Original Research Article

ON SOME ROLE OF RIEMANNIAN GEOMETRY

Abstract: This paper is study the mathematical importance of Riemannian Geometry, the surfaces, Riemannian curvature, Gauss curvature, manifolds and Geodesics, and some relation between them, as well as its applications that in some concepts in role of Riemannian geometry.

Keywords: Topological manifolds, Riemannian geometry; Riemannian curvature; Gauss curvature; Riemannian manifolds; Lie Algeebra; Geodesics.

1. Introduction

Riemannian geometry was first put forward in geometry by Bernhard Riemann in nineteenth century [1, 2, 3, 4, 5, 6, 7]. It deals with aboard range of geometries whose metric properties vary from point to point, as well as two standard types of non Euclidean geometry, as well as Euclidean geometry itself [8, 9, 10, 11, 12, 13, 14]. any smooth manifold admits a Riemannian metric [15, 16], which often helps to solve problem of differential topology [17, 18, 19, 20, 21, 22], it also serve as an entry level for more complicated structure of Poseudo-Riemannian manifolds, which (in four dimension) are the main objects of the theory of general relativity [23, 24, 25]. Other generalizations of Riemannian geometry include Riemannian geometry also study higher dimensional spaces. the universe can described as three dimensional space. However, near very heavy starts and black holes, the space is curved and bent. There are pairs of point in the universe which have more than one minimal geodesic between them, amount that space is curved can be estimated by using theorems from Riemannian geometry and measurements taken by a astronomers. Physicists believes that the curvature of space is related to gravitational field [26] of star according to partial differential equation called Einstein's Equation. Kinds of theorem Rienmannian geometry are looking for today is relationship between the curvature of space and its shape. For example, there are many different shapes that surfaces can take. They can be cylinders, spheres or paraboilds.

This paper is organized as follows. In section 2, we give a background and definitions of topological manifolds with some basic concepts. In section 3, we recall some definitions and properties for manifolds and differentiable manifolds. In section 4, we introduce Riemannian manifolds, Riemannian metrics, geometry of surface in three-dimensions, Gaussian and mean curvatures of a surfaces as

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well as basic Properties of Riemannian Curvature Tensor. In section 5, we devote to the Lie group and manifolds and show that it play central role in vectors and the distance formulas. In last section, we discuss the Geodesics curves on Riemannian manifolds.

2. Topological Manifolds

Of all the spaces which one studies in topology the Euclidean spaces and their subspaces are the most important. As we have just seen, the metric space R^n serve as a topological model for Euclidean space E^n. For finite-dimensional vector spaces over R or C and for other basic Mathematical systems which we shall encounter later. It is natural enough that we are led to study those spaces which are locally like R^n. More precisely, those spaces for which point p has a neighborhood U which is homomorphic to an open subset U of R^n, n fixed. We say that a space which this property is locally Euclidean of dimension n, and in order to Euclidean spaces, we called Manifolds, defined as follows.
Definition 2.1. A manifold M of dimension n, or $n-$manifold is a topological space with following properties: (i) M is Housdorff.
(ii) M is locally Euclidean of dimension n.
(iii) M has countable basis of open sets.

3. Manifolds and Differentiable Manifolds

A topological space is a set M together with family O of subset M satisfying the following properties:
i) $\Omega \cap \Omega \in O$,
ii) For any index set $A : (\Omega_\alpha)_{\alpha \in A} \subset O \Rightarrow \cap_{\alpha in O}$,
iii) $\phi, M \in O$.
The sets from O are called open. A topological space is called Hausdorff if for any two distinct points $p_1, p_2 \in M$ there exists open sets $\Omega_1, \Omega_2 \in O$ with $p_1 \in \Omega_1, p_2 \in \Omega_2, \Omega_1 \cap \Omega_2 = \phi$. A covering $(\Omega_\alpha)_{\alpha \in A}$ (A an arbitrary index set) is called locally finite if each $p \in M$ has a neighborhood that intersects only finitely many Ω_α. M is called paracompact if any open covering possesses a locally finite refinement. this means that for any open covering $(\Omega_\alpha)_{\alpha \in A}$ there exists a locally finite open covering $(\Omega_\beta)_{\beta \in B}$ with $\forall \beta \in B\ \exists \alpha \in A : \Omega_\beta \subset \Omega_\alpha$.

