Original Research Article

Synchronizability of Generalized Two-layer Networks with Different Layer Topologies

ABSTRACT

This paper focuses on a generalized two-layer network and its synchronizability, which randomly generate different topologies at each layer. This kind of network can better describe some irregular networks in reality. From the master stability function method of network synchronization analysis, we estimate the largest eigenvalue and the lowest nonzero eigenvalue of the Laplacian matrix. And the influence of node coupling strength on the synchronizability of generalized two-layer networks is analyzed. We obtain that the enhancement of node coupling strength can promote network synchronization in bounded and unbounded synchronization regions. In the end, we perform numerical simulations based on theoretical analysis. The numerical results also show that the more nodes, the stronger the synchronizability under the unbounded synchronization region, and the opposite is true for the bounded synchronization region. The results have a certain guiding significance for the synchronous application of general network in reality.

Keywords: generalized multiplex networks; different layer topology; eigenvalues; synchronizability

1. INTRODUCTION

Network synchronization has been always concerned by scholars and has involved in many fields, such as mathematics, physics, biological and social science [1, 2, 3]. The research on synchronization starts from single-layer network [4] and gradually expands to multi-layer network. The synchronization of regular networks first attracted people's attention such as star coupling network, ring and chain network, nearest-neighbor coupling network and globally coupling network [5, 6]. And the master stability function is used to judge the network synchronizability [7]. The proposals of the WS and NW small-world network [8, 9] provide a new research direction for network synchronization [10]. Barabási and Albert summarized the characteristics of WS small-world networks and ER random graphs, and then described a BA scale-free network for the first time [11, 12]. Due to the randomness of building a BA network, synchronizability is affected by the number of initial nodes and new node links [13]. The synchronizability of ER random graph is dependent to the connection probability of two nodes, and they determine the lowest nonzero eigenvalue by estimation [14].

Since more practical problems need to be described by multi-layer networks, the synchronization of multi-layer networks has gradually become a focus issue. At first, the synchronization of two-layer networks obtain studies, including the synchronizability and eigenvalue spectrum of networks in different structural parameter [15]. Later, people studied the influence of the number and mode of interlayer connections on the synchronizability [16, 17, 18]. On the basis of single-layer networks research, the study of network synchronizability is also extended to M-layer and more complex network structures [19, 20, 21, 22]. But the above researches all consider the same structure for different layers. The actual may be two or more layers that have different network topologies and are irregular structure [23]. Therefore, we consider the generalized two-layer network of which each layer is constructed in a random manner. We will analyse the eigenvalues and synchronizability of generalized two-layer networks. The dynamic model of two-layer network is shown in the next section. And section III estimates the eigenvalue of supra-Laplacian matrix and analyse the synchronizability in theory. Section IV makes numerical simulations for theoretical results of synchronizability. Finally, we present our results and prospects.

2. PRELIMINARIES

2.1 DYNAMIC MODEL OF TWO-LAYER NETWORKS

The dynamics of node i in a two-layer network satisfies the following equation [15, 19]:

$$\dot{x}_i^K(t) = f\left(x_i^K(t)\right) + a\sum_{j=1}^{N}\omega_{ij}^K H\left(x_j^K(t)\right) + d\sum_{L=1}^{2} d_i^{KL}\Gamma\left(x_i^K(t)\right),$$
$$i = 1,2,\cdots,N; K = 1,2,$$

where N is the number of nodes, K is the number of layers, $x_i^K(t)$ is the value of the node i in the K-th layer at time t, $f\left(x_i^K(t)\right) \in \mathbb{R}^n$ denotes a vector function of the node dynamics, $H\left(x_j^K(t)\right) \in \mathbb{R}^n$ is the coupling function between nodes in each layer and $a > 0$ denotes the coupling strength of nodes in each layer, $\Gamma\left(x_i^K(t)\right) \in \mathbb{R}^n$ is a coupling function between the same node i in two layers and $d > 0$ denotes the coupling strength of nodes between two layers.

$W^K = \left(\omega_{ij}^K\right)_{N\times N}(i, j = 1,\cdots,N; K = 1,2)$ is a coupling configuration matrix that reflects the network topology within the K-th layer and satisfies the dissipative coupling conditions $\sum_j^N \omega_{ij}^K = 0$. Here the diagonal elements ω_{ii}^K of W^K satisfy $\omega_{ii}^K = -\sum_{j=1,j\neq i}^N \omega_{ij}^K$ and the other elements take 0 or 1. If node i connects node j $(i \neq j)$, then ω_{ij}^K =1, or else ω_{ij}^K =0.

