

Out-of-plane equilibrium points in the ER3BP with triaxial-radiating primaries with a P-R drag force surrounded by a belt

Abstract

This paper studies the motion of an infinitesimal particle near the out-of-plane equilibrium points in the elliptic restricted three body problem (ER3BP) when the primaries are triaxial rigid bodies, sources of radiation with a Poyting-Robertson (P-R) drag force surrounded by a belt. It is observed that there exist two out-of-plane equilibria which lie in the $\xi\zeta$ - plane in symmetrical positions with respect to the orbital plane. The parameters involved in the system affect their positions. The position changes with an increase in triaxiality, radiation and belt in the presence of P-R drag force. We found that for the binary system the effect of triaxiality and the belt moves the out-of-plane equilibrium points in opposite directions. The position and linear stability of the out-of-plane equilibrium points are investigated numerically using first, arbitrary values for the parameters and then for the two binary systems (Xi-Bootis and Kruger 60) and they are found to be unstable in each case.

Keywords: Triaxiality, Radiation, Elliptic restricted three body problem, Stability, gravitational potential from the belt, binary systems

1. Introduction

One of the most important problem in celestial mechanics, is the three-body problem. It has been studied in many scientific researches, especially in the field of astrodynamics and

astrophysics. Renowned mathematicians and scientists have produced interesting and significant results in an attempt to understand and predict the motion of natural bodies.

Leonhard Euler in 1765 modified the three body problem into the restricted three body problem, a simplification of the general problem where one of the bodies is taken to have negligible or infinitesimal mass. The earliest and simplest particular solution was discovered by Euler in 1765. In Euler's solution, the three masses are collinear, positioned according to their masses. Then at every future time, the masses remain collinear, and the distances between them remain at the same ratio. These positions are the equilibrium points of the system and they lie on the line joining the primaries and are called collinear equilibrium points ($L_{1,2,3}$) (see Fig.1). Later on, Lagrange in 1772 discovered another particular solution. In Lagrange's solution the three masses occur at the vertices of an equilateral triangle. The masses will then follow elliptical orbits around their centre of mass while remaining in an equilateral triangle formation, the negligible mass remains stable at this point in its orbit (see Fig.2). These points are the equilibrium points and they are two called Lagrangian triangular equilibrium points ($L_{4,5}$). These equilibrium points lie on the $\xi\eta$ -plane.

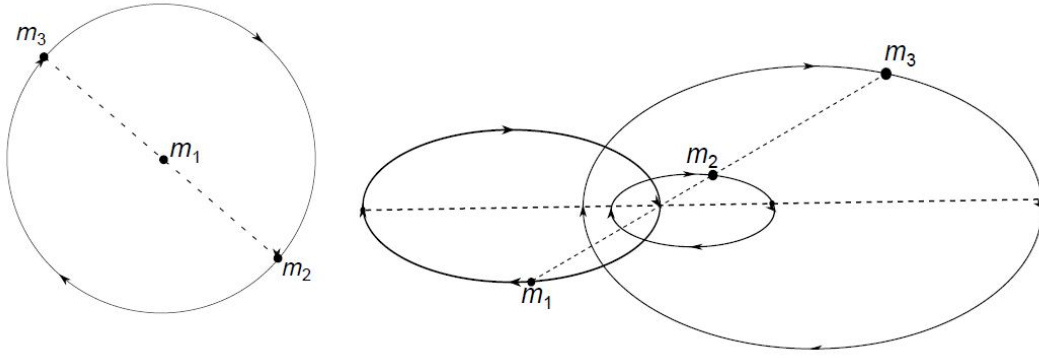


Figure 1: The Euler solution: the three bodies remain collinear at all times, in elliptical orbits around the centre of mass. Left: all masses equal. Right: unequal masses

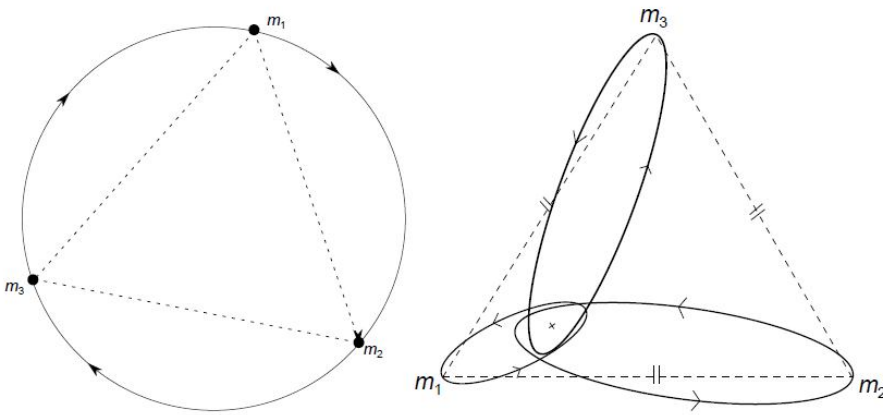


Figure 2: The Lagrange solution: the three bodies form an equilateral triangle at all times. Left: three equal masses. Right: unequal masses

The existence of out-of-plane equilibrium points (OPEPs) in the $\xi\zeta$ - plane was first pointed out by Radviesky [1,2] when studying the case of sun planet-particle and Galaxy-kernel-sun-particle and found the two equilibrium points $L_{6,7}$ on the $\xi\zeta$ - plane to be symmetrical with [4] respect to the $\xi\eta$ -plane. Since then several authors Singh [3], [4], [5], [6] and [7] based their studies on the Radviesky model under different characterisations in circular restricted three-body problem (CR3BP) or ER3BP.

On the other hand [8],[9],[10] and [11] have studied the out-of-plane points in the CR3BP or ER3BP using different models under the influence of radiation pressure or Pr-drag or oblateness or a combination of one of these forces and they found the OPEPs unstable. The basins of attraction around the OPEPs in the Copenhagen R3BP was determined by [11] using a multivariate version of the Newton-Raphson interactive method around the OPEPs when the primaries are oblate. [6] obtained OPEPs analytically and then numerically by approximation with power series expansion about the smaller primary, when one of the primaries is oblate and the other radiating and when one or two of the primaries are oblate and proved that the OPEPs exist, but they are unstable. Four additional OPEPs were obtained as result of the oblateness of the primaries. Authors like [4], [12] and [10], extend these results into the ER3BP, when one or the two primaries are oblate with or without radiation pressure and found that OPEPs exist but are unstable. A generalized out-of-plane model studied by [12] in which the primary is oblate and the secondary is triaxial and radiating in the ER3BP, shows that the OPEPs ($L_{6,7}$) are affected by the oblateness of the primary, radiation pressure and triaxiality of the secondary, semi-major axis and eccentricity. Also, [4] found that the position and stability of out-of-plane points are greatly affected by oblateness and radiation pressure of the primaries and the eccentricity of the orbits. Our work is an extension of [4] with radiating-triaxial primaries with a P-R drag force and a potential from the belt in the framework of ER3BP. This work to the best of our knowledge does not yet exist in the literature. The OPEPs has not yet been extensively researched, hence works devoted to it are few. Only recently, [13] presented a paper on OPEPs where the primaries are radiating with effective Poyting-Robertson drag force with small

perturbation in corolis and centrifugal forces and obtained four OPEPs ($L_{6,7,8,9}$) out of which two $L_{6,7}$ are stable in the absence of P-R drag.

There are several communications in both ER3BP and CR3BP. It was assumed in the classical CR3BP that only gravitational forces influence the motion of the infinitesimal mass, until [1] discovered that radiation influences the stability of the third body this gave rise to photogravitational R3BP where one of the participating bodies or both are intense emitters of radiation. It is inadequate to consider only the gravitational force in some solar or stellar dynamic problems. For instance, gravity is not the only dominant force present when a star collides with a particle, but also the repulsive forces of radiation pressure [1]. Therefore, the potential function of the CR3BP was amended so as to admit to other perturbing forces such as radiation, triaxiality, oblateness and so on. These have enabled several researchers to propose different models under different characterisations. For instance [14], [15], [16], [17] and [6] have carried out detailed investigation in CR3BP or ER3BP on the existence of the collinear and non-collinear (triangular) equilibrium points and the stability of motion around these points in $\xi\eta$ -plane and it was found to exist and the triangular equilibrium points are conditionally stable, when $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio; while the collinear equilibrium points are unstable.

