Original Research Article

Generalized Fixed Points for Four Self-Mappings with the property OWC in CMS

ABSTRACT

In this research article, we obtained fixed points for Four self – mappings with the property Occasionally Weakly Compatible (OWC) in Cone Metric Spaces (CMS). Our results are generalized, improved some of the results in this article references.

*Keywords: [*cone metric space, common fixed point, fixed point theorem and occasionally weakly compatible *]*

1. INTRODUCTION

Cone metric space concept introduced by. Huang and Zhang, [9] they replacing the real numbers by an ordered Banach space(B-space) , also proved fixed point results in this CMS, after that my authors have been generalized ,extended and improved the fixed point theorems of Huang and Zhang [9] using with different types of contractive conditions [see, for eg. 2-5, 7-11] . And recently Bhatt and Chandra [6] proved some of the results on OWC mappings in CMS. In this article we generalized and improved and extended the results of Bhatt and Chandra [6].

2. PRELIMINARIES

We have defined some the definitions and which are useful in our main results and they are existing references in this article .

Definition 2.1. Let S be a real B- space and $B \subset S$. And the set "P" is said to be cone iff

(i)., $B \neq \{\emptyset\}$ and $B \neq \{0\}$, B is closed;
(ii). $\Rightarrow \alpha x+\beta y \in B$, $\alpha , \beta \geq 0 \in \mathbb{R}$, and $x, y \in B$
(iii). If both x and $-x \in B \Rightarrow x = 0$.

Definition 2.2. Let B be a cone in a B- space S. And the partial ordering '$\leq$' w.r.t. to B by $x_1 \leq x_2 \Leftrightarrow x_2 - x_1 \in B$. We shall write $x_1 < x_2$ to indicate $x_1 \leq x_2$ but $x_1 \neq x_2$ while $x_1 << x_2$ will stand for $x_2 - x_1 \in$ interior of B. And this type of cone B is said to be an order cone.

Definition 2,3. Let S be a B-space and $B \subset S$ be an order cone. And the order cone B is said to be normal cone if there exists a constant $N > 0$ such that for all $x_1, x_2 \in S$,

$$0 \leq x_1 \leq x_2 \quad \text{implies that} \quad \|x_1\| \leq N \|x_2\|.$$

The smallest positive number N satisfies this type of inequality is said to be the normal constant of B.

Definition 2.4. Let $X_1 \neq \{\emptyset\}$ set of S. And suppose that the mapping $\rho : X_1 \times X_1 \rightarrow S$ satisfies the following properties

(d1). $0 \leq \rho\,(x_1, y_1)$ for all $x_1, y_1 \in X_1$ and $\rho\,(x_1, y_1) = 0 \Leftrightarrow x_1 = y_1$;

(d2). $\rho\,(x_1, y_1) = \rho\,(y_1, x_1)$ for all $x_1, y_1 \in X_1$;

(d3). $\rho\,(x_1, y_1) \leq \rho\,(x_1, z_1) + \rho\,(z_1, y_1)$ for all $x_1, y_1, z_1 \in X_1$.

Then ρ is said to be a cone metric on X and (X_1, ρ) is said to be aCMS.

Note that the concept of a CMS is more general than the metric space.

Example 2.5 Let $S = \mathbb{R}^2$, $B = \{(x_1, y_1) \in S$ such that : $x_1, y_1 \geq 0\} \subseteq \mathbb{R}^2$, $X_1 = \mathbb{R}$ and $\rho : X_1 \times X_1 \rightarrow S$ such that $\rho\,(x_1, y_1) = (\,|x_1 - y_1|\,, \alpha\,|x_1 - y_1|\,)$, where α is a constant and is ≥ 0 . Then (X_1, ρ) is said to be a CMS.

Definition 2.6 Suppose that (X_1, ρ) is a CMS .If we say that the sequence $\{x_{n_1}\}$ a convergent sequence

(i) for any b>>0, ,∃ a positive integer N ∋ $\rho\,(x_{n_1}, x_1) << b$ for all n>N, for some fixed point x_1 in X_1 . And denote $x_{n_1} \rightarrow x_1$,as $n_1 \rightarrow \infty$.

And (ii) If we say that the sequence $\{x_{n_1}\}$ a Cauchy sequence if for every b in S with b>>0,∃ a positive integer N ∋ for all $n_1, m_1 > N$, $\rho\,(x_{n_1}, x_{m_1}) << b$.

