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $\mathcal{I}p$-continuous map between the ideal topological spaces and let V be a closed set in Y. Since f is $\mathcal{I}p$-continuous, $f^{-1}(V)$ is a $\mathcal{I}p$-closed set in X then by Lemma 3.6, $f^{-1}(V)$ is a $\mathcal{I}rg$-closed set in X. Therefore, f is a $\mathcal{I}rg$-continuous map. □

Converse of the above lemma need not be true in general. This can be seen in the following example:

Example 3.20. *Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\mathcal{I} = \{\emptyset, \{c\}\}$, $\sigma = \{Y, \emptyset, \{b\}, \{a, b\}\}$ and $\mathcal{J} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = c$, $f(b) = a$, $f(c) = b$. Then the $\mathcal{I}rg$-closed sets are X, $\emptyset$, $\{c\}$,$\{a, b\}$,$\{a, c\}$, $\{b, c\}$ and the $\mathcal{I}p$-closed sets are X, $\emptyset$,$\{c\}$,$\{a, c\}$,$\{b, c\}$. Since for a closed set $\{$a,c$\}$ in Y, $f^{-1}(\{a, c\}) = \{a, b\}$ is not a $\mathcal{I}p$-closed set in X. Hence, f is a $\mathcal{I}rg$-continuous map but not a $\mathcal{I}p$-continuous map.*

Remark 3.2. In general, the composition of two $\mathcal{I}p$-continuous maps need not be a $\mathcal{I}p$-continuous map. This can be seen in the following example:

Example 3.21. *Let $X = \{a, b, c\}$, be with $\tau = \{X, \emptyset, \{b\}, \{c\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset, \{c\}\}$, let $Y = \{a, b, c\}$ be with $\sigma = \{Y, \emptyset, \{a, c\}, \{b\}\}$, $\mathcal{J} = \{\emptyset, \{a\}\}$ and let $Z = \{a, b, c\}$ be with $\eta = \{Z, \emptyset, \{b\}, \{c\}, \{a, c\}, \{b, c\}\}$, $\mathcal{K} = \{\emptyset\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = b$, $f(b) = c$, $f(c) = a$ and define $g : (Y, \sigma, \mathcal{J}) \to (Z, \eta, \mathcal{K})$ by $g(a) = a$, $g(b) = c$, $g(c) = b$. Then, f and g are $\mathcal{I}p$-continuous maps but the composition $g \circ f$ is not a $\mathcal{I}p$-continuous map.*

Definition 3.5. Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be two ideal topological spaces. A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called $\mathcal{I}p$-irresolute if $f^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$ for every $\mathcal{I}p$-closed set V in $(Y, \sigma, \mathcal{J})$.

Example 3.22. *Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{b\}, \{a, b\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$, $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{J} = \{\emptyset, \{c\}\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = c$, $f(b) = a$ and $f(c) = b$. Then, f is a $\mathcal{I}p$-irresolute map.*

Lemma 3.23. *Every $\mathcal{I}p$-irresolute map in an ideal topological space is a $\mathcal{I}p$-continuous map.*

Proof. Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a $\mathcal{I}p$-irresolute map and let V be any closed set in $(Y, \sigma, \mathcal{J})$. Since every closed set is a $\mathcal{I}p$-closed set, we get V is a $\mathcal{I}p$-closed set in $(Y, \sigma, \mathcal{J})$. Since f is a $\mathcal{I}p$-irresolute map, $f^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. Therefore, f is a $\mathcal{I}p$-continuous map. □

Converse of the above lemma need not be true in general. This can be seen in the following example:

Example 3.24. *let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{c\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset\}$, $\sigma = \{Y, \emptyset, \{c\}\}$, and $\mathcal{J} = \{\emptyset, \{c\}\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = b$, $f(b) = a$, $f(c) = c$. Then, f is a $\mathcal{I}p$-continuous map but not a $\mathcal{I}p$-irresolute map.*

Definition 3.6. Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be two ideal topological spaces. A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called strongly $\mathcal{I}p$-continuous map if $f^{-1}(V)$ is a closed set in $(X, \tau, \mathcal{I})$ for every $\mathcal{I}p$-closed set V in $(Y, \sigma, \mathcal{J})$.