Interest in binary systems has increased, in the last decade, this is in part because many extra solar planetary systems revealed the presence of belts of dust particles that are regarded as the young analogues of Kuiper belt. [18] and [19] suggest the position of the disc relative to the

planets when they studied the effects of belts on planetary orbits and conclude that the planets might prefer to stay near the inner part instead of outer part of the belt. Later the R3BP was modified in their paper [20] include the effect of additional gravitational force from the belt on the infinitesimal mass, which results in the formation of new libration (equilibrium) points.

In the studies conducted on belt more attention is given to motion of the particle around triangular equilibrium points very few articles are available in OPEPs. The model by [21] focus on the CR3BP when the two primaries are oblate spheroids and radiating with the gravitational potential from a belt. They obtained in addition to the usual five libration points two new collinear points as a result of the potential from the belt. The influence of the belt and non-sphericity of the primaries on the infinitesimal mass was studied by [22] . They did analytic and numerical treatment of motion of a dust grain particle around triangular equilibrium points when the bigger primary is triaxial and the smaller one an oblate spheroid with a potential from the belt. They found that triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio. It was also observed that the potential from the belt increase the range of stability.

In another study by [23] where the more massive primary is a triaxial body and less massive one an oblate spheroid emitting radiation enclosed by a circumbinary disc (belt) in the presence of Pr- drag force it was proved that the potential from the belt is a stabilizing force as it can change an unstable condition to a stable one even when the mass parameter exceeds the critical mass value ($\mu > \mu_c$).

In this paper we investigate the effect of triaxiality, radiation pressure with a P-R drag force and the potential of the belt on a test particle around the OPEPs in the framework of ER3BP.

This paper is organized in 6 sections. The first section is introduction, the equations of motion are described in section 2, locations of equilibrium points can be found in section 3, while section 4 contains the linear stability analysis of the out-of-plane equilibrium points using numerical applications, section 5 is discussion and finally section 6 is conclusion.

2. Equation of Motion

The equation of motion of an infinitesimal particle in the ER3BP when the primaries are triaxial and radiating with a P-R drag force and a gravitational potential from the belt, in a dimensionless rotating coordinate system (ξ, η, ζ) are as follows:

$$\begin{aligned}\xi'' - 2\eta' &= \Omega_\xi \\ \eta'' + 2\xi' &= \Omega_\eta \\ \zeta'' &= \Omega_\zeta\end{aligned}\tag{1}$$

$$\Omega^* = (1 - e^2)^{-1/2} \left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2} \left\{ \frac{(1-\mu)q_1}{r_1} + \frac{(1-\mu)(2\sigma_1 - \sigma_2)q_1}{2r_1^3} - \frac{3(1-\mu)(\sigma_1 - \sigma_2)q_1\eta^2}{2r_1^5} - \frac{3(1-\mu)\sigma_1 q_1 \zeta^2}{2r_1^5} + \frac{\mu q_2}{r_2} + \frac{\mu(2\sigma_3 - \sigma_4)q_2}{2r_2^3} - \frac{3\mu(\sigma_3 - \sigma_4)q_2\eta^2}{2r_2^5} - \frac{3\mu\sigma_3 q_2 \zeta^2}{2r_2^5} + \frac{M_b}{[r^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{1/2}} \right\} \right] \tag{2}$$

$$\Omega_\xi = \frac{\partial \Omega^*}{\partial \xi} - \frac{(1-e^2)^{-1/2} W_2 F_1}{n^2 r_2^2}$$

$$\Omega_\eta = \frac{\partial \Omega^*}{\partial \eta} - \frac{(1-e^2)^{-1/2} W_2 F_2}{n^2 r_2^2} \quad \text{where}$$

$$\Omega_\zeta = \frac{\partial \Omega^*}{\partial \zeta}$$

$$F_1 = \frac{(\xi + \mu) \{ (\xi + \mu) \xi' + \eta \eta' + \zeta \zeta' \} + \xi' - n \eta}{r_2^2}$$

$$F_2 = \frac{\eta \{ (\xi + \mu - 1) \xi' + \eta \eta' + \zeta \zeta' \} + \eta' + n (\xi + \mu - 1)}{r_2^2}$$

$$W_2 = \frac{\mu(1-q_2)}{c_d}$$

$$r_1^2 = (\xi + \mu)^2 + \eta^2 + \zeta^2$$

$$r_2^2 = (\xi + \mu - 1)^2 + \eta^2 + \zeta^2 \quad (3)$$

$$n^2 = \frac{1}{a} \left[1 + \frac{3}{2} e^2 + \frac{3}{2} (2\sigma_1 - \sigma_2) + \frac{3}{2} (2\sigma_3 - \sigma_4) + \frac{2M_b r_c}{[r_c^2 + T^2]^{3/2}} \right] \quad (4)$$

The effect of the gravitational potential of the belt is expressed using a model that explains a flattened potential and which best describes the gravitational potential within a system given by [24] as:

$$V(r, \zeta) = \frac{M_b}{\sqrt{r^2 + (c + \sqrt{\zeta^2 + d^2})}} \quad (5)$$

r is the radial distance of the infinitesimal mass and is given by $r^2 = \xi^2 + \zeta^2$, where c and d are the parameters which determine the density profile of the belt [24] and [25] r_c is the distance of any out-of-plane point from the origin and T is their sum, r_1 and r_2 are distances of the bigger and smaller primaries from the infinitesimal particle, respectively. q_1 and q_2 are their mass reduction factor (radiation factor), while (σ_1, σ_2) and (σ_3, σ_4) denote their triaxiality, respectively. n is the mean motion, a and e are the semi major axis and the eccentricity of the elliptic orbis respectively.

3. Location of out-of-plane equilibrium points

The equilibrium points of the infinitesimal mass is obtained if the equation $\xi' = \eta' = \xi'' = \eta'' = \zeta' = \zeta'' = 0$ satisfies Equation of motion (1); they are the solutions of the system of equations

$\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$. Hence we have:

$$\Omega_\xi =$$

$$\left[\xi - \frac{1}{n^2} \left\{ \frac{(1-\mu)(\xi+\mu)q_1}{r_1^3} + \frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5} - \frac{15(1-\mu)(\xi+\mu)\sigma_1}{2r_1^7} q_1 \eta^2 - \frac{15(1-\mu)(\xi+\mu)\sigma_1 q_1 \zeta^2}{2r_1^7} + \right. \right. \\ \left. \left. \frac{\mu(\xi+\mu-1)q_2}{r_2^3} + \frac{3\mu(\xi+\mu-1)(2\sigma_3-\sigma_4)q_2}{2r_2^5} - \frac{15\mu(\xi+\mu-1)\sigma_3}{2r_2^7} q_2 \eta^2 - \frac{15\mu(\xi+\mu-1)\sigma_3 q_2 \zeta^2}{2r_2^7} + \right. \right. \\ \left. \left. \frac{M_b \xi}{\left[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2 \right]^{\frac{3}{2}}} \right\} + \frac{W_2 \eta}{nr_2^2} \right] = 0 \quad (5)$$

$$\Omega_\eta = (1 - e^2)^{-1/2} \eta \left[1 - \frac{1}{n^2} \left\{ \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)q_1}{2r_1^5} + \frac{3(1-\mu)(\sigma_1-\sigma_2)}{r_1^5} q_1 - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^7} q_1 \eta^2 - \right. \right. \\ \left. \left. \frac{15(1-\mu)\sigma_1 q_1 \zeta^2}{2r_1^7} + \frac{\mu q_2}{r_2^3} + \frac{3\mu(2\sigma_3-\sigma_4)q_2}{2r_2^5} + \frac{3\mu(\sigma_3-\sigma_4)}{r_2^5} q_2 - \frac{15\mu(\sigma_3-\sigma_4)}{2r_2^7} q_2 \eta^2 - \frac{15\mu \sigma_3 q_2 \zeta^2}{2r_2^7} + \right. \right. \\ \left. \left. \frac{M_b}{\left[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2 \right]^{\frac{3}{2}}} \right\} - \frac{W_2(\xi+\mu-1)}{nr_2^2} \right] = 0 \quad (6)$$

$$\Omega_\zeta = (1 - e^2)^{-1/2} \left[-\frac{\zeta}{n^2} \left\{ \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} q_1 + \frac{3(1-\mu)\sigma_1}{r_1^5} q_1 - \frac{15(1-\mu)(\sigma_1-\sigma_2)}{2r_1^7} q_1 \eta^2 - \right. \right. \\ \left. \left. \frac{15(1-\mu)\sigma_1 q_1 \zeta^2}{2r_1^7} + \frac{\mu q_2}{r_2^3} + \frac{3\mu(2\sigma_3-\sigma_4)}{2r_2^5} q_2 + \frac{3\mu\sigma_3}{r_2^5} q_2 - \frac{15\mu(\sigma_3-\sigma_4)}{2r_2^7} q_2 \eta^2 - \frac{15\mu\sigma_3 q_2 \zeta^2}{2r_2^7} + \right. \right. \\ \left. \left. \frac{M_b \zeta}{\left[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2 \right]^{\frac{3}{2}}} \right\} - \frac{W_2 \zeta}{nr_2^2} \right] = 0$$

$$\left. \frac{M_b [C(\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \right\} = 0 \quad (7)$$

