# ON SOME ROLE OF RIEMANNIAN GEOMETRY

**Abstract:** This paper examines the mathematical significance of Riemannian geometry, including surfaces, Riemannian curvature, Gauss curvature, manifolds, geodesics, and the relationships between them. It also explores the applications of Riemannian geometry in various concepts.

[1] [2] [3]

## 1. Introduction

Riemannian geometry was introduced to geometry by Bernhard Riemann in the 19th century [1, 2, 3, 4, 5, 6, 7]. It encompasses a wide range of geometries whose metric properties change from one point to another, including two standard forms of non-Euclidean geometry, as well as Euclidean geometry [8, 9, 10, 11, 12, 13, 14]. Every smooth manifold is capable of admitting a Riemannian metric [15, 16], which often helps to address issues in differential topology [17, 18, 19, 20, 21, 22]It also serves as a foundational structure for the more complicated field of pseudo-Riemannian manifolds, which play a crucial role in the theory of general relativity when in four dimensions [23, 24, 25]. Riemannian geometry has been generalized to encompass higher dimensional spaces. The universe can be considered as a three-dimensional space, but near heavy stars and black holes, it becomes curved. There are points in the universe that have more than one minimal geodesic connecting them, and the amount of curvature can be estimated using principles of Riemannian geometry and data gathered by astronomers. The belief among physicists is that the curvature of space is linked to the gravitational field of a star, as described by Einstein's Equation [26] The area of focus for Riemannian geometry today is exploring the relationship between the curvature of space and its shape. For instance, surfaces can take various shapes including cylinders, spheres, or paraboloids.

This paper is structured as follows. In Section 2, we provide background information and define topological manifolds along with fundamental ideas. In section 3, we revisit the definitions and characteristics of both manifolds and differentiable manifolds. In section 4, we delve into the concepts of Riemannian manifolds, including Riemannian metrics, the geometry of three-dimensional

[1*] Corresponding author.

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surfaces, Gaussian and mean curvatures, and the fundamental properties of the Riemann curvature tensor. Section 5 covers pseudo-Riemannian manifolds and their applications in general relativity. In section 6, we discuss higher dimensional Riemannian manifolds and their properties. Finally, in section 7, we conclude with a brief summary of the main points and future directions geodesic curves on Riemannian manifolds.

## 2. Topological Manifolds

Topology is the branch of mathematics that studies the properties of spaces that are preserved under continuous transformations. In other words, it's the study of the relationships between the shape, size, and structure of spaces. This can include analyzing the properties of objects that don't change when they are stretched, bent, or distorted in some other way. Topology provides a way to describe the basic properties of objects, such as connectivity and compactness, without getting into the details of their exact shape or size. This makes it a powerful tool for modeling and understanding the behavior of real-world objects and systems, as well as for exploring abstract concepts in mathematics.

A manifold is a topological space that locally resembles Euclidean space. In other words, a manifold is a space where every point has a neighborhood that looks like a flat piece of space, there exists a neighborhood that is homeomorphic to an open subset of Euclidean space. This allows us to locally describe the geometry and topology of the space using linear algebra and calculus. Manifolds play a central role in mathematics and physics, as they can be used to model many real-world objects and systems, including curves, surfaces, and higher-dimensional spaces. There are different types of manifolds, including smooth manifolds, Riemannian manifolds, complex manifolds, etc., each with their own set of additional structures and properties.

In topology, Euclidean spaces and their subspaces are highly significant. As previously discussed, the metric space $R^n$ serves as a topological representation for Euclidean space $E^n$. This is also true for finite-dimensional vector spaces over $R$ or $C$, as well as other fundamental mathematical systems. It's only fitting that we examine spaces that resemble $R^n$ locally. More specifically, spaces that have a neighborhood U around a point $p$, which is homomorphic to an open subset $U$ of $R^n, n$ (with n fixed), are considered locally Euclidean and have a dimension of $n$. These spaces are referred to as Manifolds.

