Generalizations of Fixed Point Theorems of Zermelo, Zorn, Caristi and Others

Abstract. Motivated by the Ekeland variational principle, we obtained a Metatheorem in 1985-87 stating that some well-known existence of maximal elements can be equivalently formulated to existence theorems on fixed elements, common fixed points, common stationary points, and others. In the present article, we introduce our 2023 Metatheorem and its applications to fixed point theorems of Zermelo, Zorn, Caristi, Ekeland, and related results. In fact, this is a historical supplement of our previous article entitled "Foundations of Ordered Fixed Point Theory."

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1. Introduction

Since the appearances of the Ekeland variational principle [12-14] in 1972-79 and the Caristi fixed point theorem [6] in 1976, nearly one thousand works followed on their equivalents, generalizations, modifications, applications, and related topics. Many of them are concerned with new spaces extending complete metric spaces, new metrics or topologies on them, and new order relations extending the so-called Caristi order.

While the author was working on the Ekeland principle and the Caristi theorem in 1984-2000, in order to give some equivalents of them, we obtained a Metatheorem [24-29] in 1985-2000 on fixed point theorems related to the order theory. It claims that certain order theoretic maximal element statements are equivalent to theorems on fixed points, stationary points, common fixed points, common stationary points of families of maps or multimaps. As usual in the mathematical community, our Metatheorem was not attracted for a long period.

Later in 2022, we came back to our Metatheorem after 22 years have passed and obtained its extended versions in [30-34] with a large number of their consequences [30-40]. These are applied to the traditional order theoretic results and, consequently, there have appeared the so-called Ordered Fixed Point Theory [34]. This can be comparable to traditional several fields in the fixed point theory, that is, Analytical fixed point theory is originated from Brouwer in 1912 and concerns mainly with topological vector spaces; Metric fixed point theory is originated from Banach in 1922 and deals with generalizations of contractions and nonexpansive maps; and Topological fixed point theory relates mainly with original works of Lefschetz, Nielsen, and Reidemeister.

In our previous work entitled "Foundations of Ordered Fixed Point Theory"[34] in 2022, we established a large number of improved versions of historically well-known maximal element theorems and fixed point theorems related to order structure. It is based on our new 2023 Metatheorem and the Brøndsted-Jachymski Principle established by ourselves in 2022.

In the present article, we introduce the 2023 Metatheorem, its short history, its applications to various theorems of Zermelo, Zorn, Caristi, Ekeland, Takahashi, and related results. In fact, this is a historical supplement of [33] and organized as follows.

Section 2 is to introduce

Finally, Section 6 devotes to epilogue.

2. Extended Zermelo Fixed Point Theorem

A partial order is reflexive, antisymmetric, and transitive. A chain or a simply ordered set is a partially ordered set with extra condition that any two elements are comparable. A well order is a simple order such that every subset has the first element. From now on $\mathrm{Max}(\preceq)$ (resp. $\mathrm{Min}(\preceq)$) denotes the set of maximal (resp. minimal) elements of a partially ordered set $(X, \preceq)$, and $\mathrm{Fix}(f)$ (resp. $\mathrm{Per}(f)$) denotes the set of all fixed (resp. periodic) points of a map $f : X \to X$.

We obtained the following in [31,34]:

Brøndsted-Jachymski Principle. *Let $(X, \preceq)$ be a partially ordered set and $f : X \to X$ be a progressive map (that is, $x \preceq f(x)$ for all $x \in X$). Then we have*

$$\mathrm{Max}(\preceq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

Similarly, if $f : X \to X$ is a anti-progressive map (that is, $f(x) \preceq x$ for all $x \in X$), then

$$\mathrm{Min}(\preceq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

Let $(X, \preceq)$ be a partially ordered set and

$$S_+(x) := \{y \in X : x \preceq y\} \quad (\text{resp. } S_-(x) := \{y \in X : y \preceq x\}$$

for any $x \in X$. Then we have the following:

