ON OPTIMIZATION POLICY FOR VERMA ENTROPY BY DYNAMIC PROGRAMMING

Abstract

Varma [4, 5] entropy has attracted attention for a new class of non-linear integer programming problems that arise during the course of discussion. Our focus in this communication is to explore the techniques of dynamic programming. This process requires splitting any optimization event into a finite number of subcomponents for any occurrence of a finite generalized problem. The capacity plan should be partitioned in such a way that the expression can be optimized.

Keywords: Dynamic programming, Measure of entropy, Optimization policy etc.

1. INTRODUCTION

Figuring out the number of sub-component is only possible through powerful dynamic programming tool. In the present communication, optimization of own entropy is carried out by the author by partitioning each event into its sub-events under the finite generalized likelihood scheme.

Dynamic programming works on the principle of optimality. Principle of optimality states that in an optimal sequence of decisions or choices, each subsequences must also be optimal. An optimal policy has the property that whatever the initial states and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. Bellman's principle of optimality is related to the dynamic programming problem.

The main concept of dynamic programming is straight-forward. We divide a problem into smaller nested subproblems, and then combine the solutions to reach an overall solution. This concept is known as the principle of optimality. The term dynamic programming was first used in the 1940's by Bellman, Richard [1] to describe problems where one needs to find the best decisions one after another. In the 1950's, he refined it to describe nesting small decision problems into larger ones. The mathematical statement of principle of optimality is remembered in his name as the Bellman equation.

Dynamic programming is a useful mathematical technique for making a sequence of interrelated decisions. It provides a systematic procedure for determining the optimal combination of decisions.

In contrast to linear programming, there does not exist a standard mathematical formulation of "the" dynamic programming problem. Rather, dynamic programming is a general type of approach to problem solving, and the particular equations used must be developed to fit each situation. Therefore, a certain degree of ingenuity and insight into the general structure of dynamic programming problems is required to recognize when and how a problem can be solved by dynamic programming procedures. These abilities can best be developed by an exposure to a wide variety of dynamic programming applications and a study of the characteristics that are common to all these situations.

By the use of dynamic programming, Kapur [2] maximized measure of entropy subject to the given constraints and for optimal sub division of out-comes for obtaining maximum gain in information subject to a given budget. Kapur [2] maximized Shannon's [3] entropy

$$-\sum_{i=1}^{n} p_i \ln p_i \tag{1.1}$$

subject to the constraints

$$\sum_{i=1}^{n} p_i = c, \qquad p_i \geq 0 \tag{1.2}$$

by dynamic programming,

2. OUR RESULTS

2.1 MAXIMIZATION OF VERMA [4] MEASURES OF ENTROPY

$$H_n(P) = \sum_{i=1}^{n}[\ln(1 + ap_i) - \ln p_i - \ln(1 + a)], \quad a > 0. \tag{2.1.1}$$

For the application of this principle, we consider the maximization of

$$H_n(P) = \sum_{i=1}^{n} \ln(1 + ap_i) - \sum_{i=1}^{n} \ln p_i - \sum_{i=1}^{n} p_i \ln(1 + a)$$

subject to the constraints

$$\sum_{i=1}^{n} p_i = c, \qquad p_i \geq 0, \qquad i = 1, \ldots\ldots, n. \tag{2.1.2}$$

Let the maximum value be $H_n(c)$, then obviously

$$H_1(c) = \ln(1 + ac) - \ln c - c\ln(1 + a). \tag{2.1.3}$$

If we choose p_1 arbitrarily between 0 and c, we have to maximize

$$\sum_{i=2}^{n} \ln(1 + ap_i) - \sum_{i=2}^{n} \ln p_i - \sum_{i=2}^{n} p_i \ln(1 + a)$$

Subject to the constraints

$$\sum_{i=1}^{n} p_i = c - p_1, \qquad p_i \geq 0, \qquad i = 1, \ldots\ldots, n.$$

This maximum value will be $H_{n-1}(c-p_1)$. The principle of optimality then gives the recurrence relation

$$H_n(c) = \max_{0\le p_i\le c}[\ln(1+ap_1) - \ln p_1 - p_1\ln(1+a) + H_{n-1}(c-p_1)] \qquad (2.1.4)$$