The out-of-plane equilibrium points are the solution of above equations, when

$$\xi \neq 0, \eta = 0 \text{ and } \zeta \neq 0$$

From (7) with $\zeta \neq 0$ we get:

$$\begin{aligned} & \frac{(1-\mu)q_1}{r_1^3} + \frac{3(1-\mu)(2\sigma_1-\sigma_2)}{2r_1^5} q_1 + \frac{3(1-\mu)\sigma_1}{r_1^5} q_1 - \frac{15(1-\mu)\sigma_1 q_1 \zeta^2}{2r_1^7} + \frac{\mu q_2}{r_2^3} + \frac{3\mu(2\sigma_3-\sigma_4)}{2r_2^5} q_2 + \frac{3\mu\sigma_3}{r_2^5} q_2 - \\ & \frac{15\mu\sigma_3 q_2 \zeta^2}{2r_2^7} \\ & + \frac{M_b [C(\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} = 0 \end{aligned} \quad (8)$$

Let $Q_1 = (1 - \mu)q_1$ and $Q_2 = \mu q_2$, then (8) becomes:

$$\begin{aligned} & \frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1}{r_1^5} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} + \frac{3Q_2\sigma_3}{r_2^5} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} + \\ & \frac{M_b [C(\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} = 0 \end{aligned} \quad (9)$$

Also from Equation (5) we write:

$$n^2 \xi - \frac{Q_1(\xi+\mu)}{r_1^3} - \frac{3Q_1(\xi+\mu)(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{15Q_1(\xi+\mu)\sigma_1 \zeta^2}{2r_1^7} - \frac{Q_2(\xi+\mu-1)}{r_2^3} - \frac{3Q_2(\xi+\mu-1)(2\sigma_3-\sigma_4)}{2r_2^5} + \frac{15Q_2(\xi+\mu-1)\sigma_3 \zeta^2}{2r_2^7} - \frac{M_b \xi}{\left[\xi^2 + (c + \sqrt{\zeta^2 + b^2})^2\right]^{\frac{3}{2}}} + \frac{W_{2\eta}}{nr_2^2} = 0 \quad (10)$$

Expanding Equation (10) we obtained:

$$\xi \left\{ 1 - \frac{1}{n^2} \left(\frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1^5} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} + \frac{M_b}{\left[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2\right]^{\frac{3}{2}}} \right) \right\} - \frac{\mu}{n^2} \left(\frac{Q_1}{r_1^3} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} + \frac{Q_2}{r_2^3} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} + \frac{3Q_1(2\sigma_1-\sigma_2)}{2r_1^5} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} \right) + \frac{1}{n^2} \left(\frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3-\sigma_4)}{2r_2^5} - \frac{15Q_2\sigma_3\zeta^2}{2r_2^7} \right) + \frac{W_{2\eta}}{nr_2^2} = 0 \quad (11)$$

From (9) we have:

$$\begin{aligned}
& \frac{15Q_1(\sigma_1 - \sigma_2)\zeta^2}{2r_1^7} + \frac{15Q_2(\sigma_3 - \sigma_4)\zeta^2}{2r_2^7} = \frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1}{r_1^5} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_2^5} + \frac{3Q_2\sigma_3}{r_2^5} \\
& + \frac{M_b [c(\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \\
\zeta^2 = & \frac{2r_1^7 r_2^7}{15Q_1(\sigma_1 - \sigma_2) r_2^7 + 15Q_2(2\sigma_3 - \sigma_4) r_1^7} \left\{ \frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{Q_2}{r_2^3} + \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_2^5} + \frac{3Q_1\sigma_1}{r_1^5} + \frac{3Q_2\sigma_3}{r_2^5} + \right. \\
& \left. \frac{M_b [c(\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \right\} \quad (12)
\end{aligned}$$

Substituting Equation (9) into Equation (11) and solving we obtained:

$$\begin{aligned}
& \xi \left\{ 1 - \frac{1}{n^2} \left(-\frac{3Q_1\sigma_1}{r_1^5} - \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} - \frac{M_b [c + (\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \right) \right\} \\
& - \frac{\mu}{n^2} \left(-\frac{3Q_1\sigma_1}{r_1^5} - \frac{3Q_2\sigma_3}{r_2^5} - \frac{M_b [c + (\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \right) \\
& + \frac{1}{n^2} \left(-\frac{Q_1}{r_1^3} - \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} - \frac{3Q_1\sigma_1}{r_1^5} - \frac{3Q_2\sigma_3}{r_2^5} + \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} - \frac{M_b [c + (\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \right) + \frac{W_{2\eta}}{nr_2^2} = 0
\end{aligned}$$

$$\frac{Q_1}{r_1^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1(1-\mu)}{r_1^5} + \frac{3Q_2Q_1\sigma_3}{r_2^5} - \frac{15Q_1\sigma_1\zeta^2}{2r_1^7} +$$

$$\frac{M_b Q_1 [c + (\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} + \frac{W_{2\eta}}{nr_2^2}$$

i.e. $\xi =$ _____

$$n^2 + \frac{3Q_1\sigma_1}{r_1^5} + \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} + \frac{M_b [c (\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}}$$

$$(1 - \mu) \left\{ \frac{1}{r_1^3} + \frac{3(2\sigma_1 - \sigma_2)}{2r_1^5} + \frac{3Q_1\sigma_1}{r_1^5} + \frac{3Q_2\sigma_3}{r_2^5} - \frac{15\sigma_1\zeta^2}{2r_1^7} + \frac{M_b [c + (\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \right\} + \frac{W_{2\eta}}{nr_2^2}$$

$\xi =$ _____

$$n^2 + \frac{3Q_1\sigma_1}{r_1^5} + \frac{3Q_2\sigma_3}{r_2^5} + \frac{M_b}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} + \frac{M_b [c (\zeta^2 + d^2)^{-1/2} + 1]}{[\xi^2 + (c + \sqrt{\zeta^2 + d^2})^2]^{3/2}} \quad (13)$$

We use the initial approximation $\xi_o = (1 - \mu)$ and $\zeta_o = \sqrt{3(2\sigma_3 - \sigma_4)}$ to obtain the positions of out-of-plane points $L_{6,7}$ numerically with the aid of the software package mathematica 10.4 in

the form of power series to third order term in $(2\sigma_3 - \sigma_4)$ from Equation (12) and (13) as: (see[6] and[4]):

$$\begin{aligned}
\xi_o = & \frac{1}{2a\mu q_2} \{ [(-1 + \mu) - 3\sqrt{3}(-1 + \mu)(2 + 3e^2 - 2aq_1)] \\
& + 9W_2[(2\sigma_2 - \sigma_1)(3 - 3aq_1)(2\sigma_3 - \sigma_4)^{3/2}] \} \\
& - \frac{1}{4a\mu q_2} \{ [9W_2 \sqrt{3} (-1 + \mu) (2 + 3a(2 + 15(2\sigma_2 - \sigma_1)q_1) (2\sigma_3 - \sigma_4)^{5/2}) - [27W_2 (-1 + \\
& \mu)(2 + 3e^2 - 2aq_1 + (2\sigma_1 - \sigma_2)(3 - 3aq_1)(-2 - 3e^2 + (2\sigma_2 - \sigma_1)(-3 + 6a) \\
& (-1 + \mu)q_1))] (4(a^2\mu^2q_2^2)^{-1}(2\sigma_3 - \sigma_4)^3 + 0(2\sigma_3 - \sigma_4)^{7/2}] \} \quad (14)
\end{aligned}$$

$$\begin{aligned}
\zeta_o = & \sqrt{3}\sqrt{(2\sigma_3 - \sigma_4)} - \frac{9(-1+\mu)(2+9(2\sigma_1-\sigma_2)q_1)}{10\mu q_2} (2\sigma_3 - \sigma_4)^2 \\
& + \frac{81(-1+\mu)}{20\mu q_2} (2 + 25)(2\sigma_1 - \sigma_2)q_1(2\sigma_3 - \sigma_4)^3 - 0(2\sigma_3 - \sigma_4)^{7/2} \quad (15)
\end{aligned}$$

The equilibrium points $(\xi_o, 0, \pm\zeta_o)$ given by equations (14) and (15) are called the out-of-plane equilibrium points.