Theorem 1. *Let $(X, \preceq)$ be a partially ordered set, $x_0 \in X$ such that $(S_+(x_0), \preceq)$ (resp. $(S_-(x_0), \preceq)$) has an upper bound $v \in S_+(x_0)$ (resp. a lower bound $v \in S_-(x_0)$).*

Then the following equivalent statements hold:

(1) *v is a maximal (resp. minimal) element, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for all $w \in X \backslash \{v\}$.*

(2) *For each chain C in $S_+(v)$ (resp. $S_-(v)$), we have $\bigcap_{x \in C} S_+(x) \neq \emptyset$ (resp. $\bigcap_{x \in C} S_-(x) \neq \emptyset$).*

(3) *Any map* $f : X \to X$ *satisfying* $x \preceq f(x)$ *(resp.* $f(x) \preceq x$*) for all* $x \in S_+(x_0)$ *(resp.* $x \in S_-(x_0)$*) has a fixed point* $v \in S_+(x_0)$ *(resp.* $v \in S_-(x_0)$*) and*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preceq) \neq \emptyset \;\; (resp.\ \mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preceq) \neq \emptyset).$$

Proof. It suffices to prove the maximal case.

(1) Since $z \preccurlyeq v$ for any $z \in S_+(x_0)$, we have $x_0 \preceq z \preceq v$ and $v \in S_+(x_0)$. Hence v is a maximal element of $S_+(x_0)$.

(2) For the maximal $v \in S_+(x_0)$ in (1), we have $C = \{v\}$ is the unique chain in $S_+(v)$ and $\bigcap_{x \in C} S_+(x) = S_+(v) \neq \emptyset$, which proves (2).

(3) Since the maximal $v \in S_+(x_0)$ and $v \preceq f(v)$, we have $f(v) \in S_+(x_0)$. Therefore, $v = f(v)$ by the antisymmetry of $\preceq$. Now the equality holds by the Brøndsted-Jachymski Principle. □

Note that (3) improves the Zermelo fixed point theorem:

Theorem 2. *For every partially ordered set* $(X, \preceq)$ *if every well-ordered subset has a least upper bound then every progressive map* $f : X \to X$ *has a fixed point and*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset.$$

See Zermelo [42] in 1908, Abian [1] in 1980, and Manka [21] in 1988. Theorem 2 is equivalently formulated in (Park [32]).

Recall that a fundamental fixed point theorem of Zermelo (see, e.g., Dunford-Schwartz ([11], p.5) says that

Proposition 3. *If* $(X, \preccurlyeq)$ *is a partially ordered set in which every chain has a supremum and a selfmap* $f : X \to X$ *is progressive, then* f *has a fixed point.*

This was given implicitly in Zermelo (1904, 1908) and formulated by Bourbaki (1949-50). Later Amann (1977) derived several fixed point theorems from Proposition 3. For example, Tarski's fixed point theorem, fixed point theorems for condensing maps and nonexpansive maps.

Jachymski (2001) noted: "Under the Axiom of Choice, the assumption of Proposition 3 can be weakened to "each nonempty well-ordered subset has an upper bound. This improves Kneser's fixed point theorem (1950), which turns out to be equivalent to the Axiom of Choice as shown by Abian (1985)."

Proposition 4. *Let* $(X, \preccurlyeq)$ *be a partially ordered set in which each nonempty well-ordered subset has a supremum. Then every progressive map* $f : X \to X$ *has a fixed point.*

Propositions 3 and 4 are consequences of our Theorem 2 and their conclusions can be improved to $\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset$.

According to Toyoda (2021-22), the Zermelo fixed point theorem is also known as the Bourbaki fixed point theorem or the Bourbaki-Kneser fixed point theorem. It implies the Bernstein-Cantor-Schröder theorem, the Caristi fixed point theorem, the Ekeland variational principle, the Takahashi minimization theorem, Nadler's fixed point theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma.

3. Extended Zorn's Lemma

A partially ordered set $(X, \preceq)$ is said to be *inductive* (resp. *complete*) if every non-empty chain in X has an upper bound (resp. a least upper bound).