**Definition 2.1**. A topological space $M$ is considered an $n-$dimensional manifold (or $n-$ manifold) if it satisfies the following properties:
(i) $M$ is a Housdorff space.
(ii) $M$ is locally Euclidean with dimension $n$.
(iii) $M$ has a countable basis of open sets.

## 3. Manifolds and Differentiable Manifolds

A topological space is defined as a set $M$ along with a family $O$ of subsets of $M$ that satisfy the following properties:
(i) $\Omega \cap \Omega \in O$,
(ii) For any index set $A : (\Omega_\alpha)_{\alpha\in A} \subset O \Rightarrow \cap_{\alpha\in O}$,
(iii) $\phi, M \in O$.
A set belonging to $O$ are called open. is considered open. A topological space is considered Hausdorff if for any two distinct points $p_1, p_2 \in M$, there exist open sets $\Omega_1, \Omega_2 \in O$ with $p_1 \in \Omega_1, p_2 \in \Omega_2, \Omega_1 \cap \Omega_2 = \phi$. A covering $(\Omega_\alpha)_{\alpha\in A}$ ($A$ an arbitrary index set) is called locally finite if each $p \in M$ has a neighborhood that intersects only finitely many $\Omega_\alpha$. $M$ is referred to as paracompact if any open covering has a locally finite refinement. This means that for any open covering $(\Omega_\alpha)_{\alpha\in A}$, there exists a locally finite open covering $(\Omega_\beta)_{\beta\in B}$ with $\forall\beta \in B\ \exists\alpha \in A : \Omega_\beta \subset \Omega_\alpha$.

**Definition 3.1**. A manifold $M$, which has a dimension $d$, is a connected Hausdorff space where each point has a neighborhood $U$ that can be homeomorphically mapped onto an open subset $\Omega$ of $\mathbb{R}^d$. This mapping is called a coordinate chart and is denoted by $X : U \to \Omega$.

**Definition 3.2**. A differentiable atlas $\{U_\alpha, X_\alpha\}$ on a manifold is considered to be differentiable if all of the chart transitions,$X_\beta \circ X_\alpha^{-1} : X_\alpha(U_\alpha \cap U_\beta) \to X_\beta(U_\alpha \cap U_\beta)$ are differentiable of class $C^\infty$, provided that $U_\alpha \cap U_\beta$ is not an empty set. When a maximal differentiable atlas is formed, it is known as a differentiable structure. A differentiable manifold of dimension $d$ refers to a manifold of dimension $d$ that has a differentiable structure. If the union of two atlases results in another atlas, then they are called to be compatible.

**Definition 3.3**. If all the chart transitions have a positive functional determinant, an atlas for a differentiable manifold is referred to as oriented. If a differentiable manifold has an oriented atlas, it is called orientable. It is typical to denote the Euclidean coordinates of $\mathbb{R}^d$, where $\Omega$ is an open subset of $\mathbb{R}^d$, as $X = (x^1, \cdot\cdot\cdot, x^d)$, and these coordinates are viewed as local coordinates on the manifold $M$ if $X : U \to \Omega$ is a chart.

**Definition 3.4**. A map $h : M \to M'$ between differentiable manifold $M$ and $M'$ with charts $\{U_\alpha, X_\alpha\}$ and $\{U'_\alpha, X'_\alpha\}$ is considered differentiable if all the maps $X'_\beta, X_\alpha^{-1}$ are differentiable (class $C^\infty$, are always) where defined). If the map is bijective and differentiable in both directions, it is known as a diffeomorphism.

**Definition 3.5**. A complex manifold of complex dimension $d$ ($dim_c M = d$) is a differentiable manifold of real dimension $2d$ ($dim_R M = 2d$), where the charts map to open subsets of $C^d$ and have holomorphic chart transitions.