4.0 Linear stability of out-of-plane equilibrium points

4.1 Variational Equation

Let the positions of any out-of-plane points be denoted by (ξ_0, η_0, ζ_0) and a small displacement from this position be denoted as (α, β, γ) having a new coordinates $(\xi_0 + \alpha, \eta_0 + \beta, \zeta_0 + \gamma)$ in the neighbourhood of (ξ_0, η_0, ζ_0) .

The variations can be written as:

$$\alpha = \xi - \xi_0, \quad \beta = \eta - \eta_0, \quad \gamma = \zeta - \zeta_0 \quad (16)$$

It has the velocity

$$\dot{\xi} = \dot{\alpha}, \quad \dot{\eta} = \dot{\beta}, \quad \dot{\zeta} = \dot{\gamma}$$

and acceleration:

$$\ddot{\xi} = \ddot{\alpha}, \quad \ddot{\eta} = \ddot{\beta}, \quad \ddot{\zeta} = \ddot{\gamma}$$

Substituting these values into Equation (1) and expanding the R.H.S. by Taylor series we obtained the variational equation as:

$$\begin{aligned} \xi'' - 2\eta' &= \alpha(\Omega_{\xi\xi})^0 + \beta(\Omega_{\xi\eta})^0 + \gamma(\Omega_{\xi\zeta}) + \dot{\alpha}(\Omega_{\xi\xi})^0 + \dot{\beta}(\Omega_{\xi\eta})^0 + \dot{\gamma}(\Omega_{\xi\zeta})^0 \\ \ddot{\beta} + 2\dot{\alpha} &= \alpha(\Omega_{\eta\xi})^0 + \beta(\Omega_{\eta\eta})^0 + \gamma(\Omega_{\eta\zeta}) + \dot{\alpha}(\Omega_{\eta\xi})^0 + \dot{\beta}(\Omega_{\eta\eta})^0 + \dot{\gamma}(\Omega_{\eta\zeta})^0 \\ \ddot{\gamma} &= \alpha(\Omega_{\zeta\xi})^0 + \beta(\Omega_{\zeta\eta})^0 + \gamma(\Omega_{\zeta\zeta}) + \dot{\alpha}(\Omega_{\zeta\xi})^0 + \dot{\beta}(\Omega_{\zeta\eta})^0 + \dot{\gamma}(\Omega_{\zeta\zeta})^0 \end{aligned} \quad (17)$$

We consider only linear terms in α, β, γ . The second partial derivatives of Ω are denoted by subscript. The superscript 0, indicates that derivatives have been obtained at equilibrium point (ξ_0, η_0, ζ_0)

4.2 Characteristic Equation

Let the trial solution be as follows:

$$\begin{aligned} \alpha &= P e^{\lambda t} & \beta &= Q e^{\lambda t} & \gamma &= V e^{\lambda t} \\ \dot{\alpha} &= P \lambda e^{\lambda t} & \dot{\beta} &= Q \lambda e^{\lambda t} & \dot{\gamma} &= V \lambda e^{\lambda t} \\ \ddot{\alpha} &= P \lambda^2 e^{\lambda t} & \ddot{\beta} &= Q \lambda^2 e^{\lambda t} & \ddot{\gamma} &= V \lambda^2 e^{\lambda t} \end{aligned}$$

then substituting these values into variational Equation (17) we get:

$$P \lambda^2 e^{\lambda t} - 2 Q \lambda e^{\lambda t} = P e^{\lambda t} \Omega^0_{\xi\xi} + Q e^{\lambda t} \Omega^0_{\xi\eta} + V e^{\lambda t} \Omega^0_{\xi\zeta} + P \lambda e^{\lambda t} \Omega^0_{\xi\xi} + Q \lambda e^{\lambda t} \Omega^0_{\xi\eta} + V \lambda e^{\lambda t} \Omega^0_{\xi\zeta}$$

$$Q \lambda^2 e^{\lambda t} + 2 P \lambda e^{\lambda t} = P e^{\lambda t} \Omega^0_{\eta\xi} + Q e^{\lambda t} \Omega^0_{\eta\eta} + V e^{\lambda t} \Omega^0_{\eta\zeta} + \lambda P e^{\lambda t} \Omega^0_{\eta\xi} + \lambda Q e^{\lambda t} \Omega^0_{\eta\eta} + V \lambda e^{\lambda t} \Omega^0_{\eta\zeta}$$

$$V \lambda^2 e^{\lambda t} = P e^{\lambda t} \Omega^0_{\zeta\xi} + Q e^{\lambda t} \Omega^0_{\zeta\eta} + V e^{\lambda t} \Omega^0_{\zeta\zeta} + \lambda P e^{\lambda t} \Omega^0_{\zeta\xi} + \lambda Q e^{\lambda t} \Omega^0_{\zeta\eta} + V \lambda e^{\lambda t} \Omega^0_{\zeta\zeta}$$

Multiplying through by $e^{-\lambda t}$ and rearranging we obtained the determinant as:

$$\begin{vmatrix} \lambda^2 - \Omega^0_{\xi\xi} - \lambda \Omega^0_{\xi\xi} & -2\lambda - \Omega^0_{\xi\eta} + \lambda \Omega^0_{\xi\eta} & -\Omega^0_{\xi\eta} - \lambda \Omega^0_{\xi\zeta} \\ 2\lambda - \Omega^0_{\eta\xi} - \lambda \Omega^0_{\eta\xi} & \lambda^2 - \Omega^0_{\eta\eta} - \lambda \Omega^0_{\eta\eta} & -\Omega^0_{\eta\zeta} - \lambda \Omega^0_{\eta\zeta} \\ -\Omega^0_{\zeta\xi} - \lambda \Omega^0_{\zeta\xi} & -\Omega^0_{\zeta\eta} - \lambda \Omega^0_{\zeta\eta} & \lambda^2 - \Omega^0_{\zeta\zeta} - \lambda \Omega^0_{\zeta\zeta} \end{vmatrix} = 0$$

Expanding the determinant we have:

$$\begin{aligned}
 & (\lambda^2 - \Omega^0_{\xi\xi} - \lambda\Omega^0_{\xi\xi})[(\lambda^2 - \Omega^0_{\eta\eta} - \lambda\Omega^0_{\eta\eta})(\lambda^2 - \Omega^0_{\zeta\zeta} - \lambda\Omega^0_{\zeta\zeta}) - \{(-\Omega^0_{\zeta\eta} - \lambda\Omega^0_{\zeta\eta})(-\Omega^0_{\eta\zeta} - \lambda\Omega^0_{\eta\zeta})\} - \\
 & (-2\lambda - \Omega^0_{\xi\eta} + \lambda\Omega^0_{\xi\eta})\{(-2\lambda - \Omega^0_{\xi\eta} + \lambda\Omega^0_{\xi\eta})(\lambda^2 - \Omega^0_{\zeta\zeta} - \lambda\Omega^0_{\zeta\zeta}) - \{(-\Omega^0_{\zeta\xi} - \lambda\Omega^0_{\zeta\xi})(-\Omega^0_{\eta\zeta} - \\
 & \lambda\Omega^0_{\eta\zeta})\}] + (-\Omega^0_{\xi\eta} - \lambda\Omega^0_{\xi\eta})\{[2\lambda - \Omega^0_{\eta\xi} - \lambda\Omega^0_{\eta\xi})(-\Omega^0_{\zeta\eta} - \lambda\Omega^0_{\zeta\eta}) - \{(-\Omega^0_{\zeta\xi} - \lambda\Omega^0_{\zeta\xi})(\lambda^2 - \Omega^0_{\eta\eta} - \\
 & \lambda\Omega^0_{\eta\eta})\}] = 0 \tag{18}
 \end{aligned}$$