Note that a cone metric space (X_1, ρ) is said to be complete if every Cauchy sequence is convergent.

Definition 2.7. Let F_1 and G_1 be two self-mappings of a set X_1. If $u_1 = F_1 x_1 = G_1 x_1$ for some x_1 in X_1, then x_1 is called a coincidence point of F_1 and G_1, and u_1 is called a point of coincidence of F_1 and G_1.

Proposition 2.8 . Let F_1 and G_1 be two OWC self-mappings of a set X_1 iff there is a point x_1 in X_1 which is said to be a coincidence point of F_1 and G_1 at which F_1 and G_1 are commute.

Lemma 2.9 . Let X_1 be a set , and F_1, G_1 are two OWC self-mappings of X_1. If F_1 and G_1 have a unique point of coincidence $u_1 = F_1 x_1 = G_1 x_1$, then u_1 is said to be the unique common fixed point of F_1 and G_1.

3. Fixed Point Results

In this section we obtained a result on occasionally weakly compatible for Four self- mappings in cone metric space which generalization , extension of the results of [6].

Theorem. Let (Y, ρ) be a cone metric space and Q be a normal cone. And let four self mappings M, N , A and B are OWC mappings of X_1 and they satisfying the following conditions

(i) $\rho\,(Mx, Ny) \leq \alpha\,\rho\,(Ax, By) + \beta\,\rho(Mx, Ax) + \gamma\,\rho\,(Ny, By) + \delta[\rho\,(Mx, By) + \rho\,(Ny, Ax)]$,

for all $x, y \in Y$, for which $Mx \neq Ny$, $\alpha, \beta, \gamma, \delta \in [0, 1]$ and satisfying $\alpha + \beta + \gamma + 2\delta < 1$.

(ii) The pairs { M, A } and { N, B } are OWC.

Then M, N, A and B have a unique common fixed point.

Proof. Given {M, A } and {N, B } are OWC , then there exists $x, y \in Y$ such that Mx = Ax and Ny = By. We claim that Mx = Ny. Otherwise by (i)
$\rho (Mx, Ny) \le \alpha \rho(Ax, By) + \beta \rho (Mx, Ax) + \gamma \rho (Ny, By) + \delta[\rho (Mx, By) + \rho(Ny, Ax)]$.
Since Mx = Ax = w_1 and Ny = By = z_1 are points of coincidence of { M, A } and { N, B }respectively.
$\Rightarrow \rho (Mx, Ny) \le \alpha \rho (Mx, Ny) + \beta \rho (Mx, Mx) + \gamma \rho(Ny, Ny) + \delta[\rho (Mx, Ny) + \rho (Ny, Mx)]$,
$\le (\alpha + 2\delta) \rho (Mx, Ny)$, since $\alpha + 2\delta < 1$,
$\Rightarrow \rho (Mx, Ny) < \rho (Mx, Ny)$, which is a contradiction.
Therefore, Mx = Ny , that is, Mx = Ax = Ny = By.
Moreover, if there is another point z_1 such that $Mz_1 = Az_1$, then by (i) $Mz_1 = Az_1 = Ny = By$ or $Mx = Mz_1$. Hence, w_1 = Mx = Nx is a unique point of coincidence of M and A. Then by the (1.9) lemma w_1 is the unique common fixed point of M and A. …. (1).
Similarly, there is a unique point $z_1 \in Y$ such that $z_1 = Nz_1 = Bz_1$. We suppose that $w_1 \ne z_1$, then by (i) , we get that
$\rho (w_1, z_1) = \rho (Mw_1, Nz_1) \le \alpha \rho (Aw_1, Bz_1) +\beta \rho (Mw_1, Aw_1) + \gamma \rho (Nz_1, Bz_1) + \delta[\rho (Mw_1, Bz_1) + \rho (Mz_1, Aw_1)]$,
$\le \alpha \rho(w_1, z_1) + \beta \rho (w_1, w_1) + \gamma \rho(z_1, z_1) + \delta[\rho (w_1, z_1) + \rho(z_1, w_1)]$,
$\le (\alpha + 2\delta) \rho(w_1, z_1)$, since $\alpha + 2\delta < 1$
$\Rightarrow \rho(w_1, z_1) < \rho (w_1, z_1)$, it is a contradiction.
Hence, w_1 is a unique common fixed point of four self mappings M, N, A and B.
This completes the proof of the theorem.

3. CONCLUSION:

Our result is more general than the results of [6] and it is also improvement of the results of [6]..

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