## 4. Riemannian Manifolds

**4.1. Differentiation on Riemannian Manifolds.**
We will explore how differential calculus can be utilized to investigate the geometry of curves in Euclidean space $E^n$ or $R^n$, with a particular focus on plane curves ($n = 2$) and space curves ($n = 3$). The differentiation of vector fields along curves is employed once again to define and examine the differentiation of vector fields on a specific class of Riemannian manifolds - those that are embedded (or immersed) in $E^n$ and carry the induced Riemannian Metric. However, the primary aim is to use this scenario as a model to define differentiation of vector fields on any arbitrary manifold $M$.

**Definition 4.1.1**. A function $< -, - >$ that assigns a smooth scalar field $< X, Y >$ to every pair of smooth contravariant vector fields $X$ and $Y$ on a manifold $M$ is referred to as a smooth inner product. It must satisfy the following properties:
Symmetry:$< X, Y > = < Y, X >$ for all $X$ and $Y$.
Bilinearity: $< \alpha X, \gamma Y > = \alpha\gamma < X, Y >$ for all $X$ and $Y$ and scalars $\alpha$ and $\gamma$,
$< X, Y + Z > = < X, Y > + < X, Z >$,
$< X + Y, Z > = < X, Z > + < Y, Z >$,
Non-degeneracy: if $< X, Y > = 0$ for every $Y$ then $X = 0$.
A symmetry bilinear form is also referred to as a gizmo. A manifold that is equipped with a smooth inner product is known as a Riemannian Manifold.

**Example 4.1.1**. a. The manifold $M = E_n$ with the usual inner product, where $g_{ij} = \delta_{ij}$.
b. The Minkowski Metric is defined on the manifold $M = E_4$, with, $g_{ij}$ iven by the matrix under the identity chart.

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -C^2 \end{pmatrix},$$

We call the Riemannian manifold with a speed of light $C$ as the flat Minkowski space $M^4$. The metric plays the role of determining the length of a vector at a point, which gives us a new distance formula. This is in contrast to Euclidean 3-space:

$$d(u, v) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}.$$

Minkowski 4-space:

$$d(u, v) = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2 - C^2(v_4 - u_4)^2}.$$

**Definition 4.1.2**. A Riemannian 4-Manifold $M$ is said to be locally Minkowski if its metric has a signature. $(1, 1, 1, -c^2)$.

**Example 4.1.2** We define the scalar product of tangent vectors (origin $x$) at each point $x$ of $\mathbb{R}^n$ by

$$g_x(X,Y) = \sum_i X^i Y^i.$$

At every point $x \in \mathbb{R}^n$, a canonical metric is defined from $g_x$.

**Example 4.1.3**. If we have two Riemannian manifolds $(M_1, g_1)$ and $(M_2, g_2)$, we can construct the product manifold by taking their product, denoted as $M_1 \times M_2$, and equipping it with the Riemannian metric $g_1 \times g_2$.

$$M_1 \times M_2 = \{(x_1, x_2) : x_1 \in M_1, x_2 \in M_2$$

with a Riemannian structure.

**4.2. The Riemannian Metric**

**Definition 4.2.1**. A field $\Psi$ of $C^r-$bilinear form on a manifold $M$, with $r > 0$, is a function that assigns to each point $p$ of $M$ a bilinear form $\Psi$ on $T_p(M)$. Here, $\Psi$ is a bilinear mapping $\Psi : T_p(M) \times T_p(M) \to R$, such that for any coordinate frames $E_1, ..., E_n$ of class $C^r$-, the bilinear forms will be $C^\infty$. [to simplify notation we usually write $\Psi(X_p, Y_p)$ for $\Psi_p(X_p, Y_p)$]
Suppose $V_* : W \to U$ is a linear map of vector space and $\Psi$ is a bilinear form on $U$, then the formula $(V * \Psi)(U.W) = \Psi(V_*(U), V_*(W))$. Defined a bilinear form $V * \Psi$ on $W$. We have the following Properties:
(i) Assuming $\Psi$ is a symmetric, positive definite bilinear form on a vector space and $V_*$ is injective, then $V^*\Psi$ is symmetric, positive definite.
(ii) If $\Psi$ is symmetric (skew-symmetric), then $V^*\Psi$ is symmetric (skew-symmetric).
(iii) If $\Psi$ is bilinear form on $U$, then the linear mapping $\Psi : U \to U^*$ defined by $< W, \psi(U) >= \Psi(W, U)$ is an isomorphism onto if and only if rank $\Psi = dimU$.
(iv) Every bilinear $\Psi$ may written uniquely as the sum of symmetric and a skew-symmetric bilinear form, namely