Further expansion of Equation (18) yield the characteristic Equation:

$$\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda + a_6 = 0 \tag{19}$$

Where:

$$a_1 = -\Omega^0_{\zeta\zeta} - \Omega^0_{\eta\eta} - \Omega^0_{\xi\xi}$$

$$\begin{aligned}
 a_2 = & -\Omega^0_{\zeta\zeta} - \Omega^0_{\eta\eta} - \Omega^0_{\xi\xi} + \Omega^0_{\xi\eta}\Omega^0_{\eta\xi} + \Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - \Omega^0_{\xi\xi} - \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} + 4 + \\
 & 4\Omega^0_{\zeta\zeta} - 2\Omega^0_{\eta\xi} + 2\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta} + 4\Omega^0_{\xi\eta} - 2\Omega^0_{\xi\eta}\Omega^0_{\zeta\zeta} - \Omega^0_{\eta\xi}\Omega^0_{\xi\eta} + \Omega^0_{\xi\eta}\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta} - \\
 & \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}
 \end{aligned}$$

$$\begin{aligned}
 a_3 = & \Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - \Omega^0_{\zeta\eta}\Omega^0_{\eta\zeta} - \Omega^0_{\zeta\eta}\Omega^0_{\eta\zeta} - \Omega^0_{\zeta\eta}\Omega^0_{\eta\zeta} + \Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \\
 & \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\xi\eta}\Omega^0_{\eta\zeta} - 2\Omega^0_{\eta\xi} + \\
 & 2\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta} - 2\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta} + 2\Omega^0_{\xi\eta} - 2\Omega^0_{\xi\eta}\Omega^0_{\zeta\zeta} - \Omega^0_{\eta\xi}\Omega^0_{\xi\eta} + \Omega^0_{\xi\eta}\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta} - \\
 & \Omega^0_{\xi\eta}\Omega^0_{\eta\xi} - \Omega^0_{\xi\eta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta} - \Omega^0_{\xi\zeta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\eta}
 \end{aligned}$$

$$\begin{aligned}
a_4 = & -\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - \Omega^0_{\zeta\eta}\Omega^0_{\eta\zeta} + \Omega^0_{\zeta\zeta}\Omega^0_{\xi\xi} + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \Omega^0_{\xi\xi}\Omega^0_{\eta\dot{\eta}}\Omega^0_{\zeta\zeta} \\
& + \Omega^0_{\xi\xi}\Omega^0_{\xi\dot{\eta}}\Omega^0_{\eta\dot{\zeta}} - \Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\dot{\zeta}} - \Omega^0_{\xi\xi}\Omega^0_{\eta\dot{\eta}}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\eta}\Omega^0_{\eta\dot{\zeta}} \\
& + \Omega^0_{\xi\xi}\Omega^0_{\xi\dot{\eta}}\Omega^0_{\eta\zeta} + 2\Omega^0_{\eta\dot{\xi}}\Omega^0_{\zeta\zeta} - 2\Omega^0_{\zeta\xi}\Omega^0_{\eta\dot{\zeta}} - 2\Omega^0_{\zeta\dot{\xi}}\Omega^0_{\eta\zeta} - \Omega^0_{\xi\eta}\Omega^0_{\eta\xi} \\
& - \Omega^0_{\xi\eta}\Omega^0_{\zeta\dot{\xi}}\Omega^0_{\eta\dot{\zeta}} - 2\Omega^0_{\zeta\zeta}\Omega^0_{\xi\dot{\eta}} + \Omega^0_{\xi\dot{\eta}}\Omega^0_{\eta\xi}\Omega^0_{\zeta\dot{\zeta}} + \Omega^0_{\xi\dot{\eta}}\Omega^0_{\eta\dot{\xi}}\Omega^0_{\zeta\zeta} \\
& - \Omega^0_{\xi\dot{\eta}}\Omega^0_{\xi\zeta}\Omega^0_{\eta\dot{\zeta}} - \Omega^0_{\xi\dot{\eta}}\Omega^0_{\zeta\dot{\xi}}\Omega^0_{\eta\zeta} + 2\Omega^0_{\xi\zeta}\Omega^0_{\zeta\dot{\eta}} - \Omega^0_{\xi\zeta}\Omega^0_{\eta\dot{\xi}}\Omega^0_{\xi\dot{\eta}} \\
& - \Omega^0_{\xi\zeta}\Omega^0_{\zeta\xi} + \Omega^0_{\xi\zeta}\Omega^0_{\zeta\dot{\xi}}\Omega^0_{\eta\dot{\eta}} + 2\Omega_{\xi\dot{\zeta}}\Omega^0_{\zeta\eta} - \Omega_{\xi\dot{\zeta}}\Omega^0_{\eta\xi}\Omega^0_{\zeta\dot{\eta}} + \Omega_{\xi\dot{\zeta}}\Omega^0_{\zeta\xi}\Omega^0_{\eta\eta} \\
& + \Omega_{\xi\dot{\zeta}}\Omega^0_{\zeta\dot{\xi}}\Omega^0_{\eta\eta}
\end{aligned}$$

$$\begin{aligned}
a_5 = & -\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - \Omega^0_{\xi\xi}\Omega^0_{\eta\dot{\eta}}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\eta}\Omega^0_{\eta\dot{\zeta}} + \Omega^0_{\xi\xi}\Omega^0_{\xi\dot{\eta}}\Omega^0_{\eta\zeta} + \\
& \Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\eta}\Omega^0_{\eta\zeta} + 4\Omega^0_{\zeta\zeta} + 2\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta} - 2\Omega^0_{\zeta\xi}\Omega^0_{\eta\dot{\zeta}} - 2\Omega^0_{\zeta\zeta}\Omega^0_{\xi\eta} + \\
& \Omega^0_{\xi\eta}\Omega^0_{\eta\xi}\Omega^0_{\zeta\dot{\zeta}} + \Omega^0_{\eta\dot{\xi}}\Omega^0_{\zeta\zeta}\Omega^0_{\xi\eta} - \Omega^0_{\xi\eta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\dot{\zeta}} - \Omega^0_{\xi\eta}\Omega^0_{\zeta\dot{\xi}}\Omega^0_{\eta\zeta} - \\
& \Omega^0_{\xi\dot{\eta}}\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta} + 2\Omega^0_{\xi\zeta}\Omega^0_{\zeta\eta} - \Omega^0_{\xi\zeta}\Omega^0_{\eta\xi}\Omega^0_{\xi\dot{\eta}} - \Omega^0_{\xi\zeta}\Omega^0_{\eta\dot{\xi}}\Omega^0_{\zeta\eta} + \Omega^0_{\xi\zeta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\dot{\eta}} + \\
& \Omega^0_{\xi\zeta}\Omega^0_{\zeta\dot{\xi}}\Omega^0_{\eta\eta} - \Omega_{\xi\dot{\zeta}}\Omega^0_{\eta\xi}\Omega^0_{\zeta\eta} + \Omega_{\xi\dot{\zeta}}\Omega^0_{\zeta\xi}\Omega^0_{\eta\eta}
\end{aligned}$$

$$\begin{aligned}
a_6 = & \Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\eta}\Omega^0_{\eta\zeta} + \Omega^0_{\xi\eta}\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta} - \Omega^0_{\xi\eta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta} + \\
& \Omega^0_{\xi\dot{\eta}}\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta} + \Omega^0_{\xi\dot{\eta}}\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta} - \Omega^0_{\xi\zeta}\Omega^0_{\eta\xi}\Omega^0_{\zeta\eta} + \Omega^0_{\xi\zeta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\eta} = 0
\end{aligned}$$