The following was given in [34]:

Theorem 5. *Let $(X, \preceq)$ be a partially ordered set satisfying one of the following:*

(a) *a nonempty chain in X has an upper bound ($\Leftarrow X$ is inductive),*

(b) *a nonempty chain in X has a least upper bound ($\Leftarrow X$ is complete),*

(c) *a nonempty well-ordered subset of X has an upper bound,*

(d) *a nonempty well-ordered subset of X has a least upper bound,*

Then there exists a maximal element $v \in X$, that is, $v \not\preceq w$ for any $w \in X \backslash \{v\}$.

From the Brøndsted-Jachymski Principle and Theorem 5, we have the following improvement of Zorn's Lemma:

Theorem 6. *Let $(X, \preceq)$ be a partially ordered set satisfying one of (a)-(d) in* Theorem 5. *If $f : X \to X$ is progressive, then we have*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset.$$

Dual statements of Theorems 5 and 6 for the minimal case also hold.

4. Generalized Caristi Fixed Point Theorem

The following is well-known by Caristi [6] in 1976:

Theorem 7. (Caristi) *If (X, ρ) is a complete metric space and $\phi : X \to \mathbb{R}^+$ lower semi-continuous, then in the Brøndsted order ($x \preccurlyeq y$ iff $\rho(x, y) \le \phi(x) - \phi(y)$) every progressive map $f : X \to X$ has a fixed point.*

In 2002, Chen-Cho-Yang [8] introduced the following concept of lower semicontinuity from above:

Definition 8. [8] Let X be a metric space. A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous from above* at a point $x \in X$ if $x_n \to x$ as $n \to \infty$ and $f(x_1) \ge f(x_2) \ge \cdots \ge f(x_n) \ge \cdots$ imply that $\lim_{n\to\infty} f(x_n) \ge f(x)$.

Obviously, the usual lower semicontinuity implies lower semicontinuity from above, but the converse does not hold. In fact, Chen-Cho-Yang [8] gave an example of a function which is lower semicontinuous from above at a point, but not lower semicontinuous at that point.

Recall the following due to Chen-Cho-Yang [8]:

Theorem 9. (Caristi's Fixed Point Theorem) *Let (D, d) be a complete metric space and a function $\phi : D \to \mathbb{R}^+$ be lower semi-continuous from above. Suppose that a mapping $f : D \to D$ satisfies the following:*

$$d(x, f(x)) \le \phi(x) - \phi(f(x)) \quad \text{for all } x \in D.$$

Then there exists $x_0 \in D$ such that $f(x_0) = x_0$.

Note that (D, d) can be made into a partially ordered set by defining

$$x \preccurlyeq y \iff \phi(y) \leq \phi(x)$$

for $x,\ y \in D$.

Here we give a new proof of the Caristi Theorem 9 due to Chen-Cho-Yang [8]:

Proof. Since $\phi : D \to \mathbb{R}^+$ is l.s.c. from above at any $z \in D$, for any $\{x_n\}$ converging to z such that

$$\phi(x_1) \geq \phi(x_2) \geq \cdots \geq \phi(x_n) \geq \cdots \Longrightarrow \lim_{n\to\infty} \phi(x_n) \geq \phi(z)$$

and hence $x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq \cdots \preccurlyeq z$. Note that $C = \{z\} \subset S_+(x_1)$ is a chain in $S_+(z)$. Let $v = z \in C$. Then $C = \{v\} \subset \bigcup_{x\in C} S_+(x) \neq \emptyset$. Hence, Theorem 1(2) holds, v is maximal by (1), and our Caristi theorem (3) holds. □

The original Caristi theorem is equivalent to Zorn's Lemma. For the earlier proofs, see Kirk [18]. However, the extended version, Theorem 7, has an elementary proof as above. For, further generalizations of the Caristi theorem, Cobzaş [9] is a rich source of information.

5. Dual of Caristi Fixed Point Theorem

Until now, certain results are related to the maximality. We can obtain their dual formulations for the minimality. In this section, we obtain dual forms of the Caristi theorem.

