COMPARATIVE ANALYSIS OF ADDITIVE AND MULTIPLICATIVE ERROR TERMS OF WEIBULL, LOGISTIC GOMPERTZ, HILLS AND RICHARDS MODELS WITH FOUR PARAMETERS

Abstract

Three growth models (Richards, Gompertz, and Weibull) were estimated using a computer program employing a modified version of the Levenberg-Marquardt approach for solving non-linear regression models. With both small and high sample sizes, three data sets were collected. The most suitable model for growth studies was added by decomposing the growth models into their component parts: additive and multiplicative error factors. The analysis based on the mathematical properties was conducted using Gretl statistics software. Prior to using an iterative strategy, a third-order polynomial (cubic function) solves the issue of the initial parameters. The final estimate of the parameters, standard errors, p-values, and model adequacy standards used to choose the best growth model are included in the result. Out of the five growth models considered suitable for the agricultural data, this study was able to pinpoint the Weibull Growth Model with Multiplicative Error Term. The Hill Growth Model with Additive Error Term is the best growth model for the engineering data out of the five growth models. Among the Five Growth Models for Population Growth, the Weibull Growth Model with Additive Error Term is a viable growth model. Several growth models are suggested or recommended by this study for use in forecasting this growth behavior.

Keywords: Additive and Multiplicative Error Terms, Levenberg-Marquardt Method, Five Growth Models, Model Adequacy Criteria, Third-Order Polynomial

1. Introduction

Forecasting a non-linear model has been a significant challenge for many years, leading to the creation and transformation of numerous statistical models. Regression analysis, which includes simple multiple, multivariate, linear, and non-linear models, is the main instrument for solving the problem. Non-linear regression or growth models, such as Weibull, Logistics, Gompertz, Hill, Richard (case study), Brody, and Robertson, have been developed for prediction in order to produce more accurate findings. Several statisticians and researchers employ well-established techniques like the Gauss-Newton and Leverberge-Marguardt procedures (Biu et al, 2019).

In some circumstances, it is possible to transform a nonlinear regression function so that it is linear in the unknown parameters by appropriately transforming the response variable Yi, the predictor variable, the parameters, or any combination of these. The outcome can be applied to the converted issue if the transformed variables meet the requirements for simple or multiple linear regression. We may frequently acquire and deduce conclusions for the original problem using the results for the altered. Growth models are typically presented as differential equations that explain how populations change over time (that is temporal evolution). The pace at which population size varies is described by a differential equation. These growth models are typically created to simulate population expansion under specific assumptions (usually specified biological hypothesis). It is interesting to extract the population size at a certain time point from the rate equation once the differential equation has been constructed. One important function of growth curve models is that, despite their similarities, the Gompertz growth curve model and the logistic growth curve model have distinct fundamental features (Amiya,2019).

The evaluation of several clinical and biological investigations, as well as the evolution of mathematical functions to explain population growth, including Gompertz, Logistic, Richard, Weibull, and Von Betalanffy, made it evident how this field has changed over time. These models have been effective for a variety of growth curves. For the optimization process to commence, conventional statistical approaches for nonlinear models need a starting point (initial parameters or estimate values). The parameters must first be estimated iteratively after the nonlinear model expression has been created, the parameter names have been stated, and the initial parameter values have been supplied. The location of this beginning point in the search space frequently affects the quality of the eventual answer. The growth models have been extensively used in numerous studies to address biological growth issues (Khamis et al., 2005; Amanullah et al., 2007; Roush & Branton 2004).

Growth curves show the lifetime interactions between a person's innate desire to develop and mature all body parts and the surroundings in which this desire manifests itself. No matter their area of study, all animal biologists need to understand the growth curve (Fitzhugh, 1976). Plotting physical characteristics against age has been used to identify growth curves, including weight, height, and width. When growth is described using regression models, the data from a time series is reduced to a small number of biologically understandable parameters (Eisen, 1974). For this reason, a variety of models are available, and experimental comparisons are required to help select the model that is most suited (Brown et al., 1976).

The development of mathematical nonlinear growth models by Brody (1945), Richards (1959), Nelder (1961), and Ratkowsky (1983). These models have been used to explain weight-age connections in beef cattle (De Nise and Brinks, 1985; Nobre et al., 1987).

When analyzing the prognostic markers in patients with gastric cancer and comparing them to Cox, Zhu et al. (2011) used the Weibull model. Jaber (2017) calculated the likelihood of default, which could be used to assess the performance of a sample credit risk portfolio, using a variety of parametric models (exponential, log-normal, gamma, Weibull, log-logistic, and Gompertz) and non-parametric models (Kaplan-Meier, Nelson-Aalen).

The anti-microbial pharmacodynamics (PD) and pharmacokinetics (PC) of the treatment regimen against the disease-causing bacteria were used by Carla et al. (2018) utilizing the Hill growth model. In order to evaluate the growth performance and growth efficiency of individual plants as well as plant populations, Arne and Anders (2015) analyzed plant growth relative to plant size using absolute and relative growth rates.

Ersoy (2006) looked at early development stage parameter estimates in California turkeys for both linear and non-linear growth curve models. Several works (Ludwig, 1999; Allen and Allen, 2003; Matis and Kiffe) deal with the population's growth rate in relation to its size (2004). Ludwig (1999) uses the Gompertz and logistic models for calculating the likelihood of extinction for a single species.

Several authors have done works on the growth model. Begall, S. (1997) identified three possible mistakes that could be made when utilizing the Gompertz growth model using a case study of growth data for common mole-rats in Zambia. The mistakes include limiting growth data to the juvenile or subadult stage, confusing the mean growth rate with the (mean) maximum growth rate, and applying the growth model incorrectly when sample sizes vary. On the basis of the deterministic Gompertz law of cell proliferation, Lo (2007) suggested a stochastic model of tumor formation. In 2012, (Eberhardt and Breiwick) investigated population growth curve models, comparing integrated versions with many other varieties. In a study using R, Pain et al. (2012) revealed how to construct function-derived growth rates for a range of nonlinear models that are suitable for simulating plant growth. These growth rates enable fair comparisons between species at a given point in time or size.Amir (2013) used the Richards growth model to forecast the growth process and extract growth parameters in the green gram by combining the work of Beta, Gompertz, and Richards From a forestry perspective, Dimpal and Munindra (2014) explore some of the features of three Weibull growth models. They used the Newton-

Raphson iteration approach to estimate the parameters of their models using the mean diameter at breast height and top height growth data from the Bowmont Norwegian spruce thinning experiment, sample plot 3661. Using non-linear growth models, Karadavut & Kavurmaci (2016) identified the optimum model for plant length, dry stem and dry leaf weight in various species of bitter vetch. In their work, Dagogo et al. (2020) employed the Weilbull growth model to analyze the amount of transmitted voltage in the voltage versus time signal detection readings. Using a modified version of the Levenberg-Marquardt, the nonlinear least squares estimation approach was used by taking the derivative with respect to the parameters β_0 ,β_1 ,β_2 . By comparison of various multiple trait models (MTM) with a random regression model, PanelR et al. (2020) assessed the genetic factors and trends in the birth, weaning, and yearling weights of HC (RRM). Mugo et.al (2021) analyzed the spatial variation of rainfall and temperature in the county, and their effects on green gram production using Analytical Hierarchy Process (AHP) decision making tool in order to determine the perceived weights or influence that rainfall and temperature have on green gram production. In order to build an algorithm model to control or minimize logistics delays, Qia (2021) investigated the likelihood of numerous elements that contribute to delays in logistics. Their study developed an EY model—a BN model based on the genetic EM algorithm—based on the genetic EM algorithm and carried out associated simulation experiments based on the model to confirm its validity and viability.

In this study, the Weibull, Logistic, Gompertz, Hill, and Richard models of growth are taken into account. However, the focus of this work is on the investigation of the five growth models' addictive and multiplicative error terms (i.e. Weibull, Logistic, Gompertz, Hill and Richards models). These models, which are also known as nonlinear models, make it difficult to solve problems involving growth curves. The pertinent model parameters will be identified and contrasted. The intrinsic values of the dependent values based on the series will be obtained, and the mathematical features of these models will be deduced. There are growth models with two, three, four, etc. parameters, but this study used models with four parameters to access the relationships between the models taken into account.

This study compares the Weibull, Logistic, Gompertz, Hill, and Richards models of growth utilizing growth analysis using addictive and multiplicative error terms. The goals are to use three real data sets gathered for model comparison to generate the five growth models with additive and multiplicative error terms.

To use a third-order polynomial to determine the growth models' initial parameters (or Cubic regression techniques).

To use a modified version of the Levenberg-Marquardt algorithm to estimate nonlinear least squares (NLS) parameters for the five growth models.

The accuracy of the estimated values under the error assumptions, goodness of fit tests, and model adequacy standards will be used to compare the parameters of the five models.

To suggest/recommend the best model for growth studies.

To offer a different way to select the starting points and fit the models using decomposition techniques.

With the aid of the second order regression introduced, it will be possible to achieve the appropriate level of forecast by obtaining the initial values and solving for the additive and multiplicative error factors.

2. Materials and Methods

Three sets of data were taken into account for this research study's illustration of the fitted growth model.

1) An experiment from the Electrical/Electronic Engineering department at the University of Port Harcourt in Rivers state measuring the amount of transmitted voltage against time.

2) Results on dry matter green grain in black gram that showed sigmoidal form were reported, and statistical analysis indicated good approximation efficiency in the growth of green grain (Purnachandra R. K., & Ayele, T. G. (2013).)

The five growth model was analyzed using the Gretl statistical program and Micro-Excel.

2.1 Methods and Models

The approach used in this study was nonlinear least square estimation, which combined regression techniques with a modified Levenberg-Marguart model.

The below list of growth models is described.