The second partial derivatives evaluated at equilibrium point (ξ_0, η_0, ζ_0) as denoted by the superscript O are as follows:

$$\Omega^0_{\xi\xi} =$$

$$(1 - e^2)^{-1/2} \left[1 + \frac{1}{n^2} \left\{ \frac{3Q_1(\xi_0 + \mu)^2}{r_{10}^3} - \frac{Q_1}{r_{10}^3} + \frac{15Q_1(\xi_0 + \mu)^2(2\sigma_1 - \sigma_2)}{2r_{10}^7} - \frac{3Q_1}{2r_{10}^5} + \frac{105Q_1(\xi_0 + \mu)^2\sigma_1\zeta_0^2}{2r_{10}^9} - \frac{15Q_1\sigma_1\zeta_0^2}{2r_{10}^7} + \frac{3Q_2(\xi_0 + \mu - 1)^2}{r_{20}^5} - \frac{Q_2}{r_{20}^5} + \frac{15Q_2(\xi_0 + \mu - 1)^2(2\sigma_3 - \sigma_4)}{2r_{20}^7} - \frac{3Q_2}{2r_{20}^5} - \frac{105Q_2(\xi_0 + \mu - 1)^2\sigma_3\zeta_0^2}{2r_{20}^9} + \frac{15Q_2\sigma_3\zeta_0^2}{2r_{20}^7} + \frac{3M_b\xi_0^2}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{5/2}} - \frac{M_b}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{3/2}} \right\} \right]$$

$$\Omega^0_{\eta\eta} = (1 - e^2)^{-1/2} \left[1 - \frac{1}{n^2} \left\{ \frac{Q_1}{r_{10}^3} + \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_{10}^5} - \frac{15Q_1\sigma_1\zeta_0^2}{2r_{10}^7} + \frac{Q_2}{r_{20}^3} + \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_{20}^5} - \frac{15Q_2\sigma_3\zeta_0^2}{2r_{20}^7} - \frac{3M_b\eta_0^2}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{5/2}} + \frac{M_b}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{3/2}} \right\} \right]$$

$$\begin{aligned}
\Omega^0_{\zeta\zeta} = & (1 - e^2)^{-1/2} \left[\frac{1}{n^2} \left\{ -\frac{Q_1}{r_{10}^3} + \frac{3Q_1\zeta_0^2}{r_{10}^5} - \frac{3Q_1(2\sigma_1 - \sigma_2)}{2r_{10}^5} + \frac{15Q_1(2\sigma_1 - \sigma_2)\zeta_0^2}{2r_{10}^7} - \frac{3Q_1\sigma_1}{r_{10}^5} + \right. \right. \\
& \frac{15Q_1\sigma_1\zeta_0^2}{r_{10}^7} + \frac{45Q_1\sigma_1\zeta_0^2}{2r_{10}^7} - \frac{105Q_1\sigma_1\zeta_0^4}{2r_{10}^9} - \frac{Q_2}{r_{20}^3} + \frac{3Q_2\zeta_0^2}{r_{20}^5} - \frac{3Q_2(2\sigma_3 - \sigma_4)}{2r_{20}^5} + \frac{15Q_2(2\sigma_3 - \sigma_4)\zeta_0^2}{2r_{20}^7} - \frac{3Q_2\sigma_3}{r_{20}^5} + \\
& \frac{15Q_2\sigma_3\zeta_0^2}{2r_{20}^7} + \frac{45Q_2\sigma_3\zeta_0^2}{2r_{20}^7} - \frac{105Q_2\sigma_3\zeta_0^4}{2r_{20}^9} - \frac{M_b \left[c(\zeta_0^2 + d^2)^{-1/2} + 1 \right]}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{3/2}} + \frac{M_b c^2 \zeta_0^2 \left[(\zeta_0^2 + d^2)^{-3/2} \right]}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{3/2}} + \\
& \left. \left. \frac{3M_b \zeta_0^2 \left[c(\zeta_0^2 + d^2)^{-1/2} + 1 \right]^2}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{5/2}} \right\} \right]
\end{aligned}$$

$$\begin{aligned}
\Omega^0_{\xi\xi} = & (1 - e^2)^{-1/2} \left[\frac{3\zeta_0}{n^2} \left\{ \frac{Q_1(\xi_0 + \mu)}{r_{10}^5} + \frac{5Q_1(\xi_0 + \mu)(2\sigma_1 - \sigma_2)}{2r_{10}^7} - \frac{35Q_1(\xi_0 + \mu)\sigma_1\zeta_0^2}{2r_{10}^9} - \frac{15Q_1\sigma_1\zeta_0^2}{r_{10}^7} + \right. \right. \\
& \frac{Q_2(\xi_0 + \mu - 1)}{r_{20}^5} + \frac{5Q_2(\xi_0 + \mu - 1)(2\sigma_3 - \sigma_4)}{2r_{20}^7} - \frac{35Q_2(\xi_0 + \mu - 1)\sigma_3\zeta_0^2}{2r_{20}^9} + \frac{15Q_2\sigma_3\zeta_0^2}{r_{20}^7} + \\
& \left. \left. \frac{3M_b \xi_0^2 \left[c(\zeta_0^2 + d^2)^{-1/2} + 1 \right]}{\left[\xi_0^2 + \left(c + \sqrt{\zeta_0^2 + d^2} \right)^2 \right]^{5/2}} \right\} \right]
\end{aligned} \tag{20}$$

4. Numerical Application

We present the effect of triaxiality, belt and radiation pressure on the locations (Eqns.14 and 15) and stability (Equation 20) of OPEPs using arbitrary values In Table 1- 4 ,while in Table 6-9 the effects on the binary system (xi-Bootis and Kruger 60) are shown. The results in Table 6-9 were obtained by substituting the values of the orbital parametres (fixed) of the binary system (xi-Bootis and Kruger 60) and the varied values of triaxiality and radiation into (Equations 14 and 15) and Equation 20 for the locations and stability respectively.

Table 1: The effect of Triaxiality on the location and stability of out-of-plane equilibrium points for $e = 0.3$, $a = 0.87$, $\mu = 0.45$, $q_1 = 0.9988$, $q_2 = 0.9977$, $M_b = 0.01$, $W_2 = 1.17659 \times 10^{-12}$

S/no	Triaxiality				Out-of plane points		Roots of the characteristic equation		
	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.00	0.00	0.00	0.00	0.521012	0.311723	± 87.4420	120.2235	$\pm 33.9675i$
2.	0.02	0.015	0.003	0.002	0.477810	0.269001	± 87.5631	$\pm 120.563'$	$\pm 34.12654i$
3.	0.03	0.019	0.004	0.003	0.49061	0.24443	± 87.6615	± 120.9985	$\pm 34.241001i$

4.	0.04	0.02	0.005	0.004	0.517438	0.219650	± 88.43423	± 121.4342	$\pm 35.35790i$
5	0.05	0.03	0.006	0.005	0.539144	0.209341	± 88.9780	± 122.110	± 35.41283

Table2 : The effect of belt on the location and stability of out-of-plane equilibrium points for $e = 0.3$, $a = 0.87$, $\mu = 0.45$, $q_1 = 0.9988$, $q_2 = 0.9977$, $W_2 = 1.17659 \times 10^{-12}$

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm \zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.01	0.06735	0.741593	$-2.364473 \pm$ $0.364473i$	$\pm 1.470823i$	$2.364473 \pm$ $0.364473i$
2	0.02	0.04894	0.73965	$-6.243416 \pm$ $0.765014i$	$\pm 14.51723i$	$6.243416 \pm$ $0.765014i$
3	0.03	0.03646	0.72671	$-5.459825 \pm$ $0.886517i$	$\pm 13.44601i$	$5.459825 \pm$ $0.886517i$
4	0.04	0.03238	0.72136	$-3.556463 \pm$	$\pm 11.52649i$	$3.556463 \pm$

					0.876321i	0.876321i
5	0.05	0.02671	0.71641	-1.524192±	±8.875206i	1.524192±
					0.837649i	0.837649i

Table 3: The Effect of radiation pressure on the location and stability of out-of-plane equilibrium points for $e=0.3, a=0.87, \mu = 0.35, \sigma_1 = 0.02, \sigma_2 = 0.015, \sigma_3 = 0.003, \sigma_4 = 0.004, M_b = 0.01, W_2 = 1.17659 \times 10^{-12}$

S/no	Radiation Pressure		Out-of plane points		Roots of the characteristic equation		
	q_1	q_2	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.9960	0.9950	0.66735	0.412681	-2.54373±	±11.42462i	2.54373±
					0.543728i		0.543728i
2	0.9964	0.9954	0.67024	0.394326	-3.24342±	±16.23703i	3.24342±
					0.810034i		0.810034i
3	0.9968	0.9958	0.67646	0.343671	-6.45986±	±27.42462i	6.45986±
					0.886517i		0.886517i
4	0.9972	0.9962	0.68434	0.328763	-10.5756±	±38.57823i	10.5756±

					0.47632i		0.47632i
5	0.9976	0.9966	0.69101	0.310641	$-13.4140 \pm$	$\pm 45.41365i$	$13.4145 \pm$
					0.357649i		0.357649i