$$\Psi(U,W) = \frac{1}{2}[\Psi(U,W) + \Psi(W,U)] + \frac{1}{2}[\Psi(U,W) - \Psi(W,U)]$$

(v) If skew-symmetric form $\Psi$ has a rank equal to $dimU$, then $dimU$ is an even number.

**4.3. The geometry of surfaces in $R^3$**

In this section, we utilize the curvature of curves in Euclidean three-dimensional space to derive various quantities that measure the shape of a surface $M$ near each of its points. These measures are independent of the coordinates used on both $M$ and $E^3$, as can be seen from their definitions. We consider $M$ to be an embedded surface, and we focus only on a portion covered by a single coordinate neighborhood $U, \psi$, where $W = \psi(U)$ is a connected open subset of the $uv$-plane in $R^2$. Thus, a point $p \in U \subset M$ has coordinates $u(p), v(p) = \psi(p)$. In the Euclidean three-dimensional space with a fixed Cartesian coordinate system, we identify $E^3$ with $R^3$, and the embedding or parameter mapping $\psi^{-1} : WU \subset R^3$ is defined by $x^i = f^i(u,v), i = 1, 2, 3$.

Let $E_1 = \psi_*^{-1}(\frac{\partial}{\partial u})$ and $E_2 = \psi_*^{-1}(\frac{\partial}{\partial v})$. The unit norm vector field $N$ on $M$ is a unique unit vector at each point $p \in M$ that is orthogonal to $T_p(M) \subset T_p(R^3)$ . It is chosen such that $E_1, E_2, N$ form a frame at $p$ with the same orientation as the standard orthonormal frame of $\mathbb{R}^3 - \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$. The length and orthogonality of these vectors are defined with respect to the inner product (X,Y) of Euclidean space, which induces a Riemannian metric on $M$ by restriction.
Let $p(t)$ be any differentiable curve on $M$ with $p(0) = p$. and $p^{\cdot}(0) = X_p \in T_p(M)$. Restricting $N$ to $p(t)$ gives a vector field along a space curve, giving a derivative $\frac{dN}{dt}$ which is itself a vector field along $p(t)$. Using $(N, N) = 1$, we have $0 = \frac{d}{dt}(N, N) = 2(\frac{dN}{dt}, N)$, this means that $\frac{dN}{dt}$ is orthogonal to $N(t)$ at each point $P(t)$ and hence, is tangent to $M$, that is $\frac{dN}{dt} \in T_{p(t)}(M)$.
The vector $(\frac{dN}{dt})_{t=0}$ depends only on $X_p$ and not on the curve $p(t)$ chosen. Let $S(X_p) = -(\frac{dN}{dt})_{t=0}$. Then $X \to pS(X_p)$ is a linear map of $T_p(M) \to T_p(M)$.
$S(X)$ is a symmetric operator on the tangent space $T_p(M)$ for each $p \in M$ and $\psi(X, Y)$ is symmetric covariant tangent of order 2. The components of $S$ and $\psi$ are $C^\infty$ if $M$ is $C^\infty$ sub manifold.

### 4.4. The Gaussian and mean curvatures of a surface

The Gaussian curvature of a surface, known as $k = k_1 + k_2$, can be obtained by taking the determinant of the linear transformation $S$, which is equal to the product of its characteristic values $k_1 + k_2$ . Alternatively, it can be calculated as the negative of the trace of $S$. The mean curvature of the surface, denoted by $H = \frac{1}{2}(k_1 + k_2)$, is equal to half the sum of its characteristic values $\frac{1}{2}(k_1 + k_2)$. Regardless of the parametrization used, the fundamental forms' components allow for the direct calculation of these quantities.