The Weibull model with four parameters is expressed as

$$Y_I = \propto_0 X_i^{\propto_1 - 1} e^{-\propto_2 X_i^{-\propto_3}} \quad (1)$$

The Logistic Growth model mathematical equation is

$$y = \frac{\alpha_1 - \alpha_0}{1 + \left(\frac{x}{\alpha_2}\right)^{\alpha_3}} \quad (2)$$

The Gompertz Growth model with four parameters has the form:

$$y_i = \propto_0 e^{-\propto_1 e^{-\propto_2 X_i^{\propto_3}}} \quad (3)$$

The Hill model of growth with four parameters is expressed as

$$y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)^{\propto_3} \quad (4)$$

The Richard Model with four parameters is expressed as:

$$y_i = \propto_0 (1 - \propto_1 e^{-\propto_2 X_i})^{\propto_3} \quad (5)$$

where;

e represents Euler number (e = 2.71828)

x_i represents time

$\propto_0$, $\propto_1$, $\propto_2$, $\propto_3$ are the parameters

$\propto_0$ represents upper asymptote when time approaches positive infinite (i.e. maximum growth response or scale parameter)

$\propto_1$ represents the shape parameter related to initial time

$\propto_2$ represents growth range (or intrinsic growth range)

$\propto_3$ represents growth rate

y_i is the i^{th} observation at time

Hence, this study will centre on accessing the relationship between Weibull, Logistics, Hill, Gompertz and Richard four parameters model in growth analysis

2.1.1 The five growth model with additive error terms

Equation (1) to (5) can be expressed as

$$Y_I = \propto_0 X_i^{\propto_1 - 1} e^{-\propto_2 X_i^{-\propto_3}} + \varepsilon_i \quad (1.1)$$

$$y_i = \frac{\propto_1 - \propto_0}{1 + \left(\frac{X_i}{X_2}\right)^{\propto_3}} + \varepsilon_i \quad (1.2)$$

$$y_i = \propto_0 e^{-\propto_1 e^{-\propto_2 X_i^{\propto_3}}} + \varepsilon_i \quad (1.3)$$

$$y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)^{\propto_3} + \varepsilon_i \quad (1.4)$$

$$y_i = \propto_0 (1 - \propto_1 e^{-\propto_2 X_i})^{\propto_3} + \varepsilon_i \quad (1.5)$$

Where ε_i are the error terms (or ε_i are normal measurement errors with 0 means independent for the random effects).

2.1.2 The five Growth model with Multiplication Error Terms

Likewise the equation (1.1) to (1.5) can be expressed as

$$Y_I = \propto_0 X^{\propto_1 - 1} e^{-\propto_2 X_i^{-\propto_3}} \varepsilon_i \quad (6)$$

$$y_i = \frac{\propto_1 - \propto_0}{1 + \left(\frac{X_i}{X_2}\right)^{\propto_3}} \varepsilon_i \quad (7)$$

$$y_i = \propto_0 e^{-\propto_1 e^{-\propto_2 X_i^{\propto_3}}} \varepsilon_i \quad (8)$$

$$y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)^{\propto_3} \varepsilon_i \quad (9)$$

$$y_i = \propto_0 (1 - \propto_1 e^{-\propto_2 X_i})^{\propto_3} \varepsilon_i \quad (10)$$

2.2 Modified Levenberg – Marguardt Algorithm

employing a modified version of the Levenberg-Marguardt algorithm to accomplish non-linear square (NLS) estimation or the non-linear least squares (NLS) approach. Since a non-linear model must be developed through a succession of approximations (iterations), which may begin with trial and error, it is more difficult than a linear model to design. Many well-known techniques are employed by mathematicians, including the Gauss-Newton method and the Levenberg-Marguardt technique. The Levenberg-Marquardt Method was modified by the researcher. Which is

1. Obtain partial derivation of the model with respect to the four parameters (α_0, α_1, α_2, α_3,).
2. To develop a program in the Gretl software using equation (1) to (10) and input the initial values by fitting second order polynomial (Cubic Model), using Minitab 17 software.
3. Then substitute the second-order polynomial coefficient $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ as initial guess values for iteration process.
4. Input the data and initiate guess values on the developed program. Then, run the iteration to obtain the results.

Let $\alpha = \propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)}$ be the initial parameter. Thus, we take logarithm transformation of Equation (1)

$$y_i = \propto_0 X_i^{\propto_1 - 1} e^{-\propto_2 X_i^{-\propto_3}}$$

$$lny_i = ln \propto_0 + (\propto_1 - 1) lnX_i - \propto_2 X_i^{-\propto_3}$$

$$lny_i = ln \propto_0 + \propto_1 lnX_i - lnX_i - \propto_2 X_i^{-\propto_3} \quad (11)$$

The NLS estiation using a modified version of the Levenberg-Marguardt, we take the derivation with respect to the parameter ((α_0, α_1, α_2, α_3,) then substitute $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ as the initial guess vector for iteration process.

In Equation (3.11) let $lny_i = y_i, ln \propto_0 = \widehat{\propto}_0$

$\propto_1 = \widehat{\propto}_1, \propto_2 = \widehat{\propto}_2, \propto_3 = \widehat{\propto}_3$

$$y_i = \widehat{\propto}_0 + \widehat{\propto}_1 lnX_i - lnX_i - \widehat{\propto}_2 X_i^{-\propto_3} \quad (12)$$

By taking partial derivation of equation (12) with respect to $(\widehat{\propto}_0, \widehat{\propto}_1, \widehat{\propto}_2, \widehat{\propto}_3)$ we have,

$$\frac{\partial y_i}{\partial \hat{\alpha}_0} = 1 \quad (13)$$

$$\frac{\partial y_i}{\partial \widehat{\propto}_1} = lnX_i \quad (14)$$

$$\frac{\partial y_i}{\partial \widehat{\propto}_2} = -X_i^{-\widehat{\propto}_3} \quad (15)$$

$$\frac{\partial y_i}{\partial \propto_3} = -\propto_2 (lnX_i)(X^{-\propto_3}) \quad (16)$$

Likewise let X = $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ be the initial parameter, thus we take logarithm transformation of Equation (3)

$$y_i = \frac{\propto_i - \propto_0}{1 + \left(\frac{X_i}{\propto_2}\right)^{\propto_3}}$$

$$lny_i = ln(\propto_1 - \propto_0) - ln\left(1 + \left(\frac{X_i}{\propto_2}\right)^{\propto_3}\right) \quad (17)$$

Let $lny_i = y_i, \propto_0 = \widehat{\propto}_0, \propto_1 = \widehat{\propto}_1, \propto_2 = \widehat{\propto}_2$ and $\propto_3 = \widehat{\propto}_3$

$$Y_i = ln(\widehat{\propto}_1 - \widehat{\propto}_0) - ln\left(1 + \left(\frac{X_i}{\propto_2}\right)^{\widehat{\propto}_3}\right) \quad (18)$$

By taking partial derivation of equation (18) with respect to $(\widehat{\propto}_0, \widehat{\propto}_1, \widehat{\propto}_2, \widehat{\propto}_3)$, we have

$$\frac{\partial y_i}{\partial \widehat{\propto}_0} = \frac{-1}{\hat{\partial}_1 - \widehat{\propto}_0} \quad (19)$$

$$\frac{\partial y_i}{\partial \widehat{\propto}_1} = \frac{1}{\widehat{\propto}_1 - \widehat{\propto}_0} \quad (20)$$

$$\frac{\partial y_i}{\partial \hat{X}_2} = \frac{-X_i^{\widehat{\propto}_3}}{\widehat{\propto}_2^{\widehat{\propto}_3 + 1}\left(1 + \frac{X_i}{\propto_2}\right)} \quad (21)$$

$$\frac{\partial y_i}{\partial \hat{\propto}_3} = \frac{-1}{1+\left(\frac{X_i}{\hat{\propto}_2}\right)^{\hat{\propto}_3}}\left[\left(\frac{X_i}{\propto_2}\right)^{\propto_2} ln\left(\frac{X_i}{\hat{\propto}_2}\right)\right] \quad (22)$$

Likewise let X = $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ be the initial parameter, we take the logarithm transformation of Equation (3)

$$y_i = \propto_0 e^{-\propto_1 e^{-\propto_2 X_i^{\propto_3}}}$$

$$lny_i = ln \propto_0 - \propto_1 e^{-\propto_2 X_i^{\propto_3}} \quad (23)$$

The NLS estimation using a modified version of Levenberg – Marguadt, we take the derivative with respect to the parameter (α_0, α_1, α_2, α_3,) then substitute $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ as the initial guess vector for iteration process

Let $lny_i = y_i, ln \propto_0 = \hat{\propto}_0, \propto_1 = \hat{\propto}_1, \propto_2 = \hat{\propto}_2, \propto_3 = \hat{\propto}_3$

$$Y_i = \hat{\propto}_0 - \hat{\propto}_1 e^{-\hat{\propto}_2 X_i^{\hat{\propto}_3}}$$

By taking partial derivative of the equation

$$\frac{\partial y_i}{\partial \hat{\propto}_0} = 1 \quad (24)$$

$$\frac{\partial y_i}{\partial \hat{\propto}_1} = -e^{-\hat{\propto}_2 X_i^{\propto_3}} \quad (25)$$

$$\frac{\partial y_i}{\partial \hat{\propto}_2} = \hat{\propto}_1 X_i e^{-\hat{\propto}_2 X_i^{\hat{\propto}_3}} \quad (26)$$

$$\frac{\partial y_i}{\partial \hat{\propto}_3} = \frac{\hat{\propto}_2 X^{\hat{\propto}_3 - 1}}{\hat{\propto}^3} e^{-\propto_2 X_i^{\propto_3}} \quad (27)$$

Similarly, let $\propto = (\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ be the initial parameter. Thus, we take the logarithm transformation of the Equation (3.4)

$$y_i = \propto_0 \propto_1 \left(\frac{X_i}{\propto_2 + X_i}\right)^{\propto_3}$$

$$lny_i = ln \propto_0 + ln \propto_1 + \propto_3 ln\left(\frac{X_i}{\propto_2 + X_i}\right)$$

Let $Y_i = lny_i, \propto_i = \hat{\propto}_1, \propto_2 = \hat{\propto}_2, \propto_3 = \hat{\propto}_3$

$$Y_i = \propto_0 + ln \hat{\propto}_1 + \hat{\propto}_3 ln\left(\frac{X_i}{\hat{\propto}_2 + X_i}\right) \quad (28)$$

The NLS estimation using a modified version of Levenberg-Marguadt, we take the derivative with respect to the parameter $(\propto_0, \propto_1, \propto_2, \propto_3)$ then substitute $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ as the initial guess vector for iteration process. By partial derivative with respect to $(\hat{\propto}_0, \hat{\propto}_1, \hat{\propto}_2, \hat{\propto}_3)$, we have

$$\frac{\partial Y_i}{\partial \hat{\alpha}_0} = 1 \tag{29}$$

$$\frac{\partial Y_i}{\partial \hat{\alpha}_1} = \frac{1}{\hat{\alpha}_1} \tag{30}$$

$$\frac{\partial Y_i}{\partial \hat{\alpha}_2} = \frac{-X^2 \hat{\alpha}_3}{(\hat{\alpha}_2 + X_i)^3} \text{ or } \frac{-\hat{\alpha}_3 X_i^2}{(\hat{\alpha}_2 + X_i)^3} \tag{31}$$

$$\frac{\partial Y_i}{\partial \hat{\alpha}_3} = ln\left(\frac{X_i}{\hat{\alpha}_2 + X_i}\right) \tag{32}$$

Similarly, let $\propto = (\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ be the initial parameters, thus, we take the logarithm transformation of equation (3.5).

$$y_i = \propto_0 (1 - \propto_1 e^{-\propto_2 X_i})^{\propto_3}$$

$$lny_i = ln \propto_0 + \propto_3 ln(1 - \propto_1 e^{-\propto_2 X_i}) \tag{33}$$

The NLS estimation using a modified version of the Levenber-Marguardt, we take the derivative with respect to the parameters $(\propto_0, \propto_1, \propto_2, \propto_3)$, then substitute $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ as the initial guess vector for iteration process.