On setting $n = 2$, in (2.1.4) we achieve the following result, on using (2.1.3)

$$H_2(c) = \max_{0\le p_i\le c}[\ln(1+ap_1) - \ln p_1 - p_1\ln(1+a) + H_1(c-p_1)]$$

$$= \max_{0\le p_i\le c}\begin{bmatrix} \ln(1+ap_1) - \ln p_1 - p_1\ln(1+a) + \\ \ln\big(1+a(c-p_1)\big) - \ln(c-p_1) - (c-p_1)\ln(1+a)\end{bmatrix}$$

$$= \max_{0\le p_i\le c}\begin{bmatrix} \ln(1+ap_1) - \ln p_1 - p_1\ln(1+a) + \\ \ln(1+ac-ap_1) - \ln(c-p_1) - (c-p_1)\ln(1+a)\end{bmatrix}$$

$$= \ln\left(1+\frac{ac}{2}\right) - \ln\frac{c}{2} + \ln\left(1+\frac{ac}{2}\right) - \ln\frac{c}{2} - c\ln(1+a)$$

$$= 2\ln\left(1+\frac{ac}{2}\right) - 2\ln\frac{c}{2} - c\ln(1+a) \qquad (2.1.5)$$

we achieve the maximum value, when $p_1 = p_2 = \frac{c}{2}$.

Again, setting $n = 3$, in (2.1.4) we achieve the following result, on using (2.1.5)

$$H_3(c) = \max_{0\le p_i\le c}[\ln(1+ap_1) - \ln p_1 - p_1\ln(1+a) + H_2(c-p_1)]$$

$$= \max_{0\le p_i\le c}\begin{bmatrix} \ln(1+ap_1) - \ln p_1 - p_1\ln(1+a) + \\ 2\ln\frac{(2+a(c-p_1))}{2} - 2\ln\frac{(c-p_1)}{2} - (c-p_1)\ln(1+a)\end{bmatrix}$$

$$= \ln\left(1+\frac{ac}{3}\right) - \ln\frac{c}{3} - \frac{c}{3}\ln(1+a) + 2\ln\left(\frac{2+2a.{}^{c}/_{3}}{2}\right) - 2\ln\frac{c}{3} - \frac{2c}{3}\ln(1+a)$$

$$= 3\ln\left(1+\frac{ac}{3}\right) - 3\ln\frac{c}{3} - c\ln(1+a) \qquad (2.1.6)$$

we achieve the maximum value, when $p_1 = p_2 = p_3 = \frac{c}{3}$.

It thus seems to be

$$H_n(c) = n\ln\left(\frac{n+ac}{n}\right) - n\ln\frac{c}{n} - c\ln(1+a) \qquad (2.1.7)$$

The above result arises, when $p_1 = p_2 = \cdots = p_n = \frac{c}{n}$.

Obviously, it will be true for a specific value of n, equation (2.1.4) gives

$$H_{n+1}(c) = \max_{0\le p_i\le c}[\ln(1+ap_1) - \ln p_1 - p_1\ln(1+a) + H_n(c-p_1)]$$

$$= \max_{0 \le p_i \le c} \left[\ln(1 + ap_1) - \ln p_1 - p_1 \ln(1 + a) + n \ln\left(\frac{n + a(c - p_1)}{n}\right) - n \ln \frac{(c - p_1)}{n} \right.$$

$$\left. -(c - p_1) \ln(1 + a) \right]$$

$$= (n + 1) \ln\left(\frac{n + 1 + ac}{n + 1}\right) - (n + 1) \ln \frac{c}{n + 1} - c \ln(1 + a). \qquad (2.1.8)$$

Thus from (2.1.3), (2.1.5), (2.1.6) and (2.1.7) and the principle of mathematical induction, the result (2.1.7) and (2.1.8) are true for all value of n and c. Putting $c = 1$, we get the result that

$$\sum_{i=1}^{n} \ln(1 + ap_i) - \sum_{i=1}^{n} \ln p_i - \sum_{i=1}^{n} p_i \ln(1 + a)$$

is maximum subject to $\sum_{i=1}^{n} p_i = 1$,when $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$ and the maximum value is

$$n \ln\left(\frac{n + a}{n}\right) - n \ln \frac{1}{n} - \ln(1 + a).$$

Which is Verma's [4, 5] result.