**Theorem 4.4.1**.

$$K = \frac{Ln - m^2}{EG - F^2}, \text{ and } H = \frac{1}{2}\frac{GL - 2fm + E_n}{EG - F^2}$$

**Proof**.

$$S(X_v) = aX_v + bX_w,\ S(X_w) = cX_v + dX_w$$

We can express the components of the operator $S$ using the coordinate frames $E_1 X_v$ and $E_2 = X_w$, which are naturally given by the parameterization of $M$ near point $p$, specifically within the coordinate neighborhood $V$ and with respect to $\psi$. Therefore, we can represent $S$ as follows:

$$K = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \text{ and } 2H = a + b \text{ in term of } X_v, X_w$$

we have

$$KN = K(X_v \times X_w) = S(X_v) \times (X_w)$$

And

$$2HN = 2H(X_v \times X_w) = S(X_v) \times X_v \times X_w + X_v \times S(X_w).$$

The symbol $\times$ represents the cross product of vectors in three-dimensional Euclidean space. It should be noted that:

$$(X_v \times X_w, X_v \times X_w) = \|(X_v \times X_w)\| = EG - F^2.$$

Furthermore, we can apply the Lagrange identities, which hold for any vectors $X, Y, V, W$ in $R^3$

$$((X \times Y), (V \times W)) = \begin{vmatrix} (X,V) & (X,W) \\ (Y,V) & (Y,W) \end{vmatrix}.$$

**4.5. Basic Properties of Riemannian Curvature Tensor**
Both the curvature tensor and curvature operator satisfy the symmetry relations below at every point, and consequently, for all vector fields:
(i) $R(X,Y) \cdot Z = R(Y,X) \cdot Z = 0$,
(ii) $R(X,Y) \cdot Z = R(Y,Z) \cdot X = R(Z,X) \cdot Y = 0$,
(iii) $(R(X,Y) \cdot Z, U) = (R(X,Y) \cdot U, Z) = 0$,
(iv) $(R(X,Y) \cdot Z, U) = (R(Z,U) \cdot X, Y)$.
For all $1 \leq i \leq k, 1 \leq n$, we get
(i) $R^j_{ik_1} + R^j_{1ik} = 0$,
(ii) $R^j_{ijk_1} + R^j_{i_1k} + R^j_{ik_1} = 0$,
(iii) $R_{ijk_1} + R_{jik} = 0$,
(iv) $R_{ijk_1} = R_{kji_1}$,
(v) $R_{ijk_1} + R_{ik_1j+R_{i_1jk}}$.

**Definition 4.5.1**. If we have an orthonormal basis $x, y$ for a given section $\Pi$, then the sectional curvature $K(\Pi)$ can be defined as follows:

$$K(\Pi) = -R(x,y,x,y) - (R(x,y) \cdot x, y)$$

The symmetry and linearity properties imply that if we replace $x, y$ by any pair of vector with any pair of vectors. $x\prime, y\prime$ where $x = \alpha x\prime + \beta y\prime$ and $y = \gamma x\prime + \delta y\prime$, the relation still holds

$$(\frac{1}{\Delta^2})(R(x\prime, y\prime) \cdot x\prime, y\prime) = (R(x,y) \cdot x, y).$$

Where $\Delta = \alpha\delta - \beta\gamma$, the determinant of coefficients. If $x\prime, y\prime$ is also an orthonormal pair, then $\Delta = \pm 1$ so that the definition of $K(\Pi)$ is independent of the pair used. If it is just any arbitrary linear independent pair, the using $\Delta^2 = (x', y')(y', y') - (y', y')^2$ we have

$$K(\Pi) = \frac{(R(x', y') \cdot x', y')}{(x', y')(y', y') - (x', y')^2}.$$

in local coordinate, using $E_i, E_j = g_{ij}$ and notation above,

$$K(\Pi) = -\frac{\sum R_{ijk1} \alpha^i \beta^j \alpha^k \beta^1}{\sum (g_{ik} g_{j1} - g_{ik} g_{jk}) \alpha^i \beta^j \alpha^k \beta^1}$$