Let $lny_i = Y_i, ln \propto_0 = \hat{\alpha}_1, \propto_2 = \hat{\alpha}_2, \propto_3 = \propto_3$

$$Y_i = \propto_0 + \hat{\alpha}_3 ln\left(1 - \hat{\alpha} e^{-\hat{\alpha}_2 X_i}\right) \tag{34}$$

By taking partial derivative of equation (34) with respect to $(\hat{\alpha}_0, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ we have

$$\frac{\partial y_i}{\partial \hat{\alpha}_0} = 1 \tag{35}$$

$$\frac{\partial y_i}{\partial \hat{\alpha}_1} = \frac{-\propto_3 e^{-\hat{\alpha}_2 X_i}}{(1 - \hat{\alpha}_1 e^{-\hat{\alpha}_2 X_i})} \tag{36}$$

$$\frac{\partial y_i}{\partial \propto_2} = \frac{-\hat{\alpha}_1 \hat{\alpha}_3 e^{-\hat{\alpha}_2 X_i}}{(1 - \hat{\alpha}_1 e^{-\hat{\alpha}_2 X_i})} \tag{37}$$

$$\frac{\partial y_i}{\partial \hat{\alpha}_3} = ln\left(1 - e^{-\hat{\alpha}_2 X_i^{\propto_3}}\right) \tag{38}$$

The initial guess value $\left(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)}\right)$, we estimated by fitting a second-order polynomial (quartic model) using minitab 17 software.

$$Y_i = \propto_0^{(0)} + \propto_1^{(0)} X_i + \propto_2^{(0)} X^2 + \propto_3^{(0)} X^3 \tag{39}$$

The Gretl Statistical Software was used for the analysis, by adding the data set with additive and multiplicative error terms [e_i ~ N(0,1)] and the initial values for the parameter basis of Equation (39)

The error terms [Ei ~N(0,1)] were standardized using the equation

$$\hat{E}_i = \frac{E_i - \bar{E}}{\delta^2_{E_I}} \quad (40)$$

Thus, since the growth models considered are positive value and appreciating for increasing growth process. Therefore, we used the errors which in turn make the entire error value positive. In determining the suitable model or most appropriate model for growth studies, model selection criteria were employed.

2.3 Model Selection Criteria

2.3.1 R-Square (R^2)

R^2 is a measure of the proportion of variability in the data set that is accounted for by a regression model. It assumes that every independent variable in the model helps to explain variation in the dependent variable (y) and thus gives the percentage of explained variable if all independent variable in the model affect the dependent variable (y). The R^2 statistic is defined as

$$R^2 = 1 - \frac{SSE}{SST} \quad (41)$$

where $SS_T = \sum(y_i - \bar{y})^2$ is the total sum of square.

$SS_E = \sum(\hat{y}_i - \bar{y})^2$ is the residual sum of squares.

y_i and $\hat{y}_i$ are the original modelled data values.

2.3.2 Adjusted R-Square (R^2adj)

In least squares regression, increasing the number of regressors in the model leads to increase in R. Hence R^2 alone cannot be employed as a meaningful comparison of model. The adjusted R-square (R^2adj), tells us the percentage of variable explained by only those independent variables that truly affect the dependent variable(s) that do not belong to the model. The adjusted R-square (R^2adj) is defined as

$$R^2_{adj} = 1 - \frac{SSE\,/\,n - p}{SST\,/\,(n-1)}$$

$$= 1 - \frac{n-1}{n-p}(1 - R^2) \quad (41)$$

where n is the sample size, P is the model parameter, SSE and SST are as earlier defined.

2.3.3 Schwarz and Akaike Information Criterions (AIC)

Schwarz criterion or Bayesion information criteria (BIC) also SBC, SBIC) is a criterion for model selection among a finite set of models, the model related to the lowest B/C is preferred. The BIC is expressed as

$$\text{SBC or BIC} = \text{nln}\left(\frac{RSS}{n}\right) + Kln(n) \quad (42)$$

$$\text{AIC} = n * \ln(RSS/n) + 2*K \quad (43)$$

where n = the number of data point in x (or the number of observation)

K = the number of free parameter to be estimated

where RSS is the residual sum of squares of the model

ln is the natural logarithm (likelihood is the value of the likelihood

3. Results

To assess the outcomes of the Weilbull Logistic, Gompertz, Hill, and Richard growth model, data from the fields of agriculture, engineering, and population were gathered for this study (in terms of model building and parameter estimates). Five error terms [$\varepsilon_i \sim N(0,1)$] were stimulated and standardized using Equation (3.40) $\hat{E}_i = \frac{E-\bar{E}}{\partial^2_{E_i}}$ and Minitab 17 software.

3.1 Illustration 1

The program developed in the Grett Software using eqation (16) to (43) and inputting their initial values fitted by third-order polynomial (cubic function). Using minitab 17 software (see Appendix). where [$\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)}$] 1.56, -1.10, - 0.58, - 0.03).

1. **Weibull Model**

genr alpha zero = 1.56

genr alpha one = -1.10

genr alpha two = - 0.58

genr alpha three = - 0.03

nls Y_i = alpha zero + alpha one x $\ln X_i$ – alpha two x $\hat{X}_i$alpha three

derive alpha zero = 1

derive alpha one = log(x)

derive alpha two = - X^ ($-\hat{\propto}_3$)

derive alpha three = - $\hat{\propto}_2 \times log(X_i) \times X^{\wedge} - \propto_3$

end nls – VcV

2. **Logistic Model**

genr alpha zero = 1.56

genr alpha one = -1.10

genr alpha two = - 0.58

genr alpha three = - 0.03

nls $Yi = \log(\hat{\propto}_1 - \propto_0) - log\left(1 + \left(\frac{X_1}{\propto_2}\right)^{\hat{\propto}_3}\right)$

nls Y_i = log (alpha one – alpha zero) – log $[1 + (X \div alpha\ two)^\wedge alpha\ three]$ – 1 ÷ (alpha one – alpha two)

derive alpha zero = $\frac{-1}{alpha\ one - alpha\ zero} - 1 \div (alpha\ one - alpha\ two)$

derive alpha one = 1 ÷(alpha one –alpha two)

derive alpha two = - X_i^ alpha three ÷ (alpha two^(alpha three + 1) (1 + (X_i÷ alpha two)

derive alpha three = -1 ÷ (x_i÷alpha two)^alpha three) ((X_i÷alpha two)^ alpha three x log(X_i÷ alpha two).

3. **Compertz model**

genr alpha zero = 1.56

genr alpha one = -1.10

genr alpha two = - 0.58

genr alpha three = - 0.03

n/s Y_i = alpha zero – alpha one x exp [-alpha two x X^ alpha three]

derive alpha zero = 1

derive alpha one = (1 ÷alpha one)

derive alpha two = (alpha three x X_i^{*2}) ÷ (alpha two + X_i)*3

derive alpha three = log (xi ÷ (alpha two + x_i)

4. **HILL Models**

genr alpha zero = 1.56

genr alpha one = -1.10

genr alpha two = - 0.58

genr alpha three = - 0.03

nls Y_i = alpha zero + log(alpha one) + alpha three log(xi÷alpha two + x_i)

derive alpha zero = 1

derive alpha one = (1 ÷alpha one)

derive alpha two = (alpha three x X_i^{*2}) ÷ (alpha two + X_i)*3

derive alpha three = log (xi ÷ (alpha two + x_i)

5. **Richard Model**

genr alpha zero = 1.56

genr alpha one = -1.10

genr alpha two = - 0.58

genr alpha three = - 0.03

nls Y_i = alpha zero + alpha three (log(1 - alpha one exp[-alpha two x X]))

derive alpha zero = 1

derive alpha one = - alpha three exp [-alpha two x X_i] ÷ [1 – alpha one exp [- alpha two x X_i]

derive alpha two = [(- alpha one x alpha three exp [- alpha two x X_i) ÷ (1 – alpha one x exp[- alpha two x X_i]]

derive alpha three = log (1 – exp [-alpha two x $X^{\wedge}_i$ alpha three]]

3.1.1 Result Output for Data 1

The Result Outputs for Data 1 (in appendixes) are summarized as follows in table 1 - 3