Definition 3.1. A manifold M of dimension d is a connected Hausdorff space for every points has a neighborhood U that is homeomorphic to an open subset Ω of $\mathbb{R}^d$. Such a homeomorphism $X : U \rightarrow \Omega$ is called a coordinate chart.

Definition 3.2. An atlas $\{U_\alpha, X_\alpha\}$ on a manifold is called differentiable if all chart transitions $X_\beta \circ X_\alpha^{-1} : X_\alpha(U_\alpha \cap U_\beta) \rightarrow X_\beta(U_\alpha \cap U_\beta)$ are differentiable of class C^∞ (in case $U_\alpha \cap U_\beta \neq \phi$). A maximal differentiable atlas is called a differentiable structure, and a differentiable manifold of dimension d is a manifold of dimension d with differentiable structure. Tow atlas are called compatible if their union is again an atlas.

Definition 3.3. An atlas for a differentiable manifold is called oriented if all chart transitions have positive functional determinant. A differentiable manifold is called orientable if it possesses an oriented atlas. It is customary to write the Euclidean coordinates of $\mathbb{R}^d, \Omega \subset \mathbb{R}^d$ open as $X = (x^1, \cdot\cdot\cdot, x^d)$ and these are considered as a local coordinates on our manifold M when $X : U \to \Omega$ is a chart.

Definition 3.4. A map $h : M \to M'$ between differentiable manifold M and M' with charts $\{U_\alpha, X_\alpha\}$ and $\{U'_\alpha, X'_\alpha\}$ is called differentiable if all map X'_β, X_α^{-1} are differentiable (class C^∞, are always) where defined. Such a map is called a diffeomorphism if bijective and differentiable in both directions.

Definition 3.5. A complex manifold of complex d $(dim_c M = d)$ is a differentiable manifold of a real dimensional $2d$ $(dim_R M = 2d)$ whose charts take values in open subset of C^d with holomorphic chart transitions.

4. RIEMANNIAN MANIFOLDS

4.1. **Differentiation on Riemannian Manifolds.** We begin to show that, how differential calculus can be applied to study the geometry of curves in Euclidean space E^n or R^n especially plane curves $(n = 2)$ and space curves $(n = 3)$. In the differentiation of vector fields along curves is used again to define and study differentiation of vector fields on a special class of Riemannian manifolds-those which are imbedded (or immersed) in E^n and carry the induced Riemannian Metric. However, our main objective is to use this situation as a model in order to define differentiation of vector fields on an arbitrary Riemannian Manifolds M.
Definition 4.1.1. A smooth inner product on a manifold M is a function $< -, - >$ that associates to each pair of smooth contravariant vector field X and Y a smooth scalar field $< X, Y >$, satisfying the following properties:
Symmetry:$< X, Y >=< X, Y >$ for all X and Y.
Bilinearity: $< \alpha X, \beta Y >= \alpha\beta < X, Y >$ for all X and Y and scalars α and β,
$< X, Y + Z >=< X, Y > + < X, Z >$,
$< X + Y, Z >=< X, Z > + < Y, Z >$,
Non-degeneracy: if $< X, Y >= 0$ for every Y then $X = 0$.
We also call such a gizmo a symmetry bilinear form: A manifold endowed with a smooth inner product is called a Riemannian Manifold.

Example 4.1.1.
a. $M = E_n$, with usual inner product, $g_{ij} = \delta_{ij}$.
b. (Minkowski Metric) $M = E_4$, with, g_{ij} given under the identity chart) by the matrix,

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -C^2 \end{pmatrix},$$

where C is the speed of light. we call this Riemannian manifold flat Minkowski space M^4. The role that the metric plays is that it tell us the length of a vector, in other words, it gives us a new distance formula:
Euclidean 3-space:

$$d(x,y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}.$$

Minkowski 4-space:

$$d(x,y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 - C^2(y_3 - x_3)^2}.$$

Definition 4.1.2. A Riemannian 4-Manifolds M is called locally Minkowski if its metric has signature $(1, 1, 1, -c^2)$.

Example 4.1.2

At each point x of $\mathbb{R}^n$ we define the scalar product of tangent vectors (origin x) by

$$g_x(X,Y) = \sum_i X^i Y^i.$$

A canonical metric is so defined at every point $x \in \mathbb{R}^n$ from g_x.

Example 4.1.3.

Given two Riemannian Manifolds (M_1, g_1) and (M_2, g_2). Let us provide $t - 1$ the product manifold.

$$M_1 \times M_2 = \{(x_1, x_2) : x_1 \in M_1, x_2 \in M_2$$

with a Riemannian structure.