## 5. Lie group and manifold:

The space $\mathbb{R}^n$ is a $C^\infty$ manifold , and it has an Abelian group operation defined by component-wise addition. Additionally, the algebraic and differentiable structures are related as follows: $(X, Y) \to X + Y$ is a $C^\infty$ mapping from the product manifold $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}_n$, meaning that the group operation is differentiable. Furthermore, we observe that the mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ that maps each element $x$ to its inverse, $-x$, is also differentiable.

**Definition 5.1**. A Lie group is a group $\mathcal{G}$ that possesses the structure of a differentiable manifold, or alternatively, a disjoint union of finitely many differentiable manifolds, such that the following maps are differentiable:
$\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (Multiplication), $(g, h) \to g \cdot h$.
And $\mathcal{G} \to \mathcal{G}$ (Inverse), $g \to g^{-1}$.
We say that a differentiable manifold $M$ is left acted upon by a group $\mathcal{G}$ if there exists a differentiable map $\mathcal{G} \times M \to M, (g, x) \to gx$ , which satisfies the Lie group structure of $\mathcal{G}$ in the sense that $g(hx) = (g \cdot h)x$ for all $g, h \in \mathcal{G}, x \in M$.

**Example of Lie group** :

- The $n-$dimensional Euclidean space, denoted as $\mathbb{R}^n$, where the group operation is ordinary vector addition. It is a non-compact, abelian Lie group.
- The circle group $S'$ is a one-dimensional, compact, abelian Lie group, consisting of angles modulo $mod 2\pi$ under addition or complex numbers with an absolute value of 1 under multiplication.
- The group $\mathcal{G}l(n, R)$, which consists of invertible matrices under matrix multiplication, is a Lie group of dimension $n^2$, known as the general linear group. It has a closed and connected subgroup, denoted as $\mathcal{G}l(1, R)$, which is the special linear group consisting of matrices of dimension n with a determinant of 1. The special linear group is also a Lie group.

**Properties** :
(i) The diffeomorphism group of a Lie group acts on the Lie group.
(ii) every Lie group is parallelizable, which means it is an orientable manifold. Specifically, there exists a bundle and the product of the Lie group with its tangent space at the identity, which enables us to define a consistent notion of orientation.

## 6. Geodesic Curves on Riemannian Manifolds

A geodesics is a generalization of the notion of a straight line to curved space. We should study and define the class of curves called geodesics. Let $p(t)$ be curve on $M$ and $\frac{dp}{dt}$ its velocity vector, defined for some open interval $a < t < b$ of $R$; we assume it to be of class $C^2$ at least.

**Definition 6.1**. A curve $p(t)$ is considered to be a geodesic if its velocity vector is constant (parallel). In other words, if it satisfies the condition $(\frac{D}{dt})(\frac{dp}{dt}) = 0$, this represents the equation of a geodesic for the interval $a < t < b$.

**Definition 6.2**. A geodesic segment that has a length equal to the distance between its endpoints is referred to as a minimal geodesic.

**Theorem 6.1**. For any $q \in M$, a Riemannian manifold, there exists a neighborhood $B$ and a positive real number $\epsilon > 0$ such that for any pair of points in $B$, there exists a unique geodesic connecting them with a length less than $\epsilon > 0$ . Furthermore, the length $L^{'}$ of any piecewise $C^{'}$ curve that connects the two points must be $\geq L$, the length of the unique geodesic. The paths coincide as point sets or are identical, parameterized by arc length, if and only if $L^{'} = L$.