Table 1: Summary of the Coefficient and P-values of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_1$) for Data 1.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha	$\propto_0$ = -2.0511 ± 0.8032(0.2061)	$\propto_0$ = -0.5937± 0.6236(0.3408)	
	Beta	$\propto_1$ = -8.6250 ± 67.5328 (0.9125)	$\propto_1$ = -2.6193± 0.0417(0.0000)	
	Gamma	$\propto_2$ = 2.5601± 0.0418(0.0000)	$\propto_2$ = 0.11788± 0.0235(0.0000)	
	Omega	$\propto_3$ = 0.3908± 0.0805(0.0000)	$\propto_0$ = 0.7701± 0.04096(0.0000)	
	Iteration	26	25	
Model	BIC	15.8341	**10.6532**	MET
Selection	AIC	10.5681	**9.7283**	
Criteria	R^2	0.9245	**0.9352**	
(MSC)	R.Adj	0.9182	**0.9302**	
	SSE	1.1256	**1.0661**	
Logistic	Alpha	$\propto_0$ = 2.9911± 0.0269(0.0000)	$\propto_0$ = 1.2053± 0.5955(0.0658)	
	Beta	$\propto_1$ = -62.0432± 26.541(0.0520)	$\propto_1$ = 1.7626± 1.8519(0.3702)	
	Gamma	$\propto_2$ = 0.3908± 0.0805(0.0000)	$\propto_2$ = 0.2278± 0.0235(0.000)	
	Omega	$\propto_3$ = 7.4338 ± 0.5846 (0.0000)	$\propto_3$ = 5.2660 ±0.2273 (0.0000)	AET
	Iteration	45	46	
Model	BIC	**30.8092**	33.3299	
Selection	AIC	**33.9334**	31.4500	
Criteria	R^2	**0.71044**	0.7004	
	R-Adj	**0.6328**	0.6521	
	SSE	**4.4911**	4.5534	
I	Alpha	$\propto_0$ = 2.5062± 0.0418(0.0000)	$\propto_0$ = 3.6961± 0.4682(0.0000)	
	Beta	$\propto_1$ = 3.6283± 0.3125(0.0000)	$\propto_1$ = -2.5974± 2.6944(0.3541)	
	Gamma	$\propto_2$ = 0.1843± 0.0343(0.0002)	$\propto_2$ = -0.00253± 0.0191(0.2105)	
	Omega	$\propto_3$ =	$\propto_3$ =	
	Iteration	21	21	
	BIC	50.3821	**35.6631**	
	AIC	38.1620	**34.7420**	MET
	R^2	0.5273	**0.5892**	
	R-Adj	0.4321	**0.5892**	
	SSE	8.4957	**5.6102**	
Hill	Alpha	$\propto_0$ = 2.7214± 0.1423(0.0000)	$\propto_0$ = 5.3792± 0.5019(0.0000)	
	Beta	$\propto_1$ = 7.0254± 0.3291(0.0000)	$\propto_1$ = 5.3792± 0.5012(0.0000)	
	Gamma	$\propto_2$ = -0.5362± 0.2436(0.0566)	$\propto_2$ = 0.2915± 0.1245(0.0671)	
	Omega	$\propto_3$ = 2.4304 ± 0.0418 (0.0000)	$\propto_3$ = 2.516 ±0.0417 (0.0000)	
	Iteration	40	47	
Model	BIC	36.832	**26.8301**	MET
Selection	AIC	35.6109	**24.000**	
Criteria	R^2	0.6268	**0.8064**	
	R-Adj	0.5647	**0.7063**	
	SSE	5.9154	**3.0708**	
Richard	Alpha	$\propto_0$ = -10.6251± 78.361(0.9156)	$\propto_0$ = 3.4582± 0.46175(0.0000)	
	Beta	$\propto_1$ = -1.0873± 1.0164(0.3058)	$\propto_1$ = 1.2053± 0.5955(0.0658)	
	Gamma	$\propto_2$ = 1.2053± 0.5955(0.0652)	$\propto_2$ = 0.2278± 0.0235(0.0000)	
	Omega	$\propto_3$ = 0.4529± 0.1763(0.0247)	$\propto_3$ = 0.3548± 0.1052(0.0067)	
	Iteration	40	48	
Model	BIC	32.9334	**9.5021**	MET
Selection	AIC	30.7632	**7.2844**	
Criteria	R^2	0.71045	**0.9492**	
(MSC)	R-adj	0.6724	**0.9293**	
	SSE	4.5911	**0.9620**	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error Term

MET – Multiplicative Error Term

Table 2: Summary of the Coefficients and P-values of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_2$) for Data 1.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha	$\propto_0$ = -2.4301± 0.0154(0.3058)	$\propto_0$ = -0.3802± 0.6521(0.4782)	
	Beta	$\propto_1$ = 9.0734 ± 1.1124 (0.0000)	$\propto_1$ = 5.9910± 0.1516(0.0000)	
	Gamma	$\propto_2$ = 10.8932± 0.667(0.0000)	$\propto_2$ = 6.3492± 0.2088(0.0000)	
	Omega	$\propto_3$ = 0.01324± 0.0030(0.0000)	$\propto_0$ = 0.0756± 0.0123(0.0002)	
	Iteration	22	31	
Model	BIC	12.3870	**9.1230**	MET
Selection	AIC	9.2855	**7.0581**	
Criteria	R^2	0.9311	**0.9405**	
(MSC)	R.Adj	0.9195	**0.9307**	
	SSE	1.0944	**0.9417**	
Logistic	Alpha	$\propto_0$ = 2.9341± 0.0246(0.0000)	$\propto_0$ = 2.4367± 0.0923(0.0000)	
	Beta	$\propto_1$ = 13.0345± 0.3484(0.000)	$\propto_1$ = 12.2436± 1.7806(0.0000)	
	Gamma	$\propto_2$ = 0.0348± 0.0018(0.0000)	$\propto_2$ = 0.0745± 0.0106(0.0000)	
	Omega	$\propto_3$ = -0.9226± 0.01120(0.0000)	$\propto_3$ = 0.5562± 0.024(0.000)	
	Iteration	45	50	
Model	BIC	38.2430	**32.8370**	MET
Selection	AIC	35.9081	**30.6624**	
Criteria	R^2	0.6194	**0.6935**	
(MSC)	R-adj	0.5560	**0.6420**	
	SSE	6.032	**4.8125**	
Gompertz	Alpha	$\propto_0$ = 3.6723± 0.2715(0.0000)	$\propto_0$ = 2.5281± 0.0235(0.0000)	
	Beta	$\propto_1$ = 6.9030± 0.5230(0.0000)	$\propto_1$ = 2.4321± 0.0924(0.0000)	
	Gamma	$\propto_2$ = 0.041± 0.0003(0.0002)	$\propto_2$ = 0.0036± 0.0002(0.0003)	
	Omega	$\propto_3$ = 0.9142 ± 0.01400(0.0000)	$\propto_3$ = 0.5234 ± 0.0345(0.0000)	
	Iteration	39	38	
Model	BIC	**32.0834**	34.6781	MET
Selection	AIC	**34.1256**	31.6634	
Criteria	R^2	**0.6182**	0.6943	
(MSC)	R-Adj	**0.5343**	0.5832	
	SSE	**5.4021**	2.0342	
Hill	Alpha	$\propto_0$ = 5.6280± 0.0598(0.0000)	$\propto_0$ = -1.5083± 2.0453(0.4562)	
	Beta	$\propto_1$ = 0.8915± 0.01527(0.0000)	$\propto_1$ = 0.1407± 0.0456(0.0000)	
	Gamma	$\propto_2$ = 3.0060± 0.0621(0.0000)	$\propto_2$ = 0.01246± 0.0022(0.0000)	
	Omega	$\propto_3$ = 3.8760 ± 0.1708 (0.0000)	$\propto_3$ = 2.3826 ±0.1568 (0.0000)	
	Iteration	54	58	
Model	BIC	38.6241	**28.7351**	MET
Selection	AIC	32.7762	**25.6106**	
Criteria	R^2	0.6368	**0.8072**	
	R-Adj	0.5546	**0.76041**	
	SSE	4.9154	**2.5647**	
Richard	Alpha	$\propto_0$ = -1.0512± 0.9923(0.3102)	$\propto_0$ = -0.6932± 0.6924(0.3508)	MET
	Beta	$\propto_1$ = -0.6438± 0.2268(0.0267)	$\propto_1$ = 0.2517± 0.1108(0.0573)	
	Gamma	$\propto_2$ = 10.8966± 0.6673(0.0000)	$\propto_2$ = 6.6734± 0.2088(0.0000)	
	Omega	$\propto_3$ = 0.01376 ± 0.0030 (0.0000)	$\propto_3$ = 0.0750 ± 0.0107 (0.0003)	
	Iteration	40	43	
Model	BIC	**30.3467**	32.7140	
Selection	AIC	**32.9240**	30.9332	
Criteria	R^2	**0.7310**	0.7001	
	R-Adj	**0.6028**	0.6003	
	SSE	**3.6783**	3.6634	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error Term

MET – Multiplicative Error Term

Table 3: Summary of the Coefficient and P-value of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_3$) for Data 1.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha	$\propto_0$ = -1.1098±1.3715(0.3872)	$\propto_0$ = -0.6942± 0.6882(0.3456)	
	Beta	$\propto_1$ = 1.5627 ± 1.0534 (0.2495)	$\propto_1$ = 1.1345± 0.6437(0.0975)	
	Gamma	$\propto_2$ = 0.4682± 0.2506(0.0478)	$\propto_2$ = 0.3178± 0.1028(0.0093)	
	Omega	$\propto_3$ = 0.1324± 0.0345(0.0002)	$\propto_0$ = -0.0324± 0.0152(0.0000)	
	Iteration	22	22	
Model	BIC	**8.5364**	18.2879	AET
Selection	AIC	**7.3675**	17.9456	
Criteria	R^2	**0.9817**	0.9576	
(MSC)	R.Adj	**0.9792**	0.9455	
	SSE	**0.6303**	1.6385	
Logistic	Alpha	$\propto_0$ = 4.8734± 0.0418(0.0000)	$\propto_0$ = 36.2783± 1.8519(0.3711)	
	Beta	$\propto_1$ = 3.8706± 0.1097(0.000)	$\propto_1$ = 2.3852± 0.1586(0.0000)	
	Gamma	$\propto_2$ = 0.0119± 0.0004(0.0000)	$\propto_2$ = 0.0035± 0.0004(0.0000)	
	Omega	$\propto_3$ = 0.8934± 0.0157(0.0000)	$\propto_3$ = 0.4507± 0.03924(0.000)	
	Iteration	15	18	
Model	BIC	13.6789	**12.1628**	
Selection	AIC	10.5837	**0.0234**	MET
Criteria	R^2	0.9248	**0. 9275**	
(MSC)	R-adj	0.9102	**0.9183**	
	SSE	1.1823	**1.1578**	
Gompertz	Alpha	$\propto_0$ = 2.7341± 1.1324(0.0000)	$\propto_0$ = 2.523± 0.0234(0.0000)	
	Beta	$\propto_1$ = 1.5328± 0.00703(0.0061)	$\propto_1$ = 0.3258± 0.0675(0.0000)	
	Gamma	$\propto_2$ = 0.23452± 0.00073(0.0006)	$\propto_2$ = 0.3248± 0.0563(0.0000)	
	Omega	$\propto_3$ = 0.0143 ± 0.0334(0.0002)	$\propto_3$ = -0.0247 ± 0.020(0.2150)	
	Iteration	39	39	
Model	BIC	**32.2593**	34.9178	AET
Selection	AIC	**30.7281**	29.7102	
Criteria	R^2	**0.6908**	0.7283	
(MSC)	R-Adj	**0.6315**	0.6459	
	SSE	**4.9015**	4.5482	
Hill	Alpha	$\propto_0$ = 2.4381± 0.0269(0.0000)	$\propto_0$ = -1.6088± 2.0438(0.4746)	
	Beta	$\propto_1$ = 3.2176± 0.0931(0.0000)	$\propto_1$ = 2.3122± 1.9604(0.2063)	
	Gamma	$\propto_2$ = 0.4340± 0.0178(0.0000)	$\propto_2$ = -0.0249± 0.01223(0.0854)	
	Omega	$\propto_3$ = 1.6328 ± 0.0901 (0.0000)	$\propto_3$ = 0.760± 0.04092 (0.0000)	
	Iteration	22	30	
Model	BIC	**26.8123**	36.7281	AET
Selection	AIC	**24.6109**	35.8034	
Criteria	R^2	**0.8064**	0.6269	
	R-Adj	**0.7761**	0.5648	
	SSE	**3.0708**	5.9158	
Richard	Alpha	$\propto_0$ = 2.4907± 0.04418(0.0000)	$\propto_0$ = 2.6234± 0.0417(0.0000)	
	Beta	$\propto_1$ = 1.8763± 0.0805(0.0000)	$\propto_1$ = 0.6670 ± 0.04097(0.0000)	
	Gamma	$\propto_2$ = 0.3908± 0.03456(0.0000)	$\propto_2$ = 0.1176± 0.0235(0.0000)	
	Omega	$\propto_3$ = 0.0371 ± 0.0237(0.0000)	$\propto_3$ = 0.0360± 0.0220 (0.1455)	
	Iteration	38	44	
Model	BIC	37.2782	**4.9724**	MET
Selection	AIC	36.3705	**4.0647**	
Criteria	R^2	0.6844	**0.9875**	
	R-Adj	0.5942	**0.9839**	
	SSE	12.2039	**0.4825**	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error Term