We define the following motivated by Lin-Du [20]:

Definition 8.* Let X be a metric space. A function $f : X \to \overline{\mathbb{R}}$ is said to be *upper semicontinuous from below* at a point $x \in X$ if $x_n \to x$ as $n \to \infty$ and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n) \leq \cdots$ imply that $\lim_{n\to\infty} f(x_n) \leq f(x)$.

The following is a dual of Theorem 9; see also [37].

Theorem 9.* *Let $(X, \preccurlyeq)$ be a partially ordered complete metric space, and a function $\varphi : X \to \mathbb{R}^+$ be upper semicontinuous from below and bounded from above such that*

$$y \preccurlyeq x \quad \textit{iff} \quad d(x,y) \leq \varphi(y) - \varphi(x) \quad \textit{for} \quad x, y \in X.$$

Then the dual of Theorem 9 hold, that is,

(α) *There exists a minimal element $v \in X$; that is, $w \not\preccurlyeq v$ for any $w \in X \backslash \{v\}$.*

Theorem 9* is the dual to the Caristi fixed point theorem 9 and can be stated as follows:

Theorem 10. *Let $(X, \preccurlyeq)$ be a partially ordered complete metric space, and a function $\varphi : X \to \mathbb{R}^+$ be upper semicontinuous from below and bounded from above such that*

$$y \preccurlyeq x \quad \textit{iff} \quad d(x,y) \leq \varphi(y) - \varphi(x) \quad \textit{for} \quad x, y \in X.$$

Then every anti-progressive map $f : X \to X$ has

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset.$$

Note that there are nearly one thousand papers related to the Caristi theorem for its extensions, modifications, and applications. However, this article has something different to them.

6. Unified Generalizations

All of the key results in the above can be unified by applying our Metatheorem and its variants. In fact, from our 2023 Metatheorem in [34], we deduced the following prototype of Maximal (resp. Minimal) Element Principles in [37]:

Theorem 11. *Let $(X, \preceq)$ be a preordered set and A be a nonempty subset of X. Then the following statements are equivalent:*

(α) *There exists a maximal (resp. minimal) element $v \in A$, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \backslash F(x)$, there exists $y \in X \backslash \{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $x \preceq y$ (resp. $y \preceq x$) holds for any $x \in A$ and any $y \in F(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that, for each $x \in A \backslash Y$, there exists a $z \in X \backslash \{x\}$ satisfying $x \preceq z$ (resp. $z \preceq x$), then there exists a $v \in A \cap Y$.*

Remark 12. (1) When $\mathfrak{F}$ is a singleton, $(\beta) - (\zeta)$ are denoted by $(\beta 1) - (\zeta 1)$ and these are also equivalent to $(\alpha) - (\eta)$. Hence, Theorem 11 implies 28 equivalent statements.

(2) Note that, in Theorem 11, $(\alpha) \Longleftrightarrow (\gamma 1)$ implies the Brøndsted-Jachymski Principle.

From Metatheorem* in [40], we have the following:

Theorem 13. *Let $(X, \preceq)$ be a partially ordered set, $x_0 \in X$, and $A = S_+(x_0)$ (resp. $A = S_-(x_0)$) have an upper bound (resp. a lower bound). Then the following equivalent statements hold :*

(α) *There exists a maximal (resp. minimal) element $v \in A$, that is, $v \not\preceq w$ (resp. $w \not\preceq v$) for any $w \in X \backslash \{v\}$.*

($\theta1$) *There exists $v \in A$ such that, for each chain C in $S_+(v)$ (resp. $S_-(v)$) , we have $\bigcap_{x\in C} S_+(x) \neq \emptyset$ (resp. $\bigcap_{x\in C} S_-(x) \neq \emptyset$).*

($\theta2$) *There exist $v \in A$ and a maximal chain C^* in $S_+(v)$ (resp. $S_-(v)$), we have $\bigcap_{x\in C^*} S_+(x) \neq \emptyset$ (resp. $\bigcap_{x\in C^*} S_-(x) \neq \emptyset$).*

Remark 14. (1) For the motivation of Theorem 13 and its proof, we have a long story as shown in [454]. For the origin of maximal cases of statements ($\theta1$) and ($\theta2$), see ([3], Theorem 5.1) and ([40], Theorem 5.1*).