4.2. **The Riemannian Metric.** **Definition 4.2.1**.

a field Φ of C^r-bilinear form $r > 0$, on a manifold M consists of function, assigning to each p of M a bilinear form Φ on $T_p(M)$, that is a bilinear mapping $\Phi : T_p(M) \times T_p(M) \to R$, such that for any coordinate frames $E_1, ..., E_n$ are of class C^r-unless otherwise stated bilinear forms will be C^∞ [to simplify notation we usually write $\Phi(X_p, Y_p)$ for $\Phi_p(X_p, Y_p)$]

Suppose $F_* : W \to V$ is a linear map of vector space and Φ is a bilinear form on V, then the formula $(F * \Phi)(V.W) = \Phi(F_*(V), F_*(w))$. Defined a bilinear form $F * \Phi$ on W. We have the following Properties:

- **i:** If Φ is symmetric, positive definite, and F_* is injective, then $F^*\Phi$ is symmetric, positive definite.
- **ii:** If Φ is symmetric (skew-symmetric), then $F^*\Phi$ is symmetric (skew-symmetric).
- **iii:** If Φ is bilinear form on V, then the linear mapping $\Phi : V \to V^*$ defined by $< W, \varphi(V) = \Phi(W, V)$ is an isomorphism onto if and only if rank $\Phi = dim V$.

iv: Every bilinear Φ may written uniquely as the sum of symmetric and a skew-symmetric bilinear form, namely

$$\Phi(V,W)=\frac{1}{2}[\Phi(V,W)+\Phi(W,V)]+\frac{1}{2}[\Phi(V,W)-\Phi(W,V)]$$

v: If skew-symmetric form Φ has a rank equal to $dimV$, then $dimV$ is an even number.

4.3. **The geometry of surfaces in R^3.** In this section we use curvature for curves in Euclidean three dimensional space to obtain various quantities which measure the shape of a surface M near each its point. However they will be independent of the coordinates used both on M and on E^3, as will be seen from their definition. We suppose that M is an embedded surface of which we consider only apportion covered by a single coordinate neighborhood U, φ with $W = \varphi(U)$ an connected open subset of R^2, the uv-plane. Thus $p \in U \subset M$ has coordinates $u(p), v(p) = \varphi(p)$; and taken the Euclidean three-dimensional space with a fixed Cartesian coordinate system, that the is identifying E^3 with R^3, the imbedding or parameter mapping $\varphi^{-1} : WU \subset R^3$ is given by $x^i = f^i(u,v), i = 1,2,3$.
Let $E_1 = \varphi_*^{-1}(\frac{\partial}{\partial u})$ and $E_2 = \varphi_*^{-1}(\frac{\partial}{\partial v})$. The unit norm vector field N to M, it is the unique unit vector at each $p \in M$ which is orthogonal to $T_p(M) \subset T_p(R^3)$ and so chosen that E_1, E_2, N form a frame at p with the same orientation as $\frac{\partial}{\partial}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$ the standard orthonormal frame of $\mathbb{R}^3-$length and orthogonality are defined in terms of the inner product (X,Y) of Euclidean space which of course induces a Riemannian metric on M by restriction.
Let $p(t)$ be any differentiable curve on M with $p(0) = p.$ and $p.(0) = X_p \in T_p(M)$. Restricting N to $p(t)$ gives a vector field along a space curve, giving a derivative $\frac{dN}{dt}$ which is itself a vector field along $p(t)$. Using $(N,N) = 1$, we have $0 = \frac{d}{dt}(N,N) = 2(\frac{dN}{dt}, N)$, this means that $\frac{dN}{dt}$ is orthogonal to $N(t)$ at each point $P(t)$ and hence is tangent to M, that is $\frac{dN}{dt} \in T_{p(t)}(M)$.
The vector $(\frac{dN}{dt})_{t=0}$ depends only on X_p and not on the curve $p(t)$ chosen. Let $S(X_p) = -(\frac{dN}{dt})_{t=0}$. Then $X \to pS(X_p)$ is a linear map of $T_p(M) \to T_p(M)$.
$S(X)$ is a symmetric operator on the tangent space $T_p(M)$ for each $p \in M$ and $\varphi(X,Y)$ is symmetric covariant tangent of order 2. The components of S and φ are C^∞ if M is C^∞ sub manifold.