MET – Multiplicative Error Term

Table 1 indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Richard Growth Model with the values 9.5021,7.2844,0.9492,0.9293,and 0.9620 respectively which identified Richard Growth Model with Multiplicative Error Term as suitable Growth Model among the Five Growth Models..

Table 2 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Weibull Groth Model with the values 9.1230,7.0581,0.9405,0.9307,and 0,9417 respectively which *identified Weibull Growth Model with Multiplicative Error Term as suitable Growth Model* among the Five Growth Models.

Table 3 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Weibull Groth Model with the values 8.5364, 7.3675, 0.9576, 0.9455, and 1.6384 respectively which identified Weibull Growth Model with Additive Error Term as suitable Growth Model among the Five Growth Models.

3.2 Illustration 2

Similarly, the program developed in the Gretl Software using equation (18) to (43) and inputting the initial values fitted by second-order polynomial (quartic model). Using Minitab 17 software, see appendices C, D and E where $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ = (635.87, -51.94, 1.23, 0.008).

1. **Weibull Model**

genr alpha zero = 635.87

genr alpha one = -51.94

genr alpha two = 1.23

genr alpha three = 0.008

nls Y_i = alpha zero + alpha one x lnX_i – alpha two x $X^{\wedge}_i$ alpha three

derive alpha zero = 1

derive alpha one = log(x)

derive alpha two = - X^ (-$\hat{\propto}_3$)

derive alpha three = - $\hat{\propto}_2 \times log(X_i) \times X^{\wedge} - \propto_3$

end nls – VcV

2. **Logistics Model**

genr alpha zero = 635.87

genr alpha one = -51.94

genr alpha two = 1.23

genr alpha three = 0.008

nls Yi = $\log(\hat{\propto}_1 - \propto_0) - log\left(1 + \left(\frac{X_1}{\propto_2}\right)^{\hat{\propto}_3}\right)$

nls = Y_i = log (alpha one – alpha zero) – log $[1 + (X \div alpha\ two)\text{^}alpha\ three]$

derive alpha zero = $\frac{-1}{alpha\ one - alpha\ zero} - 1 \div (alpha\ one - alpha\ two)$

derive alpha one = 1 ÷(alpha one –alpha two)

derive alpha two = - X_i^ alpha three ÷ (alpha two^(alpha three + 1) (1 + (X_i÷ alpha two) derive alpha three = -1 ÷ (x_i÷alpha two)^alpha three) ((X_i÷alpha two)^ alpha three x log(X_i÷ alpha two).

3. **Compertz model**

genr alpha zero = 635.87

genr alpha one = -51.94

genr alpha two = 1.23

genr alpha three = 0.008

nls Y_i = alpha zero – alpha one * exp [-alpha two * X^ alpha three]

derive alpha zero = 1

derive alpha one = log(x)

derive alpha two = - X^ (-$\hat{\propto}_3$)

derive alpha three = - $\hat{\propto}_2 \times log(X_i) \times X^{\wedge} - \propto_3$

end nls – VcV

4. **HILL Models**

genr alpha zero = 635.87

genr alpha one = -51.94

genr alpha two = 1.23

genr alpha three = 0.008

nls Y_i = alpha zero + log(alpha one) + alpha three log(xi÷alpha two + x_i)

derive alpha zero = 1

derive alpha one = (1 ÷alpha one)

derive alpha two = (alpha three x X_i^{*2}) ÷ (alpha two + X_i)*3

derive alpha three = log (xi ÷ (alpha two + x_i)

5. **Richard Model**

genr alpha zero = 635.87

genr alpha one = -51.94

genr alpha two = 1.23

genr alpha three = 0.008

nls y_i = alpha zero + alpha three (log(1 - alpha one exp[-alpha two x X]))

derive alpha zero = 1

derive alpha one = - alpha three exp [-alpha two x X_i] ÷ [1 – alpha one exp [- alpha two x X_i]

derive alpha two = [(- alpha one x alpha three exp [- alpha two x X_i) ÷ (1 – alpha one x exp[- alpha two x X_i]]

derive alpha three = log (1 – exp [-alpha two x $X^{\wedge}_i$ alpha three]]

3.2.1 Result Output for Data 2

The Result Output for Data 2 (in appendices) are summarized as follow in table 4 - 6.

Table .4: Summary of the Coefficient and P-value of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_1$) for Data 2.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha Beta	$\propto_0$ = 12.9912±1.0164(0.3058) $\propto_1$ = 0..9805 ± 0.5230 (0.0000)	$\propto_0$ = 10.4800± 2.7513(0.3325) $\propto_1$ = 2.4358± 0.0924(0.0000)	

	Gamma	$\propto_2$ = 0.4810± 0.0087(0.0000)	$\propto_2$ = 0.2774± 0.0201(0.0000)	
	Omega	$\propto_3$ = 0.0018± 0.0014(0.0061)	$\propto_0$ = 0.0042± 0.0023(0.0701)	
	Iteration	41	22	
Model	BIC	**8.5364**	18.2879	
Selection	AIC	**7.3675**	17.9456	AET
Criteria	R^2	**0.9817**	0.9576	
(MSC)	R.Adj	**0.9792**	0.9455	
	SSE	**0.6303**	1.6385	
Logistic	Alpha	$\propto_0$ = 4.8734± 0.0418(0.0000)	$\propto_0$ = 36.2783± 1.8519(0.3711)	
	Beta	$\propto_1$ = 3.8706± 0.1097(0.000)	$\propto_1$ = 2.3852± 0.1586(0.0000)	
	Gamma	$\propto_2$ = 0.0119± 0.0004(0.0000)	$\propto_2$ = 0.0035± 0.0004(0.0000)	
	Omega	$\propto_3$ = 0.8934± 0.0157(0.0000)	$\propto_3$ = 0.4507± 0.03924(0.000)	
	Iteration	15	18	
Model	BIC	**5.6789**	38.5676	AET
Selection	AIC	**4.4823**	35.4362	
Criteria	R^2	**0.9845**	0.6833	
(MSC)	R-adj	**0.9826**	0.5856	
	SSE	**0.4568**	12.2039	
Gompertz	Alpha	$\propto_0$ = 3.3056± 1.1375(0.0000)	$\propto_0$ = 6.6015± 0.1640(0.0000)	
	Beta	$\propto_1$ = 0.4863± 0.0072(0.2188)	$\propto_1$ = 0.2373± 0.0152(0.0000)	
	Gamma	$\propto_2$ = 0.00832± 0.0003(0.0000)	$\propto_2$ = 0.0043± 0.0002(0.0003)	
	Omega	$\propto_3$ = 0.0013 ± 0.0022(0.015)	$\propto_3$ = 0.2345 ± 0.0132(0.0000)	
	Iteration	39	17	
Model	BIC	**7.6456**	17.1974	
Selection	AIC	**6.72836**	16.2897	AET
Criteria	R^2	**0.9856**	0.9455	
(MSC)	R-Adj	**0.9780**	0.9456	
	SSE	**5.6340**	1.6482	
Hill	Alpha	$\propto_0$ = 2.9407± 0.0416(0.0000)	$\propto_0$ = 2.51932± 0.0435(0.0000)	
	Beta	$\propto_1$ = 15.0556± 1.1657(0.0000)	$\propto_1$ = 6.9456± 0.0086(0.0016)	
	Gamma	$\propto_2$ = 0.0371± 0.0054(0.0000)	$\propto_2$ = 0.00250± 0.022(0.1456)	
	Omega	$\propto_3$ = -0.5361 ± 0.2378 (0.0586)	$\propto_3$ = 1.7696 ± 1.8502 (0.3711)	
	Iteration	38	43	
Model	BIC	**0.9312**	36.5982	AET
Selection	AIC	**0.0345**	35.6027	
Criteria	R^2	**0.9983**	0.7081	
	R-Adj	**0.9892**	0.63451	
	SSE	**0.3214**	11.4036	
Richard	Alpha	$\propto_0$ = 7.4483± 0.5486(0.0000)	$\propto_0$ = 5.9312± 0.1515(0.0000)	
	Beta	$\propto_1$ = 0.0013± 0.0011(0.2188)	$\propto_1$ = 0.0004 ± 0.0028(0.8982)	
	Gamma	$\propto_2$ = 0.4686± 0.0062(0.0000)	$\propto_2$ = 0.2379± 0.0143(0.0000)	
	Omega	$\propto_3$ = 0.9226 ± 0.00111 (0.0000)	$\propto_3$ = 0.5372 ± 0.0246 (0.0000)	
	Iteration	47	40	
Model	BIC	34.5628	**26.5728**	MET
Selection	AIC	32.6061	**23.6042**	
Criteria	R^2	0.7606	**0.9028**	
	R-Adj	0.6732	**0.8732**	
	SSE	9.2587	**1.7628**	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error Term