Example 3.25. *Let $X = Y = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b, c\}\}$, $\mathcal{I} = \{\emptyset\}$, $\sigma = \{Y, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$, and $\mathcal{J} = \{\emptyset, \{a\}\}$. Define $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then, f is a strongly $\mathcal{I}p$-continuous map.*

Lemma 3.26. *Every continuous map in an ideal topological space is a strongly $\mathcal{I}p$-continuous map.*

Proof. Let $(X,\tau,\mathcal{I})$ and $(Y,\sigma,\mathcal{J})$ be two ideal topological spaces. Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ be a continuous map and V be a closed set in Y then it is a $\mathcal{I}p$-closed set in Y. Since f is continuous, $f^{-1}(V)$ is closed in X. Therefore, f is a strongly $\mathcal{I}p$-continuous map. □

Converse of the above lemma need not be true in general. This can be seen in the following example:

Example 3.27. *let $X=Y=\{a,b,c\}$, $\tau=\{X,\emptyset,\{b\},\{c\},\{b,c\}\}$, $\mathcal{I}=\{\emptyset\}$, $\sigma=\{Y,\emptyset,\{b\}\{a,b\}\}$, and $\mathcal{J}=\{\emptyset,\{a\}\}$. Define $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ by $f(a)=a$, $f(b)=c$, $f(c)=b$. Then, f is a strongly $\mathcal{I}p$-continuous map but not a continuous map.*

Proposition 3.4. *Composition of two strongly $\mathcal{I}p$-continuous maps in an ideal topological spaces is a strongly $\mathcal{I}p$-continuous map.*

Proof. Let $(X,\tau,\mathcal{I})$, $(Y,\sigma,\mathcal{J})$ and $(Z,\eta,\mathcal{K})$ be an ideal topological spaces. Let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ and $g:(Y,\sigma,\mathcal{J})\to(Z,\eta,\mathcal{K})$ be a strongly $\mathcal{I}p$-continuous map. Let V be a closed set in $(Z,\eta,\mathcal{K})$ then it is a $\mathcal{I}p$-closed set in $(Z,\eta,\mathcal{K})$. Since g is a strongly $\mathcal{I}p$-continuous map, $g^{-1}(V)$ is a closed set in $(Y,\sigma,\mathcal{J})$ then it is a $\mathcal{I}p$-closed set in $(Y,\sigma,\mathcal{J})$. Now as f is a strongly $\mathcal{I}p$-continuous map, we get $f^{-1}(g^{-1}(V))$ is a closed set in $(X,\tau,\mathcal{I})$. So $(g\circ f)^{-1}(V)$ is a closed set in $(X,\tau,\mathcal{I})$, thus $g\circ f$ is a strongly $\mathcal{I}p$-continuous map. □

Theorem 3.28. *Let $(X,\tau,\mathcal{I})$, $(Y,\sigma,\mathcal{J})$ and $(Z,\eta,\mathcal{K})$ be an ideal topological spaces and let $f:(X,\tau,\mathcal{I})\to(Y,\sigma,\mathcal{J})$ and $g:(Y,\sigma,\mathcal{J})\to(Z,\eta,\mathcal{K})$ be two mappings.*

1. If f is a $\mathcal{I}p$-continuous map and g is a continuous map, then $g\circ f$ is a $\mathcal{I}p$-continuous map.
2. If f is a strongly $\mathcal{I}p$-continuous map and g is a $\mathcal{I}p$-continuous map, then $g\circ f$ is a continuous map.
3. If f and g are strongly $\mathcal{I}p$-continuous maps, then $g\circ f$ is a $\mathcal{I}p$-irresolute map.
4. If f is a strongly $\mathcal{I}p$-continuous map and g is a $\mathcal{I}p$-irresolute map, then $g\circ f$ is a continuous map.
5. If f and g are $\mathcal{I}p$-irresolute maps, then $g\circ f$ is a $\mathcal{I}p$-irresolute map.
6. If f is a $\mathcal{I}p$-irresolute map and g is a continuous map, then $g\circ f$ is a $\mathcal{I}p$-irresolute map.
7. If f is a $\mathcal{I}p$-irresolute map and g is a $\mathcal{I}p$-continuous map, then $g\circ f$ is a $\mathcal{I}p$-continuous map.

Proof.