4.4. **The Gaussian and mean curvatures of a surface.** The negative of the trace and determinant of any matrix of the linear transformation S, the determinant is $k = k_1k_2$ the product of the characteristic values; it is called the Gaussian Curvature of the surface. The trace is $k_1 + k_2$ the sum of characteristic values; and $H = \frac{1}{2}(k_1 + k_2)$ is called Mean Curvature of the surface. These quantities may be computed directly from the components of the fundamental forms, using any parametrization of the surface. This we now proceed to d.
Theorem 4.4.1.

$$K=\frac{Ln-m^2}{EG-F^2}, \text{ and } H=\frac{1}{2}\frac{GL-2fm+E_n}{EG-F^2}$$

Proof.

$$S(X_u) = aX_u + bX_v,\ S(X_v) = CX_u + dX_v$$

Give the components of the operator S in term of the coordinate frames $E_1 X_u$ and $E_2 = X_v$ naturally given by parameterization of M near p, that is on the coordinate neighborhood U, φ.
Thus, we may write

$$K = \begin{vmatrix} a & b \\ c & d \end{vmatrix},\ \text{and } 2H = a + b \text{ in term of } X_u, X_v$$

we have

$$KN = k(X_u \times X_v) = S(X_u) \times (X_v)$$

And

$$2HN = 2H(X_u \times X_v) = S(X_u) \times X_u \times X_v + X_u \times S(X_v).$$

Where × denote the cross product of vector in three-dimensional Euclidean space.
Now note that

$$(X_u \times X_v, X_u \times X_v) = \|(X_u \times X_v)\| = EG - F^2.$$

And use the fact that for any vector X, Y, U, V of R^3 we have the Lagrange identities

$$((X \times Y), (U \times V)) = \begin{vmatrix} (X, U) & (X, V) \\ (Y, U) & (Y, V) \end{vmatrix}.$$

4.5. **Basic Properties of Riemannian Curvature Tensor.** The following symmetry relations hold for the curvature tensor and curvature operator at each point, and hence for all vector fields.

(1) $R(X, Y) \cdot Z = R(Y, X) \cdot Z = 0$
(2) $R(X, Y) \cdot Z = R(Y, Z) \cdot X = R(Z, X) \cdot Y = 0$
(3) $(R(X, Y) \cdot Z, W) + = (R(X, Y) \cdot W, Z) = 0$
(4) $(R(X, Y) \cdot Z, W) + = (R(Z, W) \cdot X, Y)$.

For all $1 \leq i \leq k, 1 \leq n$, we have
$i.$ $R^j_{ik_1} + R^j_{1ik} = 0.$
$ii.$ $R^j_{ijk_1} + R^j_{i_1k} + R^j_{ik_1} = 0.$
$iii.$ $R_{ijk_1} + R_{jik} = 0.$
$iv.$ $R_{ijk_1} = R_{kji_1}.$
$v.$ $R_{ijk_1} + R_{ik_1j+R_{i_1jk}}$

Definition 4.5.1.
The sectional curvature $K(\Pi)$ of the section Π with orthonormal basis x, y if defined as

$$K(\Pi) = -R(x, y, x, y) - (R(x, y) \cdot x, y)$$

From the symmetry and linearity properties it is to see that replacing x, y by any pair of vector $x\prime, y\prime$ where $x = \alpha x\prime + \beta y\prime$ and $y = \gamma x\prime + \delta y\prime$ gives the relation

$$(\frac{1}{\Delta^2})(R(x\prime, y\prime) \cdot x\prime, y\prime) = (R(x, y) \cdot x, y).$$

Where $\Delta = \alpha\gamma - \beta\delta$, the determinant of coefficients. If $x\prime, y\prime$ is also an orthonormal pair, then $\Delta = \pm 1$ so that the definition of $K(\Pi)$ is independent of the pair used. If it is just any arbitrary linear independent pair, the using $\Delta^2 = (x', y')(y', y') - (y', y')^2$ we have

$$K(\Pi) = \frac{(R(x', y') \cdot x', y')}{(x', y')(y', y') - (x', y')^2}.$$

in local coordinate, using $E_i, E_j = g_{ij}$ and notation above,

$$K(\Pi) = -\frac{\sum R_{ijk1}\alpha^i\beta^j\alpha^k\beta^1}{\sum(g_{ik}g_{j1} - g_{ik}g_{jk})\alpha^i\beta^j\alpha^k\beta^1}$$

5. Lie group and manifold:

The space $\mathbb{R}^n$ is a C^∞ manifold and at the same an Abelian group operation given by component wise addition. Moreover, the algebric and differentiable structure are related :$(X, Y) \to X + Y$ is a C^∞ mapping of the product manifold $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}_n$, that is the group operation is differentiable. We also see that the mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ given by taking each element x to its inverse $-x$ is differentiable.