MET – Multiplicative Error Term

Table 5: Summary of the Coefficient and P-value of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_2$) for Data 2.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha	$\propto_0$ = 7.4436±0.5487(0.0000)	$\propto_0$ = 5.3343± 0.1432(0.0000)	
	Beta	$\propto_1$ = 4.4567 ± 1.2345 (0.0015)	$\propto_1$ = 1.7834± 0.7621(0.0051)	
	Gamma	$\propto_2$ = 0.01567± 0.2341(0.0000)	$\propto_2$ = 0.00127± 0.0125(0.0025)	
	Omega	$\propto_3$ = 1.4568± 0.4562(0.0561)	$\propto_0$ = 0.4867± 0.2341(0.0000)	
	Iteration	47	45	
Model	BIC	**33.6848**	38.0502	AET
Selection	AIC			

Criteria (MSC)	R^2 R.Adj SSE	**32.7771** **0.7492** **0.6734** **0.6834**	37.7429 0.6734 0.5768 12.7103	
Logistic Model Selection Criteria (MSC)	Alpha Beta Gamma Omega Iteration BIC AIC R^2 R-adj SSE	$\propto_0$ = 8.0627± 1.1124(0.0000) $\propto_1$ = 10.3467± 0.6778(0.000) $\propto_2$ = 0.6343± 0.2268(0.0000) $\propto_3$ = 0.0275± 0.0074(0.0070) 38 **7.3426** **6.8273** **0.9853** **0.9812** **0.5672**	$\propto_0$ = 5.9324± 0.1516(0.0000) $\propto_1$ = 6.6892± 0.2066(0.0000) $\propto_2$ = 0.2517± 0.1108(0.0577) $\propto_3$ = 0.0205± 0.0178(0.000) 20 18.7632 17.7063 0.9624 0.9548 1.3962	AET
Gompertz Model Selection Criteria	Alpha Beta Gamma Omega Iteration BIC AIC R^2 R-Adj SSE	$\propto_0$ = 9.0734± 1.1012(0.0000) $\propto_1$ = 0.0147± 0.0030(0.000) $\propto_2$ = 10.7361± 0.6697(0.0000) $\propto_3$ = 8.3312 ± 1.6172 (0.0010) 58 **25.0621** **20.678** **0.7862** **0.7571** **1.6789**	$\propto_0$ = 5.8362± 0.1502(0.0000) $\propto_1$ = 0.0750 ± 0.0107(0.0000) $\propto_2$ = 6.653± 0.2088(0.0020) $\propto_3$ = 1.4321 ± 0.2182 (0.0001) 35 30.0125 36.6787 0.5568 06767 9.2435	AET
Hill Model Selection Criteria	Alpha Beta Gamma Omega Iteration BIC AIC R^2 R-Adj SSE	$\propto_0$ = 11.098± 2.5628(0.0038) $\propto_1$ = -0.63436± 0.2268(0.0263) $\propto_2$ = -70.9340± 35.0630(0.00701) $\propto_3$ = 0.001256 ± 0.1281 (0.0000) 45 **6.5436** **5.8676** **0.9853** **0.9802** **0.5673**	$\propto_0$ = -3.6862± 2.3456(0.2130) $\propto_1$ = 0.251± 2.1374(0.2370) $\propto_2$ = 0.2571± 2.1374(0.2372) $\propto_3$ = 0.0043 ± 0.627 (0.0000) 50 16.6102 15.5464 0.9630 0.9556 1.398	AET
Richard Model Selection Criteria (MSC)	Alpha Beta Gamma Omega Iteration BIC AIC R^2 R-Adj SSE	$\propto_0$ = 4.3438± 0.00156(0.0002) $\propto_1$ = 0.0004± 0.028(0.8982) $\propto_2$ = 0.0162± 0.00011(0.0000) $\propto_3$ = 0.0002 ± 0.0256(0.0060) 34 35.2677 36.3599 0.7634 0.6973 9.1303	$\propto_0$ = 7.8246± 1.1373(0.0000) $\propto_1$ = 0.4673± 0.0062(0.0000) $\propto_2$ = 4.2431± 0.0004(0.0025) $\propto_3$ = 0.0025 ± 0.1245(0.0000) 28 **4.3766** **3.7639** **0.9896** **0.9873** **0.4036**	AET

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error Term

MET – Multiplicative Error Term

Table 6: Summary of the Coefficient and P-value of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_3$) for Data 2.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull Model Selection Criteria (MSC)	Alpha Beta Gamma Omega Iteration BIC AIC R^2 R.Adj SSE	$\propto_0$ = 10.3562±1.0345(0.0000) $\propto_1$ = 0.0275 ± 0.0073 (0.0072) $\propto_2$ = -0.6088± 0.0305(0.0450) $\propto_3$ = 0.0141± 0.0032(0.0032) 47 19.2456 17.1324 0.9449 0.9293 2.1293	$\propto_0$ = 7.75612± 0.1640(0.0000) $\propto_1$ = 0.0205 ± 0.0178(0.0008) $\propto_2$ = 0.29948± 0.14618(0.0832) $\propto_3$ = 0.03361± 0.0047(0.0002) 40 **6.0406** **5.3267** **0.9847** **0.9804** **0.5963**	MET

Logistic	Alpha	$\propto_0$ = 7.4320± 2.7203(0.0047)	$\propto_0$ = -2.5321± 2.3033(0.3078)	
	Beta	$\propto_1$ = 10.9331± 0.6640(0.0000)	$\propto_1$ = 6.9416± 0.2800(0.0000)	
	Gamma	$\propto_2$ = 0.0141± 0.0032(0.0032)	$\propto_2$ = 0.0336± 0.04110.0003)	
	Omega	$\propto_3$ = 8.6732± 1.6631(0.0012)	$\propto_3$ = 1.6146± 0.2851(0.0007)	
	Iteration	47	40	
Model	BIC	33.6048	**24.8432**	
Selection	AIC	32.5124	**23.7604**	MET
Criteria	R^2	0.7606	**0. 9027**	
(MSC)	R-adj	0.6921	**0.8748**	
	SSE	9.2587	**3.7628**	
Gompertz	Alpha	$\propto_0$ = 8.7691± 1.6634(0.0012)	$\propto_0$ = 2.6146± 0.2486(0.0002)	
	Beta	$\propto_1$ = 3.8706± 0.1097(0.0000)	$\propto_1$ = 2.4358± 0.0923(0.0000)	
	Gamma	$\propto_2$ = 0.9226± 0.0111(0.0000)	$\propto_2$ = 0.5572± 0.0246(0.0000)	
	Omega	$\propto_3$ = 0.4686 ± 0.0062(0.0000)	$\propto_3$ = 0.2379 ± 0.0143(0.000)	
	Iteration	12	40	
Model	BIC	**7.0405**	20.8134	
Selection	AIC	**6.1923**	17.1327	AET
Criteria	R^2	**0.9853**	0.9462	
(MSC)	R-Adj	**0.9801**	0.9193	
	SSE	**0.5683**	2.1456	
Hill	Alpha	$\propto_0$ = 13.0375± 0.3486(0.0000)	$\propto_0$ = 12.3923± 1.7806(0.0000)	
	Beta	$\propto_1$ = 0.0013± 0.0011(0.2188)	$\propto_1$ = 0.0004 ± 1.0028(0.8987)	
	Gamma	$\propto_2$ = 0.4821± 0.0061(0.0000)	$\propto_2$ = -0.0249± 0.0234(0.0000)	
	Omega	$\propto_3$ = 0.0083 ± 0.0003 (0.0000)	$\propto_3$ = 0.760± 0.0002 (0.0000)	
	Iteration	39	17	
Model	BIC	**-19.8240**	2.6403	
Selection	AIC	**-15.0024**	3.5720	AET
Criteria	R^2	**0.7450**	0.8268	
	R-Adj	**0.7358**	0.8206	
	SSE	**2.8794**	1.9558	
Richard	Alpha	$\propto_0$ = 8.3564± 0.5708(0.0000)	$\propto_0$ = 5.5051± 0.5054(0.0000)	
	Beta	$\propto_1$ = 0.0475± 0.0063(0.0072)	$\propto_1$ = 0.0205 ± 0.0178(0.2860)	
	Gamma	$\propto_2$ = 0.0141± 0.00326(0.0012)	$\propto_2$ = 0.0339± 0.042(0.0002)	
	Omega	$\propto_3$ = 8.3428 ± 1.5628(0.0002)	$\propto_3$ = 1.5628± 0.2857 (0.0000)	
	Iteration	40	**12**	
Model	BIC	30.8671	**4.4109**	
Selection	AIC	31.5082	**3.7856**	
Criteria	R^2	0.7865	**0.9887**	MET
	R-Adj	0.7258	**0.9845**	
	SSE	8.3456	**0.4320**	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error Term
MET – Multiplicative Error Term

Table 4 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Hill Growth Model with the values 0.9312, 0.0345, 0.9983, 0.9892, and 0.3214 respectively which *identified Hill Growth Model with Additive Error* Term *as suitable Growth Model* among the Five Growth Models.

Table 5 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is is the Richard Growth Model with the values 4.3766, 3.7639, 0.9896, 0.9873, and 0.4036 respectively which identified Richard Growth Model with Multiplicative Error Term as suitable Growth Model among the Five Growth Models.

Table 6 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Richard Growth Model with the values 4.4109, 3.7856, 0.9887, 0.9845, and 0.4320 respectively which identified Richard Growth Model with Multiplicative Error Term as suitable Growth Model among the Five Growth Models.