$D = \left(d_i^{KL}\right)_{2\times 2}(i = 1,\cdots,N)$ is a coupling configuration matrix of node i in a two-layer network reflecting the interlayer network topology and satisfying the dissipative coupling condition $\sum_{L=1}^2 d_i^{KL} = 0$, here the diagonal elements d_i^{KL} of D satisfy $d_i^{LL} = -\sum_{L=1,K\neq L}^2 d_i^{KL}$ $(i = 1,\cdots,N)$ and the other elements take 0 or 1. If a node i at K-th layer links the node i at L-th layer $(K \neq L)$, then d_i^{KL} =1, or else d_i^{KL} =0.

Let $M^K = -aW^K$, denoting the intralayer Laplacian matrix in K-th layer. Thus, we can use the the straight sum of the intralayer Laplacian matrix for K layers to express the intralayer supra-Laplacian matrix $\mathcal{M}_I$:

$$\mathcal{M}_I = \begin{pmatrix} M^1 & 0 \\ 0 & M^2 \end{pmatrix} = M^1 \oplus M^2,$$

reflecting the intralayer topology of two-layer networks.

Take $M_n = -dD$, representing the Laplacian matrix of nodes connection in different layer. The interlayer super-Laplacian matrix $\mathcal{M}_L$ becomes the Kronecker product of M_n and I_N, $\mathcal{M}_L = M_n \otimes I_N$, reflecting the interlayer topology of two-layer networks. Here I_N is a Nth-order identity matrix. Therefore, we have the supra-Laplacian matrix $\mathcal{M} = \mathcal{M}_I + \mathcal{M}_L$.

We assume that the dynamic model of nodes and the coupling function between nodes are all same. For a undirected connected network, we know its eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N$. On the basis of the master stability function, the ratio $R = \lambda_N/\lambda_2$ of the largest eigenvalue λ_N and the lowest nonzero eigenvalue λ_2 for Laplacian matrix is smaller, the network synchronizability is stronger in bounded synchronized region. The lowest nonzero eigenvalue λ_2 is lager, the synchronizability of networks is stronger in unbounded synchronized region.

2.2 GENERALIZED TWO-LAYER NETWORKS

We focus on the generalized multiplex network with two layers which have different topologies but the same number of nodes, as shown in Fig.1.

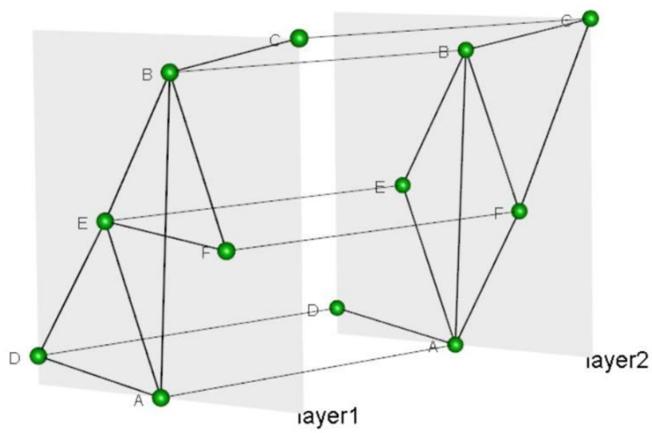


Fig. 1. The generalized two-layer network composed of two different topologies. The thin lines are interlayer links, and the thick lines are intralayer links.

3. SYNCHRONIZABILITY OF GENERALIZED TWO-LAYER NETWORKS

For the eigenvalues of generalized multiplex networks, we used a simplified method to make a approximate estimation. We assume that the eigenvalues are $\lambda_i (i = 1, \cdots, 2N)$, so we have

$$|\lambda\mathcal{M} - I| = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \cdots\cdots (\lambda - \lambda_{2N}),$$

Where N is the number of nodes in each layer.