1. Let V be a closed set in $(Z,\eta,\mathcal{K})$. Since g is a continuous map, $g^{-1}(V)$ is a closed set in $(Y,\sigma,\mathcal{J})$. Now as f is a $\mathcal{I}p$-continuous map, we get $f^{-1}(g^{-1}(V))$ is a $\mathcal{I}p$-closed set in $(X,\tau,\mathcal{I})$. So $(g\circ f)^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X,\tau,\mathcal{I})$. Thus $g\circ f$ is a $\mathcal{I}p$-continuous map.
2. Let V be a closed set in $(Z,\eta,\mathcal{K})$. Since g is a $\mathcal{I}p$-continuous map, $g^{-1}(V)$ is $\mathcal{I}p$-closed in $(Y,\sigma,\mathcal{J})$. Now as f is a strongly $\mathcal{I}p$-continuous map, we get $f^{-1}(g^{-1}(V))$ is a closed set in $(X,\tau,\mathcal{I})$. So $(g\circ f)^{-1}(V)$ is a closed set in $(X,\tau,\mathcal{I})$, thus $g\circ f$ is a continuous map.
3. Let V be a closed set in $(Z,\eta,\mathcal{K})$ then it is a $\mathcal{I}p$-closed set in $(Z,\eta,\mathcal{K})$. Since g is a strongly $\mathcal{I}p$-continuous map, $g^{-1}(V)$ is a closed set in $(Y,\sigma,\mathcal{J})$ then it is a $\mathcal{I}p$-closed set in $(Y,\sigma,\mathcal{J})$. Now as f is a strongly $\mathcal{I}p$-continuous map, we get $f^{-1}(g^{-1}(V))$ is a closed set in $(X,\tau,\mathcal{I})$ then it is a $\mathcal{I}p$-closed set in $(X,\tau,\mathcal{I})$. So $(g\circ f)^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X,\tau,\mathcal{I})$, thus $g\circ f$ is a $\mathcal{I}p$-irresolute map.
4. Let V be a closed set in $(Z,\eta,\mathcal{K})$ then it is a $\mathcal{I}p$-closed set in $(Z,\eta,\mathcal{K})$. Since g is a $\mathcal{I}p$-irresolute map, $g^{-1}(V)$ is a $\mathcal{I}p$-closed in $(Y,\sigma,\mathcal{J})$. Now as f is a strongly $\mathcal{I}p$-continuous map, we get $f^{-1}(g^{-1}(V))$ is a closed set in $(X,\tau,\mathcal{I})$. So $(g\circ f)^{-1}(V)$ is a closed set in $(X,\tau,\mathcal{I})$, thus $g\circ f$ is a continuous map.

5. Let V be a $\mathcal{I}p$-closed set in $(Z, \eta, \mathcal{K})$. Since g is a $\mathcal{I}p$-irresolute map, we get $g^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(Y, \sigma, \mathcal{J})$. Again, f is a $\mathcal{I}p$-irresolute map, we get $f^{-1}(g^{-1}(V))$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. That is, $(g \circ f)^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. Hence $g \circ f$ is a $\mathcal{I}p$-irresolute map.
6. Let V be a closed set in $(Z, \eta, \mathcal{K})$. Since g is continuous, $g^{-1}(V)$ is a closed set in $(Y, \sigma, \mathcal{J})$. Since f is a $\mathcal{I}p$-irresolute map, we get $f^{-1}(g^{-1}(V))$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. That is, $(g \circ f)^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. So $g \circ f$ is a $\mathcal{I}p$-irresolute map.
7. Let V be a closed set in $(Z, \eta, \mathcal{K})$. Since g is a $\mathcal{I}p$-continuous map, $g^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(Y, \sigma, \mathcal{J})$. But f is a $\mathcal{I}p$-irresolute map, we get $f^{-1}(g^{-1}(V))$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. That is, $(g \circ f)^{-1}(V)$ is a $\mathcal{I}p$-closed set in $(X, \tau, \mathcal{I})$. Hence $g \circ f$ is a $\mathcal{I}p$-continuous map.

□

Definition 3.7. Let $(X, \tau, \mathcal{I})$ and $(Y, \sigma, \mathcal{J})$ be two ideal topological spaces. A map $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called $\mathcal{I}p$-closed map [(13)] if the image of every closed set in X is a $\mathcal{I}p$-closed set in Y.

Some of the important features of the $\mathcal{I}p$-closed maps and the relations with this and some of the other generalized closed maps, such as $\mathcal{I}g$-closed maps, $\alpha\mathcal{I}g$-closed maps, and $\mathcal{I}rg$-closed maps in ideal topological spaces, are established by the authors in [(13)].

4 CONCLUSIONS

In this paper, we defined new class of generalized closed sets called $\mathcal{I}p$-closed sets in ideal topological spaces. We looked at the main characteristics of this new class of generalized closed sets and compared it to other classes of generalized closed sets that are already exist in ideal topological spaces. By utilizing these newly found $\mathcal{I}p$-closed sets, we discovered a $\mathcal{I}p$-continuous map, an $\mathcal{I}p$-irresolute map, and a strongly $\mathcal{I}p$-continuous map. In each segment, we took a look at some of their most significant features. In future, we can enrich the concept of generalized $\mathcal{I}p$-closed sets in ideal topological spaces by extending it to fuzzy topological spaces. Overall, the generalized $\mathcal{I}p$-closed set in ideal topological spaces is likely to involve a combination of theoretical developments, applications to other areas of mathematics, and the construction of new examples and counterexamples.

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