3.3 Illustration 3

Likewise for Data 3, the program developed in the Grett Software using equation 3.16 to 3.43 in section 3.5 and inputting values fitted by second-order polynomial (quartic model) using minitab 17 software. See Appendices C, D and E where $(\propto_0^{(0)}, \propto_1^{(0)}, \propto_2^{(0)}, \propto_3^{(0)})$ = (38401, 1666, 5.72, 0.38)

1. **Neibull Model**

   genr alpha zero = 38401

   genr alpha one = 1666

   genr alpha two = - 5.72

   genr alpha three = 0.38

   nls y_i = alpha zero + alpha one x $\ln X_i$ – alpha two x $X^{\wedge}_i$alpha three

   derive alpha zero = 1

   derive alpha one = log(x)

   derive alpha two = - X^ ($-\hat{\propto}_3$)

   derive alpha three = - $\hat{\propto}_2 \times log(X_i) \times X^{\wedge} - \propto_3$

   end nls – VcV

2. **Logistics Model**

   genr alpha zero = 38401

   genr alpha one = 1666

   genr alpha two = - 5.72

   genr alpha three = 0.38

   n/s yi = $\log(\hat{\propto}_1 - \propto_0) - log\left(1 + \left(\frac{X_1}{\propto_2}\right)^{\hat{\propto}_3}\right)$

   nls = y_i = log (alpha one – alpha zero) – log $[1 + (X \div alpha\ two)^{\wedge}alpha\ three]$

   derive alpha zero = $\frac{-1}{alpha\ one - alpha\ zero} - 1 \div (alpha\ one - alpha\ two)$

   derive alpha one = 1 ÷(alpha one –alpha two)

   derive alpha two = - X_i^ alpha three ÷ (alpha two^(alpha three + 1) (1 + (X_i÷ alpha two)

   derive alpha three = -1 ÷ (x_i÷alpha two)^alpha three) ((X_i÷alpha two)^ alpha three x log(X_i÷ alpha two).

3. **Compertz model**

   genr alpha zero = 38401

   genr alpha one = 1666

   genr alpha two = - 5.72

   genr alpha three = 0.38

n/s Y_i = alpha zero – alpha one x exp [-alpha two x X^ alpha three] derive alpha zero = 1

derive alpha one = (1 ÷alpha one)

derive alpha two = (alpha three x X_i^{*2}) ÷ (alpha two + X_i)*3

derive alpha three = log (xi ÷ (alpha two + x_i)

4. **HILL Models**

   genr alpha zero = 38401

   genr alpha one = 1666

   genr alpha two = - 5.72

   genr alpha three = 0.38

n/s y_i = alpha zero + log(alpha one) + alpha three log(xi÷alpha two + x_i)

derive alpha zero = 1

derive alpha one = (1 ÷alpha one)

derive alpha two = (alpha three x X_i^{*2}) ÷ (alpha two + X_i)*3

derive alpha three = log (xi ÷ (alpha two + x_i)

5. **Richard Model**

   genr alpha zero = 38401

   genr alpha one = 1666

   genr alpha two = - 5.72

   genr alpha three = 0.38

   n/s y_i = alpha zero + alpha three (log(1 - alpha one exp[-alpha two x X]))

   derive alpha zero = 1

   derive alpha one = - alpha three exp [-alpha two x X_i] ÷ [1 – alpha one exp [- alpha two x X_i]

   derive alpha two = [(- alpha one x alpha three exp [- alpha two x X_i) ÷ (1 – alpha one x exp[- alpha two x X_i]]

   derive alpha three = log (1 – exp [-alpha two x $X^{\wedge}_i$ alpha three]]

3.3.1 Result Output for Data 3

The Result Output for Data 3 (in Appendices) are summarized as follows in table 7 - 9.

Table 7: Summary of the Coefficient and P-value of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_1$) for Data 3.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha	$\propto_0$ = 15.2436±0.094(0.000)	$\propto_0$ = 15.2449± 0.0260(0.0000)	
	Beta	$\propto_1$ = 0.0257 ± 0.0120 (0.0000)	$\propto_1$ = 0.0755± 0.0105(0.0000)	
	Gamma	$\propto_2$ = 0.8243± 0.0223(0.0000)	$\propto_2$ = 0.5673± 0.0248(0.0000)	
	Omega	$\propto_3$ = 0.4684± 0.0063(0.0001)	$\propto_0$ = 0.7738± 0.0356(0.0000)	
	Iteration	18	21	
Model Selection Criteria (MSC)	BIC	**-456.4380**	-323.3064	AET
	AIC	**-481.6343**	-356.1231	
	R^2	**0.9896**	0.9658	
	R.Adj	**0.9896**	0.9824	
	SSE	**0.0057**	0.051	
Logistic	Alpha	$\propto_0$ = 13.5148± 0.0659(0.0000)	$\propto_0$ = 13.2042± 1.8519(0.0000)	
	Beta	$\propto_1$ = 3.0081± 0.621(0.0000)	$\propto_1$ = 2.458± 0.1586(0.0000)	
	Gamma	$\propto_2$ = 0.0119± 0.004(0.0000)	$\propto_2$ = 0.0035± 0.0004(0.0000)	
	Omega	$\propto_3$ = 0.0083± 0.0003(0.0000)	$\propto_3$ = 0.4507± 0.0038(0.000)	
	Iteration	44	48	
Model Selection Criteria	BIC	**-21.2024**	6.8498	AET
	AIC	**-27.3962**	0.6623	
	R^2	**0.8302**	0.7328	

(MSC)	R-adj	**0.8215**	0.7157	
	SSE	**1.8025**	3.0976	
Gompertz	Alpha	$\propto_0 = 16.4122 \pm 0.1134(0.0000)$	$\propto_0 = 12.3928 \pm 1.7806\ (0.0000)$	
	Beta	$\propto_1 = 4.8706 \pm 0.1082(0.0000)$	$\propto_1 = 0.0005 \pm 0.0023(0.8982)$	
	Gamma	$\propto_2 = 0.2345 \pm 0.0003(0.0000)$	$\propto_2 = 0.2358 \pm 0.0123(0.0003)$	
	Omega	$\propto_3 = 0.0071 \pm 0.0043(0.0000)$	$\propto_3 = 0.5234 \pm 1.3801(0.0000)$	
	Iteration	49	50	
Model	BIC	**-24.873**	26.8001	
Selection	AIC	**-35.0534**	21.7982	AET
Criteria	R^2	**0.8326**	0.6234	
(MSC)	R-Adj	**0.8326**	0.5982	
	SSE	**1.8238**	2.8793	
Hill	Alpha	$\propto_0 = 13.2560 \pm 0.3486(0.0000)$	$\propto_0 = 14.3981 \pm 1.7806(0.0000)$	
	Beta	$\propto_1 = 0.0025 \pm 0.0023(0.2187)$	$\propto_1 = 0.0028 \pm 0.0028\ (0.8982)$	
	Gamma	$\propto_2 = 0.4635 \pm 0.0058(0.0000)$	$\propto_2 = 3.8706 \pm 0.1097(0.0000)$	
	Omega	$\propto_3 = 0.4635 \pm 0.0143\ (0.0000)$	$\propto_3 = 0.0038 \pm 0.0002\ (0.0000)$	
	Iteration	47	46	
Model	BIC	**-25.8209**	3.6082	
Selection	AIC	**-38.267**	-4.5720	AET
Criteria	R^2	**0.8268**	0.7540	
	R-Adj	**0.8305**	0.7682	
	SSE	**1.9660**	2.6783	
Richard	Alpha	$\propto_0 = 18.2561 \pm 0.0233(0.0000)$	$\propto_0 = 15.3920 \pm 1.6821(0.0000)$	
	Beta	$\propto_1 = 3.7602 \pm 0.1052(0.2188)$	$\propto_1 = 0.556 \pm 0.0246(0.0000)$	
	Gamma	$\propto_2 = 0.5642 \pm 0.0004(0.0000)$	$\propto_2 = 0.0762 \pm 0.01021(0.0000)$	
	Omega	$\propto_3 = 0.00235 \pm 0.0142\ (0.0000)$	$\propto_3 = 0.4521 \pm 0.0005\ (0.0000)$	
	Iteration	41	44	
Model	BIC	**-21.8304**	4.6704	
Selection	AIC	**-36.1243**	-8.4563	AET
Criteria	R^2	**0.7624**	0.7328	
	R-Adj	**0.7824**	0.7543	
	SSE	**1.9543**	2.346	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error term

MET – Multiplicative Error Term

Table 8: Summary of the Coefficient and P-value of the Five Growth Model with Additive and Multiplicative Error Terms($\hat{\varepsilon}_2$) for Data 3.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha	$\propto_0 = 20.5361 \pm 0.0084(0.0000)$	$\propto_0 = 16.6650 \pm 0.008(0.0000)$	
	Beta	$\propto_1 = 0.0645 \pm 0.0017\ (0.0000)$	$\propto_1 = 2.4907 \pm 0.0414(0.0000)$	
	Gamma	$\propto_2 = 0.4567 \pm 0.0245(0.0000)$	$\propto_2 = 0.3908 \pm 0.0805(0.0000)$	
	Omega	$\propto_3 = 0.3456 \pm 0.09532(0.0061)$	$\propto_0 = 1.8756 \pm 0.2301(0.0000)$	
	Iteration	15	24	
Model	BIC	**-320.0771**	-177.3891	
Selection	AIC	**-302.2580**	-187.0782	AET
Criteria	R^2	**0.9993**	0.9905	
(MSC)	R.Adj	**0.9993**	0.9902	
	SSE	**0.0080**	0.1078	
Logistic	Alpha	$\propto_0 = 17.3383 \pm 0.5486(0.0000)$	$\propto_0 = 11.0089 \pm 2.5977(0.0038)$	
	Beta	$\propto_1 = 10.8900 \pm 0.6697(0.000)$	$\propto_1 = 0.2517 \pm 2.1374(0.00573)$	
	Gamma	$\propto_2 = 0.0137 \pm 0.0030(0.0000)$	$\propto_2 = 6.6892 \pm 2.1374(0.0000)$	
	Omega	$\propto_3 = 0.0750 \pm 0.0107(0.0000)$	$\propto_3 = 0.2182 \pm 2.0486(0.0003)$	
	Iteration	47	42	
Model	BIC		-150.7201	
Selection	AIC	**-452.3821**	-182.5671	AET
Criteria	R^2	**-448.6702**	0.9862	
(MSC)	R-adj	**0.9983**	0.9857	
	SSE	**0.9983**	0.1556	
		0.0068		

