Original Research Article

A Generalized Unique Common Fixed Point Theorem for Occasionally Weakly Compatible Four Self-Mappings in Cone Metric Space

ABSTRACT

In this, paper we prove a unique common fixed point theorem for occasionally weakly compatible four self – mappings in cone metric space. Our result generalized and improved the recent results existing in the literature.

*Keywords: [*Fixed point, cone metric space, and occasionally weakly compatible *]*

1. INTRODUCTION AND PRELIMINARIES

L.G. Huang and X. Zhang, [7] was generalized the concept of a metric space, and they replacing the real numbers by and ordered Banach space and proved fixed theorems in cone metric space, later on my Mathematicians were inspired these results and generalized ,extended and improves with different types of contractive conditions [see, eg. 2, 3, 4, 6,] . Recently Bhatt and Chandra [5] were proved some results on occasionally weakly compatible mappings. Our result generalized and improved the results of [5].

The following definitions are useful to our main results which are due to in [7].

Definition 1.1. Let B be a real Banach space and P be a subset of B. The set P is called a cone if and only if:
(a). P is closed, non–empty and $P \neq \{0\}$;
(b). $a,b\in\mathbb{R}$, $a,b\geq 0$, $x,y\in P \Rightarrow ax+by\in P$;
(c). $x\in P$ and $-x\in P \Rightarrow x = 0$.

Definition 1.2. Let P be a cone in a Banach space B, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if $y-x\in P$. We shall write $x<y$ to indicate $x \leq y$ but $x \neq y$ while $x<<y$ will stand for $y-x\in \text{int } P$, where int P denotes the interior of the set P. This cone P is called an order cone.

Definition 1,3. Let B be a Banach space and $P \subset B$ be an order cone. The order cone P is called normal if there exists $L>0$ such that for all $x, y\in B$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq L \|y\|.$$

The least positive number L satisfying the above inequality is called the normal constant of P.

Definition 1.4[7] Let X be a nonempty set of B. Suppose that the map d: $X \times X \rightarrow B$ satisfies:

(d1). $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3). $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is clear that the concept of a cone metric space is more general than that of a metric space.

Example 1.5 [7] Let $B = \mathbb{R}^2$, $P = \{(x, y) \in B$ such that : $x, y \geq 0\} \subseteq \mathbb{R}^2$, $X = \mathbb{R}$ and d: $X \times X \rightarrow B$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.6[7] Let (X, d) be a cone metric space .We say that $\{x_n\}$ is
(i) a convergent sequence if for any c>>0, there is a natural number N such that for all n>N, $d(x_n, x) << c$, for some fixed x in X .We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$) .

(ii) a Cauchy sequence if for every c in B with c>>0, there is a natural number N such that for all n, m>N, $d(x_n, x_m) << c$
A cone metric space (X, d) is said to be complete if every Cauchy sequence is convergent.

Definition 1.7[1]. Let f and g be self-mappings of a set X. If w = fx = gx for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Proposition 1.8 [5]. Let f and g be occasionally weakly compatible (owc)self-mappings of a set X iff there is a point x in X which is coincidence point of f and g at which f and g are commute.

Lemma 1.9[5] . Let X be a set , f, g are owc self-mappings of X. If f and g have a unique point of coincidence w = fx = gx, then w is the unique common fixed point of f and g.

2. Main Results

In this section we have generalized the Theorem 3.1. of [5].

Theorem. Let (X ,d) be a cone metric space and P be a normal cone. Suppose that f,g , S and T are occasionally weakly compatible self mappings of X and satisfy the following conditions
(i) $d(fx, gy) \leq p\, d(Sx, Ty) + q\, d(fx, Sx) + r\, d(gy, Ty) + t[d(fx, Ty) + d(gy, Sx)]$,

for all $x, y \in X$, for which $fx \neq gy$, $p, q, r, t \in [0, 1]$ and satisfying $p + q + r + 2t < 1$.
(ii) The pairs { f, S } and { g, T } are occasionally weakly compatible.

Then f, g, S and T have a unique common fixed point.

Proof. Given { f, S } and { g, T } are occasionally weakly compatible , then there exists $x, y \in X$ such that fx = Sx and gy = Ty. We claim that fx = gy. Otherwise by (i)
$d(fx, gy) \leq p\, d(Sx, Ty) + q\, d(fx, Sx) + r\, d(gy, Ty) + t[d(fx, Ty) + d(gy, Sx)]$.
Since fx = Sx =w and gy = Ty = z are points of coincidence of { f, S } and { g, T }respectively.
$\Rightarrow d(fx, gy) \leq p\, d(fx, gy) + q\, d(fx, fx) + r\, d(gy, gy) + t[d(fx, gy) + d(gy, fx)]$,

$\leq (p + 2t)\, d(fx, gy)$, since $p + 2t < 1$,
$\Rightarrow d(fx, gy) < d(fx, gy)$, which is a contradiction.
Therefore, $fx = gy$, that is, $fx = Sx = gy = Ty$.
Moreover, if there is another point z such that $fz = Sz$, then by (i) $fz = Sz = gy = Ty$ or $fx = fz$.
Hence, $w = fx = gx$ is a unique point of coincidence of f and S. Then by the Lemma (1.9)
w is the unique common fixed point of f and S. …. (1).
Similarly, there is a unique point $z \in X$ such that $z = gz + Tz$. We suppose that $w \neq z$, then by (i) , we get that

$$d(w, z) = d(fw, gz) \leq p\, d(Sw, Tz) + q\, d(fw, Sw) + r\, d(gz, Tz) + t[\, d(fw, Tz) + d(gz, Sw)],$$
$$\leq p\, d(w, z) + q\, d(w, w) + r\, d(z, z) + t[\, d(w, z) + d(z, w)] ,$$
$$\leq (p + 2t)\, d(w, z), \text{ since } p + 2t < 1$$

$\Rightarrow d(w, z) < d(w, z)$, which is a contradiction.
Therefore, w is the unique common fixed point of f, g, S and T.

3. CONCLUSION:
Our result is more general than the results of [5].

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