**Proof**

To prove this theorem we can use theorem 6.2, lemma 6.1 and lemma 6.2. By Theorem 6.2, for any $q \in M$ , there exists a neighborhoods $B$ and $\varepsilon > 0$ such that any pair of points$p, p'$ of $B$ can be connected by a unique geodesic of length $L < \varepsilon$. The equation of this geodesic is given by $p(t) = Exp\ tX_p, 0 \leq t \leq 1$ where $T_p(M)$ with $\|X_p\| = L$. The open set $B$ is contained in a coordinate neighborhood $U, \varphi$ that contains this geodesic. Furthermore,$Exp_p$ is a diffeomorphism between the open ball of vectors $X_p$ of $T_p(M)$ of length $\|X_p\| < \varepsilon$ and an open set $N_p$ of $U$ containing $B$. This implies that any sphere $\{X_p | \|X_p\| = r < \varepsilon\}$ is mapped diffeomorphically to a submanifold of $U$ denoted by $S_r$ (also called a geodesic sphere).

**Theorem 6.2**. Given a Riemannian manifold $M$ and a coordinate neighborhood $U$ with coordinate chart $\varphi$, for any $q \in U$, there exists a neighborhood $B \subset U$ of $q$ and a positive constant $\varepsilon > 0$ such that for any two points $p, p'$ of $B$, there exists a unique geodesic joining them whose length is less than $\varepsilon > 0$. This geodesic can be expressed as $Exp\ tX_p, 0 \leq t \leq 1$ and lies entirely in $U$. For each $p \in B$, $Exp_p$ maps the set of vectors $\{X_p | \|X_p\| < \varepsilon\}$ diffeomorphically onto an open set $N_p$ such that $B$ is contained in $N_p$ which is contained in $U$.

**Remark 6.1**. Due to the limitations of our chosen neighborhood $N_B$, it is not possible to infer that if $(p, X_p) \in N_B$,, then $(p, tX_p) \in N_B$ for all values of $0 < t < 1$. Therefore, it cannot be generally assumed that if $p, p' \in B$, they will be connected by a geodesic entirely contained within $B$. Our choices have been specifically made to ensure that for each point $p \in V$, $Exp_p$ will diffeomorphically map the $\varepsilon$ ball $\{X_p | \|X_p\| < \varepsilon\}$ to $U$ and clearly include $B$ in its image, thereby providing each point $p \in B$ with a normal neighborhood $N_p$ that contains $B \subset N_p \subset U$.

**Lemma 6.1**. Suppose $M$ is a $C^{\infty}$-manifold with dimension $n$. There exists a unique topology on $T(M)$ that satisfies the following conditions: for each coordinate neighborhood $U, \varphi$ of $M$, the set Ũ $=\pi^{-1}(U)$ is an open set in $T(M)$, and the mapping ˜$\varphi$: Ũ$\rightarrow \varphi(U) \times R^n$, defined as above, is a homomorphism. Using this topology, $T(M)$ becomes a topological manifold with a dimension of $2n$ , and the neighborhood Ũ and ˜$\varphi$ together provide a $C^{\infty}$-structure relative to which $\pi$

is an (open) $C^\infty$-mapping of $T(M)$ onto $M$.

**Proof**

Suppose $U, \varphi$ and $U', \varphi'$ are coordinate neighborhoods on $M$ such that $U \cap U' \neq \phi$. Then, it follows that $\tilde{U} \cap \tilde{U}' \neq \phi$. By comparing the coordinates of a point $p \in U \cap U'$, and the components of any $X_p \in T(M)$ relative to the two coordinate systems, we can derive the formulas for the change of coordinates in $\tilde{U} \cap \tilde{U}'$:

$\tilde{\varphi}' \circ \tilde{\varphi}^{-1}(x^1, ..., x^n; y^1, ..., y^n) = (f^1(x), ..., f^n(x); \sum_{i=1}^{n} y^i \frac{\partial f^1}{\partial x^i}, ..., \sum_{i=1}^{n} y^i \frac{\partial f^n}{\partial x^i}),$