By observing the synchronizability of multi-layer networks [15-17, 19-21] and repeating calculations, we found the eigenvalues of Laplacian matrix having the form of $0, Md, \lambda_i = c_i a (i = 1, \cdots, N-1), \lambda_i = c_i a + Md, i = N-2, \cdots, 2N-2$, where $c_i (i = 1, \cdots, 2N-2)$ belong to complex number field. So we can obtain $|\lambda\mathcal{M} - I| = \lambda(\lambda - 2d)\varphi(\lambda)$, where $\varphi(\lambda)$ is a 10-order polynomial. Let $\lambda_1 = 0, \lambda_2 = 2d$ and take $\lambda_i \approx \mathrm{Re}(\lambda_i), i = 3, \cdots, 2N$. we assume $\mathrm{Re}(\lambda_i) = r_i a, i = 3, \cdots, N+1$ and $\mathrm{Re}(\lambda_i) = r_i a + 2d, i = N+2, \cdots, 2N$, where $r_i (i = 3, \cdots, 2N)$ are real numbers. Therefore,

$$\begin{aligned}|\lambda\mathcal{M} - I| &= \lambda(\lambda - 2d)(\lambda - \lambda_3)(\lambda - \lambda_4) \cdots\cdots (\lambda - \lambda_{2N}) \\ &\approx \lambda(\lambda - 2d)(\lambda - \mathrm{Re}(\lambda_3))(\lambda - \mathrm{Re}(\lambda_4)) \cdots\cdots (\lambda - \mathrm{Re}(\lambda_{2N})) \\ &= \lambda(\lambda - 2d)(\lambda - r_3 a)(\lambda - r_4 a) \cdots\cdots (\lambda - r_{N+1} a)[\lambda - (r_{N+2} a + 2d)] \\ &\quad [\lambda - (r_{N+3} a + 2d)] \cdots\cdots [\lambda - (r_{2N} a + 2d)].\end{aligned}$$

We assume $r_3 \le r_4 \le \cdots \le r_{N+1}, r_{N+2} \le \cdots \le r_{2N}$. Thus, the lowest nonzero eigenvalue λ_2 of $\mathcal{M}$ takes $\min\{2d, r_3 a\}$ and the largest eigenvalue λ_{2N} of $\mathcal{M}$ takes $r_{2N} a + 2d$. Then $R = \frac{\lambda_{2N}}{\lambda_2} = \frac{r_{2N} a + 2d}{\min\{2d, r_3 a\}}$.

In bounded synchronized region, when $r_3 a > 2d$, $R = \frac{r_{2N} a + 2d}{2d}$; when $r_3 a < 2d$, $R = \frac{r_{2N} a + 2d}{r_3 a}$. For unbounded synchronized region, when $r_3 a > 2d$, $\lambda_2 = \min\{2d, r_3 a\} = 2d$; when $r_3 a < 2d$, $\lambda_2 = r_3 a$. Due to the master stability function method, we get the variation of synchronizability , as shown in Table 1.

Table 1. The variation of synchronizability with coupling strength a, d

The generalized two-layer network with N nodes			
synchronizability			
		$\mathbf{a}\uparrow$	$\mathbf{d}\uparrow$
bounded synchronized region: $R = \frac{r_{2N} a + 2d}{\min\{2d, r_3 a\}}$	$r_3 a > 2d$	$\downarrow$	$\uparrow$
	$r_3 a < 2d$	$\uparrow$	$\downarrow$
unbounded synchronized region:	$r_3 a > 2d$	-	$\uparrow$
	$r_3 a < 2d$	$\uparrow$	-

$\lambda_2 = \min\{2d, r_3 a\}$

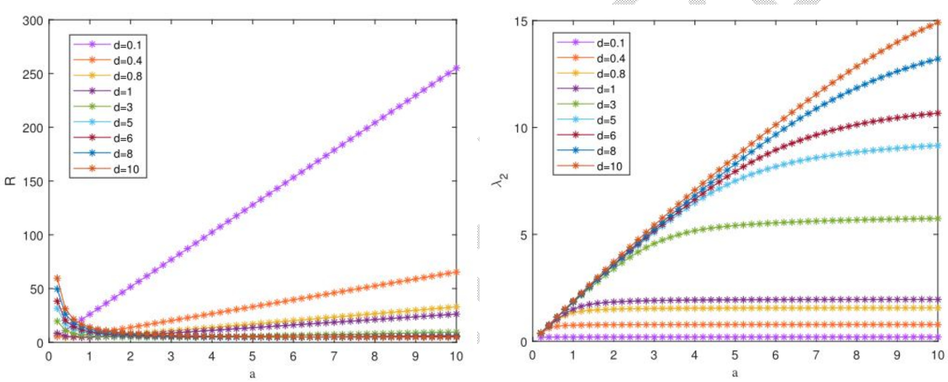
-: unchange, ↑: increase, ↓: decrease

4. NUMERICAL SIMULATIONS

We randomly constructed two N × N adjacency matrices to represent the two single-layer network topology. The corresponding nodes in two single-layer networks are connected to form a multiplex network with two layers. We constructed 1000 two-layer networks by MATLAB and recorded eigenvalues of $\mathcal{M}$ for each network. Finally, we calculated the mean eigenvalues to shown the variation of the synchronizability with coupling strength a and d.