(2) Because of Theorem 13, we can add 4 more equivalent statements to the 28 statements generated by Theorem 11.

(3) From Theorem 11($\gamma1$), we can deduce many examples of maps $f : X \to X$ satisfying $\mathrm{Per}(f) = \mathrm{Fix}(f) \neq \emptyset$; see [453]. Such sets X can have more rich properties by the following main theorem of Jachymski ([16], Theorem 2):

Theorem 15. [16] *Let X be a nonempty abstract set and $f : X \to X$. The following statements are equivalent:*

(a) $\mathrm{Per}(f) = \mathrm{Fix}(f) \neq \emptyset$.

(b) (Zermelo) *There exists a partial order $\preccurlyeq$ such that every chain in $(X, \preccurlyeq)$ has a supremum and f is progressive with respect to $\preccurlyeq$.*

(c) (Caristi) *There exists a complete metric d and a lower semicontinuous function $\varphi : X \to \mathbb{R}^+$ such that f satisfies Caristi's condition.*

(d) *There exists a complete metric d and a d-Lipschitzian function $\varphi : X \to \mathbb{R}^+$ such that f satisfies Caristi's condition and f is nonexpansive with respect to d; i.e.*

$$d(fx, fy) \leq d(x, y) \quad \text{for all} \quad x, y \in X.$$

(e) (Hicks-Rhoades) *For each $\alpha \in (0, 1)$, there exists a complete metric d such that f is nonexpansive with respect to d and*

$$d(fx, f^2x) \leq \alpha d(x, fx) \quad \text{for all} \quad x \in X.$$

(f) *There exists a complete metric d such that f is continuous with respect to d and for each $x \in X$, the sequence $(f^n x)_{n=1}^{\infty}$ is convergent (the limit may depend on x).*

(g) *There exists a partition of X, $X = \bigcup_{\gamma\in\Gamma} X_\gamma$, such that all the sets X_γ are nonempty, f-invariant and pairwise disjoint, and for all $\gamma \in \Gamma$, $f|_{X_\gamma}$ has a unique periodic point.*

(h) *For each $\alpha \in (0, 1)$, there exists a partition of X, $X = \bigcup_{\gamma\in\Gamma} X_\gamma$, and complete metrics d_γ on X_γ such that all the sets X_γ are nonempty; f-invariant and pairwise disjoint; and*

$$d_\gamma(fx, fy) \leq \alpha d_\gamma(x, y) \quad \text{for all } x, y \in X.$$

7. Ekeland Principle

In order to obtain some equivalents of the well-known central result of Ivar Ekeland [12-14] on the variational principle for approximate solutions of minimization problems, we obtained a Metatheorem in [24-29] and related works in 1983-2000. Later in 2022 we found an extended version of the Metatheorem and, finally, the 2023 version in [34]. Our Theorem 11 can be applied to give equivalencies for various situations as we have shown in our previous works. Motivated by this, we derive the following; see also [39]:

Theorem 16. *Let (X, d) be a complete metric space and a proper function $\varphi : X \to \overline{\mathbb{R}}$ l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). Let $A = \operatorname{dom}\varphi = \{x \in X : -\infty < \varphi(x) < \infty\}$.*

Then the following equivalent statements hold:

(α) *There exists a maximal (resp. minimal) element $v \in A$, that is,*

$$d(v, w) > \varphi(v) - \varphi(w) \quad (resp.\ \ d(v, w) > \varphi(w) - \varphi(v))$$

for any $w \in X\backslash\{v\}$.

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x, y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \quad (resp.\ \ d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for all $x \in A\backslash\{f(x)\}$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.