Table 4: The Combined effect of the perturbations on the location and stability of out-of-plane equilibrium points for $e = 0.3, a = 0.34, W_2 = 1.17659 \times 10^{-12}$

(a)

S/no.	Triaxiality				Radiation	Factors	Belt	Mass ratio
	σ_1	σ_2	σ_3	σ_4	q_1	q_2	M_b	μ
1.	0.02	0.01	0.002	0.001	0.9980	0.9976	0.01	0.0375
2.	0.03	0.02	0.003	0.002	0.9984	0.9980	0.02	0.0380
3.	0.04	0.03	0.004	0.003	0.9988	0.9984	0.03	0.0385
4.	0.05	0.04	0.005	0.004	0.9992	0.9988	0.04	0.0390
5.	0.06	0.05	0.006	0.005	0.9996	0.9992	0.05	0.0395

(b)

out-of-plane ξ	points $\pm \zeta$	The characteristic Roots		
		$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
0.633412	0.197065	± 227.584	± 19.2802	$\pm 86.7122i$
0.633011	0.205634	± 331.385	± 90.4429	$\pm 87.2517i$

0.632785	0.218767	± 379.863	± 90.547	$\pm 88.5313i$
0.632145	0.224261	± 463.07	± 90.5989	$\pm 88.9132i$
0.631004	0.234659	± 88.9780	± 122.110	$\pm 35.41283i$

In Table 5 below we present the numerical data of the binary system xi-Bootis and Kruger 60.

Table 5: Numerical data for the Binary System

Binary system	Masses (M_{\odot})		Eccentricity (e)	Semi-major axis (a)	Luminosity L_{\odot}		Spectral Types
	M_1	M_2			L_1	L_2	
Xi Bootis	0.9	0.66	0.5117	4.9044	0.49	0.061	G8/k4
Kruger 60	0.271	0.176	0.4100	2.3830	0.01	0.0034	M3/M4

Source:NASA ADS

Table 6: The effect of triaxiality on the location and stability of out-of-plane equilibrium points of xi-Bootis for $e = 0.5117$, $a = 0.7304$, $\mu = 0.4231$, $q_1 = 0.9988$, $q_2 = 0.9998$, $W_2 = 1.17659 \times 10^{-12}$

S/no	Triaxiality				Out-of plane points		Roots of the characteristic equation		
	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.015	0.011	0.002	0.001	0.466010	0.275418	± 610.524	$-173.012 \pm 184.316i$	$173.012 \pm 184.316i$
2.	0.02	0.015	0.003	0.002	0.477810	0.269001	± 814.061	$-175.981 \pm 182.895i$	$175.981 \pm 182.895i$
3.	0.03	0.019	0.004	0.003	0.49061	0.24443	± 998.23	$-174.887 \pm$	$174.887 \pm$

4.	0.04	0.02	0.005	0.004	0.517438	0.219650	± 1627.48	$180.49i$	$180.49i$
								$-175.13\pm$	$175.13\pm$
5.	0.05	0.03	0.006	0.005	0.539144	0.209341	± 1321.43	$176.832i$	$176.832i$
								$-173.39\pm$	$173.39\pm$
								$176.972i$	$176.972i$

Table 7: The effect of belt (M_b) on the location and stability of out-of-plane equilibrium points of xi-Bootis for $e = 0.5117$, $a = 0.7304$, $\mu = 0.4231$, $q_1 = 0.9988$, $q_2 = 0.9998$. $\sigma_1=0.02$, $\sigma_2=0.015$, $\sigma_3=0.003$, $\sigma_4=0.002$, $W_2 = 1.17659 \times 10^{-12}$

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.02	0.521012	0.211723	± 47.347306	$\pm 113.5678 i$	± 33.98450
2	0.03	0.513422	0.211965	± 47.748921	$\pm 120.5631i$	± 34.12654
3	0.04	0.51200	0.221343	± 48.256439	$\pm 120.9985i$	± 34.241001
4	0.05	0.51042	0.229867	± 48.84320	$\pm 121.43423i$	± 35.35790
5	0.06	0.50964	0.239341	± 49.22418	$\pm 122.1101i$	± 35.4128321

Table 8: The effect of triaxiality on the location and stability of out-of-plane equilibrium points of Kruger 60 for $e = 0.4100$, $a = 0.5894$, $\mu = 0.3937$, $q_1 = 0.9992$ and $q_2 = 0.9996$, $W_2 = 1.17659 \times 10^{-12}$

S/No	Triaxiality				Out-of-plane points		Roots of the characteristic equation		
	σ_1	σ_2	σ_3	σ_4	ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1.	0.02	0.002	0.002	0.001	0.946710	0.241070	-37.2193± 21.4739i	0±42.9477i	37.2193±21. 47396i
2.	0.03	0.025	0.003	0.002	0.951193	0.216110	-38.567± -22.0922i	0±44.1875i	38.567± 22.0922i
3.	0.04	0.035	0.004	0.003	0.958414	0.213279	-51.476± 28.7198i	0±57.5122i	51.476± 28.7198i
4.	0.05	0.045	0.005	0.004	0.959130	0.207454	-100.461± 55.2628i	0±110.803i	100.461± 55.2628i
5.	0.06	0.055	0.006	0.005	0.960314	0.204511	-154.48±	0±164.244i	154.48±

Table 9: The effect of belt (M_b) on the location and stability of out-of-plane equilibrium points of Kruger-60 for $e = 0.4100$, $a = 0.5894$, $\mu = 0.3937$, $q_1 = 0.9992$ and $q_2 = 0.9996$. $\sigma_1 = 0.02$, $\sigma_2 = 0.015$, $\sigma_3 = 0.003$, $\sigma_4 = 0.002$, $W_2 = 1.17659 \times 10^{-12}$

S/no	M_b	Out-of plane points		Roots of the characteristic equation		
		ξ	$\pm\zeta$	$\lambda_{1,2}$	$\lambda_{3,4}$	$\lambda_{5,6}$
1	0.01	0.321012	0.200534	± 54.223624	± 19.2802	$\pm 86.7122i$
2	0.02	0.321342	0.201823	± 54.534534	± 90.4429	$\pm 87.2517i$
3	0.03	0.321440	0.201944	± 54.655978	± 90.547	$\pm 88.5313i$
4	0.04	0.321452	0.202112	± 55.232720	± 90.5989	$\pm 88.9132i$
5	0.05	0.3214634	0.202472	± 55.703529	± 122.110	$\pm 35.41283i$