Fig. 2 shows the relationship between synchronizability and coupling strength when nodes N=6 in each layer. When $r_3 a > 2d$, $\lambda_2 = 2d$, λ_2 is relevant to d. The synchronizability remain unchanged for different a. As shown in Fig. 2(b), λ_2 is going to flatten out. At the same time, $R = \frac{r_{2N} a + 2d}{2d}$, the larger a, the larger R and the weaker the synchronizability. Conversely, the larger d, the smaller R and the stronger the synchronizability. As shown in Fig. 2(a), when R is linearly increase with a, the slope of increasing is decreased with a larger d.

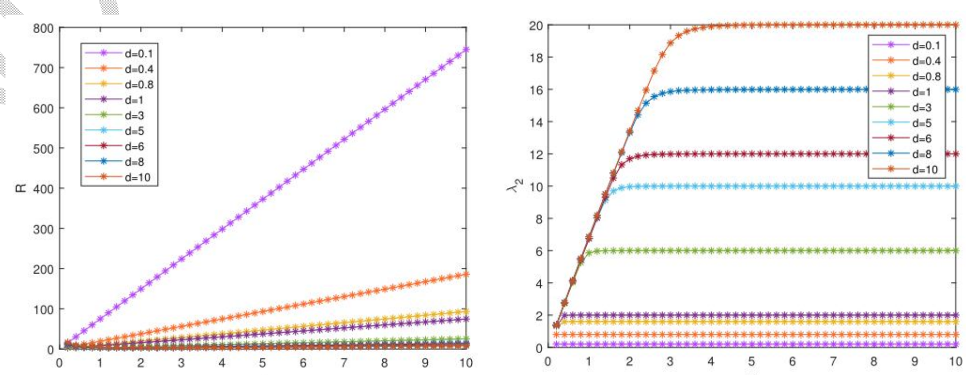
When $r_3 a < 2d$, $\lambda_2 = r_3 a$. The synchronizability is linearly increase with a and the slope keeps same for different d, as shown in Fig. 2(b). When $r_3 a < 2d$, $R = \frac{r_{2N} a + 2d}{r_3 a}$. The larger a, the smaller R and the stronger the synchronizability. The larger d, the larger R and the weaker the synchronizability. As shown in Fig. 2(a), it occurs only when a is small.



(a) (b)

Fig. 2. The trend of synchronizability with coupling strength a, d (N= 6). (a) The variation of R with a and d; (b) The variation of λ_2 with a and d.

Taking N= 20, we obtain the result shown in Fig. 3. The trend of synchronizability is similar to Fig. 2 (N= 6). In other words, the synchronizability varies with the a and d in the same way under different network sizes. However, in larger networks, the increasing rate of λ_2 is faster and the value of R is significantly greater than that of smaller networks. Thus, increasing network size can enhance the synchronizability for unbounded synchronized region. But the result is opposite for bounded synchronized region.



(a) (b)

Fig. 3. The trend of synchronizability with coupling strength a, d (N= 20). (a) The variation of R with a and d; (b) The variation of λ_2 with a and d.

5. CONCLUSION AND PROSPECT

We analyzed the effects of coupling strength (a, d) and node number (N) on the synchronizability of generalized two-layer networks. In bounded synchronized region, when interlayer coupling d is stronger than intralayer coupling a, the synchronizability is strengthened with increasing a. And the increasing of d will weaken the synchronizability. When interlayer coupling d is weaker than intralayer coupling a, the synchronizability is weakened with increasing a. And the increasing of d will strengthen the synchronizability. In unbounded synchronized region, when interlayer coupling d is stronger than a, the synchronizability is strengthened with increasing a. Until λ_2 reaches $2d$, the synchronizability is unchanged and is only related to interlayer coupling. And a larger interlayer coupling will enhance the synchronizability. Besides, the synchronizability is also affected by the number of nodes (N). Under the same coupling strength, the more N, the stronger the synchronizability in the unbounded synchronization region, and the weaker the synchronizability in the bounded synchronization region. The results will be helpful to explore the network synchronization phenomenon in the actual background.

There are still some unexplored questions here. For example, how about the synchronizability of generalized multiplex networks with more layers? How does the number of layers affect the synchronizability of generalized multiplex networks? These issues still need further study.

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