(δ) *Let $\mathfrak{F}$ be a family of multimaps $T : A \multimap X$ such that, for any $x \in A\backslash T(x)$, there exists $y \in X\backslash\{x\}$ satisfying*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x, y) \leq \varphi(y) - \varphi(x)),$$

then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in T(v)$ for all $T \in \mathfrak{F}$.

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ such that*

$$d(x, y) \leq \varphi(x) - \varphi(y) \quad (resp.\ \ d(x, y) \leq \varphi(y) - \varphi(x))$$

holds for any $x \in A$ and any $y \in T(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.

(η) *If Y is a subset of X such that, for each $x \in A\backslash Y$, there exists a $z \in X\backslash\{x\}$ satisfying*

$$d(x, z) \leq \varphi(x) - \varphi(z) \quad (resp.\ \ d(x, z) \leq \varphi(z) - \varphi(x)),$$

then there exists a $v \in A \cap Y$.

(θ1) *There exists* $v \in A$ *such that, for each chain* C *in* $S(v)$*, we have* $\bigcap_{x \in C} S(x) \neq \emptyset$.

(θ2) *There exist* $v \in A$ *and a maximal chain* C^* *in* $S(v)$*, we have* $\bigcap_{x \in C^*} S(x) \neq \emptyset$.

In Theorem 16, (X, d) can be made a partially ordered set $(X, \preccurlyeq)$ by defining

$$x \preccurlyeq y \Longleftrightarrow \varphi(y) \leq \varphi(x) \ \ (\text{resp. } x \preccurlyeq y \Longleftrightarrow \varphi(x) \leq \varphi(y))$$

for $x,\ y \in X$.

Theorem 16 includes various earlier results and is very useful as shown in [39]. Especially, (α) for maximal case is called the *weak Ekeland Principle* and that $(\gamma1)$ extends the Caristi theorem.

In 1989, by using Ekeland's variational principle, Mizoguchi-Takahashi [22] derived the following Caristi-Kirk's theorem [7], which is the set-valued version of the Caristi fixed point theorem:

Theorem 17. (Caristi-Kirk) *Let* (X, d) *be a complete metric space and* $T : X \multimap X$ *be a multimap with nonempty values such that for each* $x \in X$*, there exists* $y \in T(x)$ *satisfying* $d(x, y) + \varphi(y) \leq \varphi(x)$*, where* $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ *is a proper, lower semicontinuous and bounded below functional. Then,* T *has a fixed point, that is, there exists* $\bar{x} \in X$ *such that* $\bar{x} \in T(\bar{x})$.

Here the lower semicontinuous function can be replaced by the one *from above*. The extended form of this follows from the first form of Theorem D(ζ).

Moreover, consider the following extended form of Takahashi [41]:

Theorem 18. (Takahashi Principle) *Let* (X, d) *be a complete metric space and* $\varphi : X \to \mathbb{R} \cup \{\infty\}$ *a proper function lower semicontinuous from above and bounded from below.*

If for every $x \in \operatorname{dom}\varphi$ *with* $\varphi(x) > \inf \varphi(X)$ *there exists an element* $y \in \operatorname{dom}\varphi \backslash \{x\}$ *such that* $\varphi(y) + d(x, y) \leq \varphi(x)$*, then* φ *attains its minimum on* X*, i.e., there exists* $z \in \operatorname{dom}\varphi$ *such that* $\varphi(z) = \inf \varphi(X)$.

Further, from Theorem 16(α), (γ) and the Brøndsted-Jachymski Principle, we have the following:

Theorem 19. *Let* (X, d) *be a complete metric space and* $\varphi : X \to \mathbb{R} \cup \{\infty\}$ *a proper function l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). If* $f : X \to X$ *is a map such that*

$$d(x, f(x)) \leq \varphi(x) - \varphi(f(x)) \ \ (\textit{resp. } d(x, f(x)) \leq \varphi(f(x)) - \varphi(x))$$

for any $x \in X$*. Then we have*

$$\operatorname{Max}(\preccurlyeq) = \operatorname{Fix}(f) = \operatorname{Per}(f) \ \ (\textit{resp. } \operatorname{Min}(\preccurlyeq) = \operatorname{Fix}(f) = \operatorname{Per}(f)).$$

Consequently, this section demonstrates the usefulness of our Metatheorem. Until now, we gave more than one hundred examples or applications of our Metatheorem, and each of them might have useful consequences.