2.2 MAXIMIZATION OF MODIFIED VERSION OF VERMA [4, 5] MEASURES OF ENTROPY

$$H_n(P) = \sum_{i=1}^{n} [\ln(1 + ap_i) - p_i \ln p_i - \ln(1 + a)], \quad a > 0. \qquad (2.2.1)$$

For the application of this principle, we consider the maximization of

$$H_n(P) = \sum_{i=1}^{n} \ln(1 + ap_i) - \sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n} p_i \ln(1 + a)$$

subject to the constraints

$$\sum_{i=1}^{n} p_i = c, \qquad p_i \ge 0, \qquad i = 1, \dots \dots, n. \qquad (2.2.2)$$

Let the maximum value be $H_n(c)$, then obviously

$$H_1(c) = \ln(1 + ac) - c \ln c - c \ln(1 + a). \qquad (2.2.3)$$

If we choose p_1 arbitrarily between 0 and c, we have to maximize

$$\sum_{i=2}^{n} \ln(1 + ap_i) - \sum_{i=2}^{n} p_i \ln p_i - \sum_{i=2}^{n} p_i \ln(1 + a)$$

Subject to the constraints

$$\sum_{i=1}^{n} p_i = c - p_1, \qquad p_i \ge 0, \qquad i = 1, \dots \dots, n.$$

This maximum value will be $H_{n-1}(c - p_1)$. The principle of optimality then gives the recurrence relation

$$H_n(c) = \max_{0 \le p_i \le c} [\ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + H_{n-1}(c - p_1)] \qquad (2.2.4)$$

On setting $n = 2$, in (2.2.4) we achieve the following result, on using (2.2.3)

$$H_2(c) = \max_{0\le p_i\le c}[\ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + H_1(c - p_1)]$$

$$= \max_{0\le p_i\le c}\begin{bmatrix} \ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + \\ \ln\big(1 + a(c - p_1)\big) - (c - p_1)\ln(c - p_1) - (c - p_1)\ln(1 + a) \end{bmatrix}$$

$$= \max_{0\le p_i\le c}\begin{bmatrix} \ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + \\ \ln(1 + ac - ap_1) - (c - p_1)\ln(c - p_1) - (c - p_1)\ln(1 + a) \end{bmatrix}$$

$$= \ln\left(1 + \frac{ac}{2}\right) - \frac{c}{2}\ln\frac{c}{2} + \ln\left(1 + \frac{ac}{2}\right) - \frac{c}{2}\ln\frac{c}{2} - c\ln(1 + a)$$

$$= 2\ln\left(1 + \frac{ac}{2}\right) - c\ln\frac{c}{2} - c\ln(1 + a) \qquad (2.2.5)$$

we achieve the maximum value, when $p_1 = p_2 = \frac{c}{2}$.

Again, setting $n = 3$, in (2.2.4) we achieve the following result, on using (2.2.5)

$$H_3(c) = \max_{0\le p_i\le c}[\ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + H_2(c - p_1)]$$

$$= \max_{0\le p_i\le c}\begin{bmatrix} \ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + \\ 2\ln\frac{(2 + a(c - p_1))}{2} - (c - p_1)\ln\frac{(c - p_1)}{2} - (c - p_1)\ln(1 + a) \end{bmatrix}$$

$$= \ln\left(1 + \frac{ac}{3}\right) - \frac{c}{3}\ln\frac{c}{3} + 2\ln\left(\frac{2 + 2a.{}^{c}/_{3}}{2}\right) - \frac{2c}{3}\ln\frac{c}{3} - c\ln(1 + a)$$

$$= 3\ln\left(1 + \frac{ac}{3}\right) - c\ln\frac{c}{3} - c\ln(1 + a) \qquad (2.2.6)$$

we achieve the maximum value, when $p_1 = p_2 = p_3 = \frac{c}{3}$.

It thus seems to be

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The above result arises, when $p_1 = p_2 = \cdots = p_n = \frac{c}{n}$.