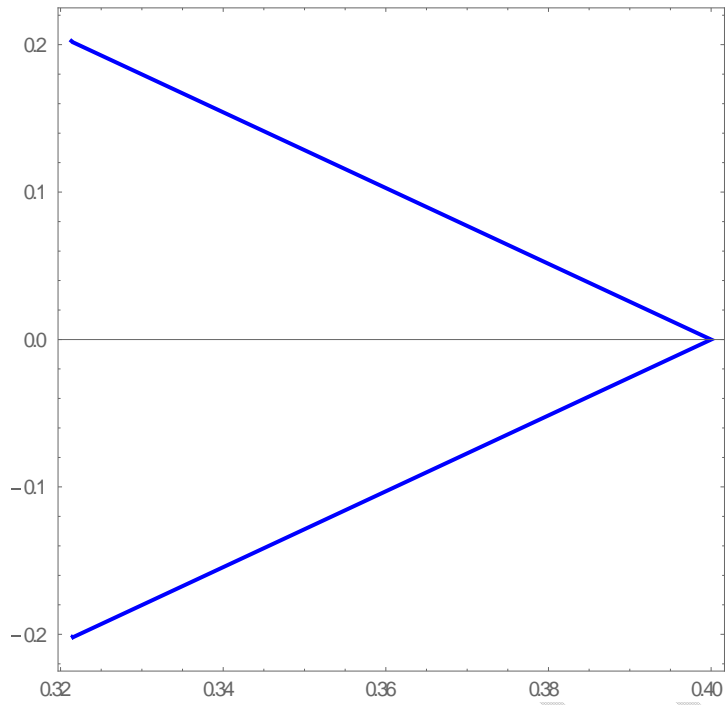


Fig.3 Graph showing the effect of triaxiality on the OPEPs of XI-Bootis

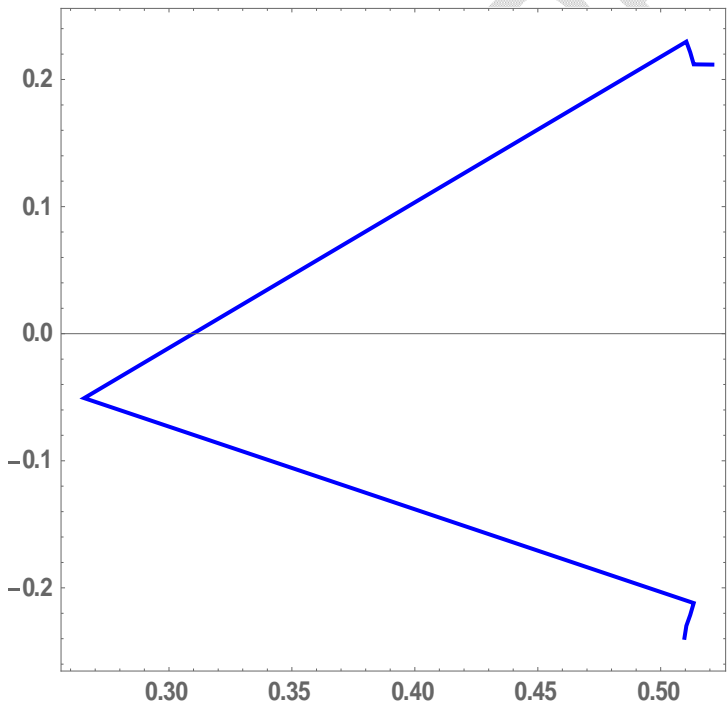


Fig.4 Graph showing the effect of the belt on the OPEPs of XI-Bootis

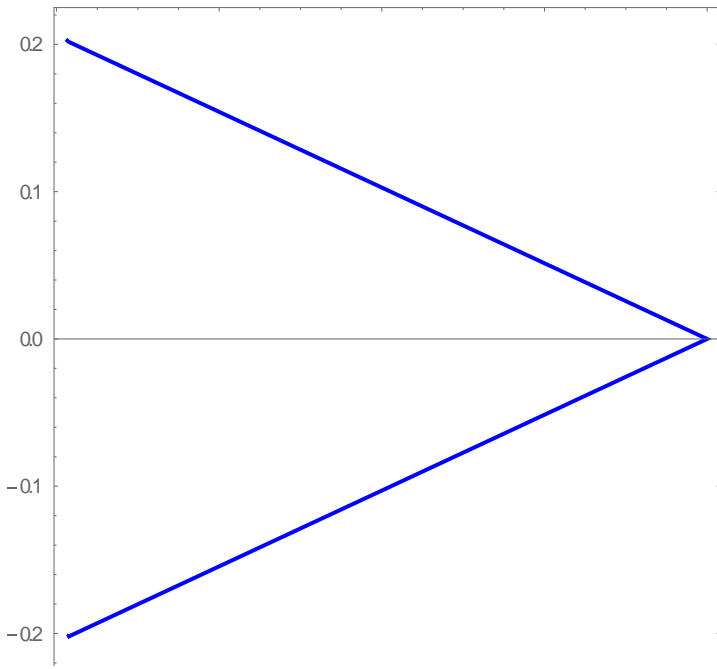


Fig.5 Graph showing the effect of triaxiality on the OPEPs of Kruger-60

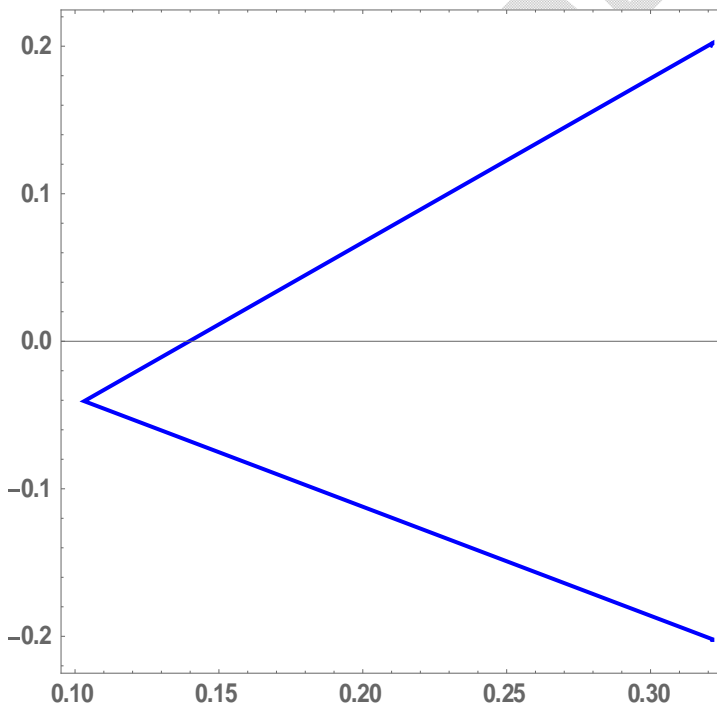


Fig.6 Graph showing the effect of the belt on the OPEPs of Kruger-60

5. Discussion

The motion of a third body under the influence of triaxial and radiating primaries together with a circumbinary disc has been described in equation (1)-(4). The positions of out-of-plane equilibrium points are given in equations 14 and 15 and are first obtained analytically and then numerically by power series expansion about the triaxiality coefficient of the smaller primary in Equations 14 and 15 to third order term with the aid of the software MATHEMATICA 10.4. The stability of these points are obtained by solving the roots of Equation (19) numerically. The positions of out-of-plane points and the characteristic roots obtained using arbitrary values for the parameters are shown in Tables 1-4. Generally, EPs are stable only if the six roots λ_i ($i=1,2,3,4,5,6$) are purely imaginary roots or complex roots with negative real parts and are unstable if λ_i ($i=1,2,3,4,5,6$) are complex or real roots [26].

Table 1 and 2, shows that the point $L_{6,7}$ shifts towards the line joining the primaries as the effects of triaxiality and belt are being increased respectively, while in Table 3 $L_{6,7}$ is seen to move away from the line joining the primaries as the radiation factors is increasing. The combined effects of all the parameters are shown in Table 4. The arbitrary values assign to the parameters are shown in Table 4a. Table 4b shows their effects on OPEPs and its stability, In all cases the out-of-plane equilibrium points moves away from the ξ -axis when the values of the parameters were increased. The roots (λ_i ($i=1,2,3,4,5,6$)) in Tables 1-4 are complex or real roots, hence the OPEPs are unstable. The numerical data of the binary systems (xi-bootis and kruger-60) are shown in Table 5. The effects of triaxiality and the belt on the binary systems

can be observed in Table 6-9 and Fig.3-6. These Tables and the graphs shows that increasing the values of triaxiality and belt,while keeping the orbital parametres of the Xi-bootis and Kruger-60 constant, results in a shift of the OPEPs.It can be seen in Table 6 that OPEPs shifts towards the ξ -axis this can be seen clearly in Fig.3,this in contrast to the effect of the belt on OPEPs of Xi-bootis in Table 7, where OPEPs shifts away from the the line joining the primaries (see also Fig.4) .The OPEPs in both Tables are unstable due to nature of their roots which are complex or real roots. The effects of triaxiality and the belt on Kruger-60 is similar to their effects on Xi-bootis.The effects of triaxiality moves the OPEPs towards the line joining the primaries(see Table 8 and Fig. 5) ,while the effect of the belt moves OPEPs away from the ξ -axis (See Table 9 and Fig. 6).Similar to what we obtained in the case of xi-Bootis,the roots obtained for OPEPs of Kruger-60 are either complex or real as such OPEPs are unstable . The changes in the positions of OPEPs are as shown in the graphs (Fig.2-6) below.The instability OPEPs have been confirmed by [6], [25] and [4].

6. Conclusion

We have obtained the out of plane equilibrium points and their stability in the framework of ER3BP when the primaries are triaxial, radiating with P-R drag force and surrounded by a belt. It is found that the positions are affected by triaxiality, radiation and the belt. We found that for the binary systems the effect of triaxiality and the belt moves OPEPs in opposite directions-while the effect of triaxiality moves OPEPs towards the ξ -axis, the belt moves OPEPs away from the ξ -axis. It was also observed that using arbitrary values and the values of the binary systems the OPEPs still remain unstable. Our OPEPs (Equations 14 and 15) tally with that of (Singh and Umar 2013) when $(2\sigma_1 - \sigma_2) = A_1$ and $(2\sigma_3 - \sigma_4) = A_2$.

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