The formulas for the change of coordinates $\varphi' \circ \varphi^{-1}$ on $U \cap U'$ are given by $h'^i = f^i(x^1, ..., x^n)$, where , $i = 1, ..., n$. It is easy to see that the given formulas represent diffeomorphisms of $\tilde{\varphi}(\tilde{U} \cap \tilde{U}')$ onto $\tilde{\varphi}'(\tilde{U} \cap \tilde{U}')$. Additionally, note that locally, on the domain $\tilde{U}$ of each coordinate neighborhood of the given type, $T(M)$, $\tilde{U}$ is diffeomorphic to $\varphi(U) \times R^n$. In the case of Euclidean space, $U, \varphi$ can be chosen to be all of $M = R^n$, resulting in $T(R^n)$ being diffeomorphic to $R^n \times R^n$. It is evident that for any manifold $M$, $T(M) = 2$ dim $M$. Furthermore, it is worth noting that in local coordinates, $\pi$ corresponds to the projection of $R^n \times R^n$ onto its first factor.

**Lemma 6.2**. Assume that $q \in M$ and $X_q T_q(M)$ is such that Exp $X_q$ defined, then Exp $tX_q$ is defined for all $t$ such that $|t| \leq 1$ and $q(t) =$ Exp $tX_q$ is the geodesic through $q$ at $t = o$ with $(\frac{dq}{dt})_0 = X_q$.

**Proof**

The statement above follows from considering the unique geodesic $q(t)$ with $q(0) = q$ and $(\frac{dq}{dt})_0 = X_q$ so that Exp $X_q = q(1)$. Let $c$ with $|c| < 1$ be a scalar such that $\tilde{q}(t) = q(ct)$. We have $\tilde{q}(0) = q$ and $(\frac{dq}{dt})_{t=0} = cX_q$, which implies that $\exp cX_q = \tilde{q}(1) = q(c)$. Replacing $c$ by $t$ in this equation yields the desired result.

### 6.1. Metric Geometry

A geodesic in a metric space $M$ is a curve $\gamma : I \to M$, where $I$ is an interval in the real numbers, such that for any point $p$ on the curve, there exists a neighborhood $J$ of $p$ in $I$, such that for any pair of points $t_1, t_2 \in J$, the distance between $\gamma(t_1)$ and $\gamma(t_2)$ is minimized by the curve $\gamma$. More precisely, there is a constant $\gamma > 0$ such that for any $t \in I$ and any $t_1, t_2 \in J$, we have:

$$d(\gamma(t_1), \gamma(t_2)) = \gamma|t_1 - t_2|.$$

Where $d$ denotes the distance function on $M$, and $\gamma(t)$ denotes the tangent vector to $\gamma$ at $t$. In other words, $\gamma$ is the shortest path between any pair of points on it. This generalizes the notion of geodesic for Riemannian manifold. furthermore, in metric geometry, the geodesic considered is often equipped with natural parameterization i.e. in the above identity

$$\gamma = 1 \text{ and } d(\gamma(t_1), \gamma(t_2)) = \gamma|t_1 - t_2|.$$

If the last equality is satisfied $t_1, t_2 \in I$, the geodesic is called a minimizing geodesic or shortest path.

**Example 6.1.1**. The most well-known instances of this concept include straight lines in Euclidean geometry and on a sphere, where the images of geodesics are represented by great circles. On a sphere, the shortest route between two points, $A$ and $B$, can be calculated by determining the shorter piece of the great circle that passes through both $A$ and $B$. If $A$ and $B$ are antipodal points, there are infinitely many shortest paths connecting them.

## 7. Conclusion

This paper concludes that Riemannian geometry plays a vital role in mathematics, particularly in the areas of covariance derivative, exterior derivative, and general curvature. Its applications extend to various scientific disciplines, including physics and mechanics. Geodesics, a fundamental mathematical tool, also hold significant importance in mathematics and theoretical physics, particularly in the theory of general relativity.

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