Obviously, it will be true for a specific value of n, equation (2.2.4) gives

$$H_{n+1}(c) = \max_{0\le p_i\le c}[\ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + H_n(c - p_1)]$$

$$= \max_{0\le p_i\le c}\Big[\ln(1 + ap_1) - p_1 \ln p_1 - p_1 \ln(1 + a) + n\ln\left(\frac{n + a(c - p_1)}{n}\right) - (c - p_1)\ln\frac{(c - p_1)}{n}$$

$$-(c - p_1)\ln(1 + a)\Big]$$

$$= (n+1)\ln\left(\frac{n+1+ac}{n+1}\right) - c\ln\frac{c}{n+1} - c\ln(1+a). \qquad (2.2.8)$$

Thus from (2.2.3), (2.2.5), (2.2.6) and (2.2.7) and the principle of mathematical induction, the result (2.2.7) and (2.2.8) are true for all value of n and c. Putting $c = 1$, we get the result that

$$\sum_{i=1}^{n} \ln(1+ap_i) - \sum_{i=1}^{n} p_i \ln p_i - \sum_{i=1}^{n} p_i \ln(1+a)$$

is maximum subject to $\sum_{i=1}^{n} p_i = 1$,when $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$ and the maximum value is

$$n\ln\left(\frac{n+a}{n}\right) - \ln\frac{1}{n} - c\ln(1+a).$$

Which is modified Verma's [4, 5] result.

2.3 MAXIMIZATION OF VERMA [5] MEASURES OF PROBABILISTIC ENTROPY INVOLVING TWO PARAMETERS

$$V_{\alpha,\beta}(P) = \frac{1}{1-\alpha}\ln\frac{\sum_{i=1}^{n} {p_i}^{\alpha+\beta-1}}{\sum_{i=1}^{n} {p_i}^{\beta}}, \quad \alpha \neq 1, \beta \geq 0, \alpha+\beta-1 > 0 \qquad (2.3.1)$$

For the application of this principle, we consider the maximization of

$$H_n(P) = \frac{1}{1-\alpha}\left[\ln\sum_{i=1}^{n} {p_i}^{\alpha+\beta-1} - \ln\sum_{i=1}^{n} {p_i}^{\beta}\right]$$

subject to the constraints

$$\sum_{i=1}^{n} p_i = c, \qquad p_i \geq 0, \qquad i = 1, \dots \dots, n. \qquad (2.3.2)$$

Let the maximum value be $H_n(c)$, then obviously

$$H_1(c) = \frac{1}{1-\alpha}\left[\ln c^{\alpha+\beta-1} - \ln c^{\beta}\right]. \qquad (2.3.3)$$

If we choose p_1 arbitrarily between 0 and c, we have to maximize

$$\frac{1}{1-\alpha}\left[\ln\sum_{i=2}^{n} {p_i}^{\alpha+\beta-1} - \ln\sum_{i=2}^{n} {p_i}^{\beta}\right]$$

Subject to the constraints

$$\sum_{i=1}^{n} p_i = c - p_1, \qquad p_i \geq 0, \qquad i = 1, \dots \dots, n.$$

This maximum value will be $H_{n-1}(c - p_1)$. The principle of optimality then gives the recurrence relation

$$H_n(c) = \max_{0 \leq p_i \leq c}\left[\frac{1}{1-\alpha}\left\{\ln {p_1}^{\alpha+\beta-1} - \ln {p_1}^{\beta}\right\} + H_{n-1}(c-p_1)\right] \qquad (2.3.4)$$

On setting $n = 2$, in (2.3.4) we achieve the following result, on using (2.3.3)

$$H_2(c) = \max_{0 \le p_i \le c} \left[\frac{1}{1-\alpha}\left\{\ln p_1{}^{\alpha+\beta-1} - \ln p_1{}^{\beta}\right\} + H_1(c - p_1)\right]$$

$$= \max_{0 \le p_i \le c} \frac{1}{1-\alpha}\left[\ln p_1{}^{\alpha+\beta-1} - \ln p_1{}^{\beta} + \ln(c - p_1)^{\alpha+\beta-1} - \ln(c - p_1)^{\beta}\right]$$

$$= \frac{2}{1-\alpha}\left[\ln\left(\frac{c}{2}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{2}\right)^{\beta}\right] \tag{2.3.5}$$

we achieve the maximum value, when $p_1 = p_2 = \frac{c}{2}$.