Definition 5.1. A Lie group is a group $\mathcal{G}$ carrying the structure of a differentiable manifold or, more generally, of a disjoin union of finitely many differentiable manifolds for which the following maps are differentiable:
$\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ (Multiplication), $(g, h) \to g \cdot h$.
And $\mathcal{G} \to \mathcal{G}$ (Inverse), $g \to g^{-1}$.
We say that $\mathcal{G}$ acts on a differentiable manifold M from the left if there is a differentiable map $\mathcal{G} \times M \to M, (g, x) \to gx$.
That respects the lie group structure of $\mathcal{G}$ in sense that $g(hx) = (g \cdot h)x$ for all $g, h \in \mathcal{G}, x \in M$.

Example of Lie group :

- Euclidean space $\mathbb{R}^n$ with ordinary vector addition as the group operation becomes an $n-$dimensional non compact abelian Lie group.
- The circle group $S\prime$ consisting of angle $mod 2\pi$ under addition or complex numbers with absolute value 1 under multiplication is one dimensional compact abelian Lie group.
- The group $\mathcal{G}ln(R)$ of invertible matrices (under matrix multiplication) is Lie group n^2, called the general linear group, it has a closed connected subgroup $\mathcal{G}ln(R)$, the special linear group, consisting of matrices of dimensional 1. Which is also a Lie group.

Properties :

1. The diffeomorphism group of a Lie group act on the Lie Group.
2. Every Lie group is parallizable, and hence an orientable manifold (there is a bundle and the product of itself with the tangent space at identity).

6. Geodesic Curves on Riemannian Manifolds

A geodesics is a generalization of the notion of a ”straight line” to ”curved space”.
We shall define and study the class of curves called on M and $\frac{dp}{dt}$ its velocity vector, defined for some open interval $a < t < b$ of R; we suppose it to be of class C^2 at least.
Definition 6.1. The (parameterized) curve $p(t)$ is said to be a geodesic if its velocity vector is constant (parallel), that is, if it satisfies the condition $(\frac{D}{dt})(\frac{dp}{dt}) = 0$, the equation of a geodesic, for $a < t < b$.
Definition 6.2. A geodesic segment whose length is the distance between its endpoints is called a minimal geodesic.
Theorem 6.1. For each $q \in M$ a Riemannian Manifold, there exists a neighborhood B and an $\epsilon > 0$ such that each pair of points of B can be joined by a unique geodesic of length $L < \epsilon$, and the length $L^{'}$ of any piecewise $C^{'}$ curve joining these two points is $\geq L$. Moreover, $L^{'} = L$ if and only if these paths coincide as a point sets, or equivalently, when parameterized by arc length, are identical.

6.1. **Metric Geometry.** in metric geometry, a geodesic is a curve which is everywhere locally distance minimizer, a curve $\gamma : I \rightarrow M$ from interval I of the real to the metric space M is a geodesic if there is a constant $\gamma > 0$ such that for any $t \in I$ there is neighborhood J of t in I such that for any $t_1, t_2 \in J$ we have

$$d(\gamma(t_1), \gamma(t_2)) = \gamma|t_1 - t_2|$$

The generalizes the notion of geodesic for Riemannian manifold however, in metric geometry the geodesic considered is often equipped with natural parameterization i.e. in the above identity

$$\gamma = 1 \text{ and } d(\gamma(t_1), \gamma(t_2)) = \gamma|t_1 - t_2|.$$

If the last equality is satisfied $t_1, t_2 \in I$ the geodesic is called a minimizing geodesic or shortest path.
Example 6.1.1.
The most familiar examples are straight lines in Euclidean geometry on a sphere, the images of geodesics are the great circles. The shorter path from point A to point B on a sphere is given by shorter piece of the great circle passing through A and B. If A and B are antipodal points, then there infinitely many shortest paths between them.

7. Conclusion

In conclusion of this paper, we can say that, the role of Riemannian geometry in mathematics is very important especially in covariance derivative and exterior derivative and curvature as general, with many applications in physics,mechanics and various scientific areas. Also the geodesics are very important mathematical tool in mathematics and theoretical physics.

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