8. Analysis of Applications of Principles

In this section, we analyze certain situations in the present article such that Theorems 11 and 13 are applicable:

(1) (Zermelo type) Let $x_0 \in X$ and $A = S_+(x_0)$ (resp. $A = S_-(x_0)$).
Theorems 1, 2 and Propositions 3, 4 belong to this case.
The Zermelo fixed point theorem implies the Caristi fixed point theorem, the Bernstein-Cantor-Schröder theorem, the Ekeland variational principle, the Takahashi minimization theorem, and others. Moreover, under the Axiom of Choice, it implies Zorn's Lemma.

(2) (Zorn type) Let $X = A$ satisfy (a)-(d) in Theorem 5.
Theorems 5, 6 and their minimal cases belong to this case.
The 29 references of [27] show the origins, variants, consequences, applications of a particular form of Theorem 3, and we will only indicate names of their authors like Abian (1971), Ekeland [14](1979), Bishop-Phelps (1961), Turinici (1980-1984), Smithon (1971, 1973), Hoft-Hoft (1976), Tuy (1981), Kasahara (1975), Maschler-Peleg (1976), Phelps (1964), Caristi [6](1976), Banach (1922), and others. Recall that Taskovic (1986) showed an equivalent form of Zorn's lemma.

(3) (Caristi type) Let (X, d) be a complete metric space and a function $\phi : X \to \mathbb{R}^+$ be lower semi-continuous from above. Define partial order on X using ϕ.
Examples are Theorems 7 and 9.
Equivalent formulations of the Caristi theorem were originally given in Park [24, 25] in 1985-1986. The Caristi theorem implies the Banach contraction principle and numerous applications.

(4) (dual Caristi type) Let (X, d) be a complete metric space and a function $\phi : X \to \mathbb{R}^+$ be upper semi-continuous from below. Define partial order on X using ϕ.
Theorems 9^* and 10 are examples of this case.
Extensions of the dual Caristi theorem were given by Lin-Du [20] and their equivalent formulations by Park [31, 39].

(5) (Ekeland type) Let (X, d) be a complete metric space and $\varphi : X \to \mathbb{R} \cup \{\infty\}$ a proper function l.s.c. from above and bounded from below (resp. u.s.c. from below and bounded from above). Define partial order on X using ϕ.
Theorems 16–20 are examples of this case.
The original Theorem 16 was given in [24, 25] in 1985-1986. It implies the variational principle of Ekeland (1979), works of Tuy (1981), Kasahara (1975), Mascher-Peleg (1976), etc. Classical applications of Theorem 16 are numerous in vast fields of mathematical sciences.

By applying Theorems 11 and 13 to each of such types, we can obtain more than one hundred true statements. Only some of them are known as famous theorems; see our previous works in the references. We will not trace all of them.

8. Epilogue

As we have seen in Metatheorem [34], the maximal element v can be fixed point, stationary point, common fixed point, common stationary point, etc. Some authors seem to be not recognized this fact yet. Its applications are numerous.

In this article, we adopted some consequences of our 2023 Metatheorem as basis of our stdy. We showed that the maximal elements in certain preordered sets can be reformulated to fixed points or stationary points of maps or multimaps and to common fixed points or common stationary points of a family of maps or multimaps, and conversely. Actually such points are same as we have seen in the proof of Metatheorem Therefore, if we have a theorem on any of such points, it can be converted to at least six equivalent theorems on other types of points without any serious argument.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or various fixed points that can be applicable our Metatheorem. Some of such theorems can be seen in our previous works and the present article. Therefore, a metatheorem like Theorems 11 and 13 are machines to expand our knowledge easily. In this article we presented relatively old and well-known examples.

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