Again, setting $n = 3$, in (2.3.4) we achieve the following result, on using (2.3.5)

$$H_3(c) = \max_{0 \le p_i \le c} \left[\frac{1}{1-\alpha}\left\{\ln p_1{}^{\alpha+\beta-1} - \ln p_1{}^{\beta}\right\} + H_2(c - p_1)\right]$$

$$= \max_{0 \le p_i \le c} \left[\frac{1}{1-\alpha}\left\{\ln p_1{}^{\alpha+\beta-1} - \ln p_1{}^{\beta}\right\} + \frac{2}{1-\alpha}\left\{\ln\left(\frac{c-p_1}{2}\right)^{\alpha+\beta-1} - \ln\left(\frac{c-p_1}{2}\right)^{\beta}\right\}\right]$$

$$= \frac{1}{1-\alpha}\left[\ln\left(\frac{c}{3}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{3}\right)_1^{\beta} + 2\left\{\ln\left(\frac{c}{3}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{3}\right)^{\beta}\right\}\right]$$

$$= \frac{3}{1-\alpha}\left[\ln\left(\frac{c}{2}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{2}\right)^{\beta}\right] \tag{2.3.6}$$

we achieve the maximum value, when $p_1 = p_2 = p_3 = \frac{c}{3}$.

It thus seems to be

$$H_n(c) = \frac{n}{1-\alpha}\left[\ln\left(\frac{c}{n}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{n}\right)^{\beta}\right] \tag{2.3.7}$$

The above result arises, when $p_1 = p_2 = \cdots = p_n = \frac{c}{n}$.

Obviously, it will be true for a specific value of n, equation (2.3.4) gives

$$H_{n+1}(c) = \max_{0 \le p_i \le c} \left[\frac{1}{1-\alpha}\left\{\ln p_1{}^{\alpha+\beta-1} - \ln p_1{}^{\beta}\right\} + H_n(c - p_1)\right]$$

$$= \max_{0 \le p_i \le c} \left[\frac{1}{1-\alpha}\left\{\ln p_1{}^{\alpha+\beta-1} - \ln p_1{}^{\beta}\right\} + \frac{n}{1-\alpha}\left\{\ln\left(\frac{c-p_1}{2}\right)^{\alpha+\beta-1} - \ln\left(\frac{c-p_1}{2}\right)^{\beta}\right\}\right]$$

$$= \frac{1}{1-\alpha}\left[\ln\left(\frac{c}{3}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{3}\right)_1^{\beta} + n\left\{\ln\left(\frac{c}{3}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{3}\right)^{\beta}\right\}\right]$$

$$= \frac{(n+1)}{1-\alpha}\left[\ln\left(\frac{c}{2}\right)^{\alpha+\beta-1} - \ln\left(\frac{c}{2}\right)^{\beta}\right]. \tag{2.3.8}$$

Thus from (2.3.3), (2.3.5), (2.3.6) and (2.3.7) and the principle of mathematical induction, the result (2.3.7) and (2.3.8) are true for all value of n and c. Putting $c = 1$, we get the result that

$$\frac{1}{1-\alpha}\left[\ln \sum_{i=1}^{n} p_i{}^{\alpha+\beta-1} - \ln \sum_{i=1}^{n} p_i{}^{\beta}\right]$$

is maximum subject to $\sum_{i=1}^{n} p_i = 1$,when $p_1 = p_2 = \cdots = p_n = \frac{1}{n}$ and the maximum value is

$$\frac{n}{1-a}\left[\ln\left(\frac{1}{n}\right)^{\alpha+\beta-1} - \ln\left(\frac{1}{n}\right)^{\beta}\right].$$

Which is Verma's [5] result.

CONCLUDING REMARKS

In this communication, when we maximize the Verma [4, 5] entropy by splitting any optimization event into finite number of subevents $i.e.$ by dynamic programming, then we conclude the following results for all cases:

(i) as the probability increases, the allotment of subevents also increases. This is again expected for let $p_1, \dots\dots\dots, p_n$ be arranged in ascending order and let $m_1, \dots\dots\dots, m_n$ be the optimal assignments so that $\max\left(\frac{p_i}{m_i}, \frac{p_j}{m_j}\right) \leq \max\left(\frac{p_i}{m_j}, \frac{p_j}{m_i}\right)$. Since $p_j > p_i$, this implies that $m_j \geq m_i$ for if $m_j < m_i$, then $\frac{p_j}{m_j} > \frac{p_i}{m_i}$ and this contradicts our assumption that the given allotment is optimal.

(ii) as we subdivide the events, the value of the maximum entropy increases or remains unchanged.

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