Gompertz	Alpha	$\propto_0$ = 13.8231± 0.3280(0.0000)	$\propto_0$ = 12.3924± 1.7806(0.0000)	
	Beta	$\propto_1$ = 0.0025± 0.0014(0.2188)	$\propto_1$ = 0.0004± 0.0028(0.8982)	
	Gamma	$\propto_2$ = 0.49322± 0.0969(0.0000)	$\propto_2$ = 0.4686± 0.0062(0.0000)	
	Omega	$\propto_3$ = 0.2734 ± 0.0023(0.015)	$\propto_3$ = 0.0037 ± 0.0002(0.0000)	
	Iteration	43	48	
Model	BIC	**-19.8214**	2.7082	
Selection	AIC	**-26.1243**	-4.5732	AET
Criteria	R^2	**0.8268**	0.7450	
(MSC)	R-Adj	**0.8205**	0.7814	
	SSE	**1.944**	2.8793	
Hill	Alpha	$\propto_0$ = 12.4188± 0.1128(0.0000)	$\propto_0$ = 13.4321± 0.1010(0.0000)	
	Beta	$\propto_1$ = 3.4270± 0.0260 (0.0000)	$\propto_1$ = 5.004± 0.0143(0.0200)	
	Gamma	$\propto_2$ = 0.8246± 0.0111(0.0000)	$\propto_2$ = 0.0134± 0.1097(0.0000)	
	Omega	$\propto_3$ = 0.0348 ± 0.0003 (0.0000)	$\propto_3$ = 0.4810 ± 0.0087 (0.0000)	
	Iteration	40	46	
Model	BIC	**-240.5672**	-160.6305	AET
Selection	AIC	**-200.0351**	-170.8018	
Criteria	R^2	**0.8996**	0.7867	
	R-Adj	**0.8952**	0.7552	
	SSE	**1.8234**	2.8703	
Richard	Alpha	$\propto_0$ = 16.5148± 0.0658(0.0000)	$\propto_0$ = 13.6578± 0.008(0.0000)	
	Beta	$\propto_1$ = 5.0069 ± 0.0631(0.0000)	$\propto_1$ = 0.0162 ± 0.0011(0.0000)	
	Gamma	$\propto_2$ = 6.9805± 0.5230(0.0000)	$\propto_2$ = 0.2773± 0.0202(0.0701)	AET
	Omega	$\propto_3$ = 0.4810 ± 0.0087 (0.0000)	$\propto_3$ = 0.0037 ± 0.0002 (0.0002)	
	Iteration	48	35	
Model	BIC	**-20.8213**	-5.6083	
Selection	AIC	**-30.245**	-3.5720	
Criteria	R^2	**0.8993**	0.7534	
	R-Adj	**0.8903**	0.7358	
	SSE	**1.5628**	2.356	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error term

MET – Multiplicative Error Term

Table 9: Summary of the Coefficient and P-value of the Five Growth Model with Additive and Multiplicative Error Terms ($\hat{\varepsilon}_3$) for Data 3.

		Estimated coefficient (P-values)		Remark
Model	Model Statistics	Additive Error Terms	Multiplicative Error terms	
Weibull	Alpha	$\propto_0$ = 12.3671±0.3294 (0.3058)	$\propto_0$ = 11.2025± 0.0969(0.3325)	
	Beta	$\propto_1$ = 0.0017 ± 0.0013 (0.0000)	$\propto_1$ = 0.0054± 0.00023(0.0000)	
	Gamma	$\propto_2$ = 0.4862± 0.0245(0.0000)	$\propto_2$ = 0.2773± 0.0203(0.0000)	
	Omega	$\propto_3$ = 0.83400± 0.5230 (0.0061)	$\propto_0$ = 2.4307± 0.0945(0.0701)	
	Iteration	**13**	22	
Model	BIC	**-339.0726**	-168.273	
Selection	AIC	**-345.6205**	-185.8024	AET
Criteria	R^2	**0.9993**	0.9862	
(MSC)	R.Adj	**0.9892**	0.9857	
	SSE	**0.080**	0.1565	
Logistic	Alpha	$\propto_0$ = 13.5148± 0.0658(0.0000)	$\propto_0$ = 11.3456± 0.667(0.0000)	
	Beta	$\propto_1$ = 3.425± 0.0621(0.000)	$\propto_1$ = 0.0162± 0.0011(0.0000)	
	Gamma	$\propto_2$ = 0.01199± 0.0004(0.0000)	$\propto_2$ = 0.3054± 0.0069(0.0000)	
	Omega	$\propto_3$ = 0.0421± 0.00290(0.0000)	$\propto_3$ = 0.4507± 0.0396(0.000)	
	Iteration	43	43	
Model	BIC	**-238.720**	40.5977	
Selection	AIC	**-30.0534**	50.6119	AET
Criteria	R^2	**0.8384**	0.5966	
(MSC)	R-adj	**0.8326**	0.6119	
	SSE	**1.7625**	2.8793	
Gompertz	Alpha	$\propto_0$ = 10.5559± 0.0137(0.0000)	$\propto_0$ = 10.4568± 0.0711(0.0000)	
	Beta	$\propto_1$ = 0.0402± 0.0028(0.2188)	$\propto_1$ = 0.1456± 0.0432(0.0000)	
	Gamma	$\propto_2$ = 0.8912± 0.0245(0.0000)	$\propto_2$ = 0.4562± 0.0249(0.0003)	
	Omega	$\propto_3$ = 0.9104 ± 0.0140(0.015)	$\propto_3$ = 0.5224 ± 0.0245(0.0000)	
	Iteration	42	36	
Model	BIC			MET

Selection	AIC	26.8720	**-28.9774**	
Criteria	R^2	21.0534	**-35.7890**	
(MSC)	R-Adj	0.6119	**0.8384**	
	SSE	0.5977	**0.8329**	
		2.873	**1.8238**	
Hill	Alpha	$\propto_0$ = 13.5184± 0.0658(0.0000)	$\propto_0$ = 10.7281± 0.0083(0.0000)	
	Beta	$\propto_1$ = 3.8270± 0.1097(0.0000)	$\propto_1$ = 2.4831± 0.1586(0.0000)	
	Gamma	$\propto_2$ = 0.8910± 0.0146(0.0000)	$\propto_2$ = 0.4506± 0.03961(0.0002)	
	Omega	$\propto_3$ = -0.0401 ± 0.0293 (0.0586)	$\propto_3$ = 0.0035 ± 0.0004 (0.0000)	
	Iteration	42	36	
Model	BIC	**-24.9774**	6.6205	
Selection	AIC	**-29.7960**	0.9824	AET
Criteria	R^2	**0.8372**	0.7256	
	R-Adj	**0.8342**	0.7202	
	SSE	**1.8248**	3.0976	
Richard	Alpha	$\propto_0$ = 12.5148± 0.0558(0.0000)	$\propto_0$ = 11.6658± 0.0082(0.0000)	
	Beta	$\propto_1$ = 4.2050± 0.0621(0.0000)	$\propto_1$ = 0.0162 ± 0.00011(0.0000)	
	Gamma	$\propto_2$ = 0.0119± 0.0004(0.0000)	$\propto_2$ = 0.3064± 0.0069(0.0000)	
	Omega	$\propto_3$ = 0.0085 ± 0.0003 (0.0000)	$\propto_3$ = 2.5382 ± 0.1586 (0.0000)	
	Iteration	22	**13**	
Model	BIC	-178.4862	**-462.8935**	MET
Selection	AIC	-183.6706	**-470.0748**	
Criteria	R^2	0.9905	**0.9994**	
	R-Adj	0.9902	**0.9994**	
	SSE	0.1078	**0.0063**	

Footnote: Sig at *0.10, **0.05, ***0.01. AET – Additive Error Term

MET – Multiplicative Error Term

Table 7 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Weibull Growth Model with the values -456.4380, -481.6343, 0.9896, 0.9896, and 0.0057 respectively which *identified Weibull Growth Model with Additive Error Term as suitable Growth Model* among the Five Growth Models.

Table 8 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Logistic Growth Model with the values -452.3821, -448.6702, 0.9983, 0.9983, and 0.0068 respectively which identified Logistic Growth Model with Additive Error Term as suitable Growth Model among the Five Growth Models.

Table 9 also indicate that the lowest values of Model Selection Criteria BIC, AIC and SSE, then highest values in terms of R^2 and R^2-Adjusted is the Richard Growth Model with the values -462.8935, -470.0748, 0.9994, 0.9994 and 0.0063 respectively which identified Richard Growth Model with Multplicative Error Term as suitable Growth Model among the Five Growth Models.

4. Summary and conclusion

The five growth models wasdecomposed by additive and multiplicative error terms. The problem of the initial parameters is addressed by second-order regression techniques before an iterative approach in this study. The aim of this work was to compare among the five growth models; Weibull, Logistic, Gompertz, Hill and Richard using addictive and multiplicative error term, the best model for growth analysis. Then the five growth models was derived and their mathematical properties. Using three real data with additive and multiplicative error terms for model comparison. To obtain the initial parameters of the growth models by second-order regression techniques [using Minitab 17 statistical software].To determine the parameters of the five growth models by Nonlinear Least Square (NLS) estimation using a modified version of the Levenberg-Marquardt algorithm [Gretl statistical software]. To compare the parameter of the five models by model adequacy criteria under the error assumptions. To recommend/suggest the most appropriate model for growth studies.

This study identified the Weibull Growth Model with Multiplicative Error Term as suitable growth model among the five growth models for the Agricultural Data. The Hill Growth Model with Additive Error Term as suitable growth model among the five growth models for the Engineering Data. The Weibull Growth Model with Additive Error Term as suitable growth model among the Five Growth Models for the Population Growth.

This study was able to show the use of the five growth model using real data sets from Population, Engineering and Agricultural product. The modified version of the Levenberg-Marquardt method for solving non-linear regression model was used. A suitable Weibull growth model with Additive error term and Richard growth model with multiplicative error term was identified as the most adequate model for growth analysis using model selection criteria; like Mean squared error, R^2, R-adjusted, BIC and AIC and can be used for forecasting.

This research work recommend/suggest that for further studies on growth analysis Weibull Growth Model with addictive error term and Richard Growth Model with multiplicative error term should be considered for growth studies.

This research work was able to identify the Weibull Growth Model with Additive Error Terms and Richard Growth Model with multiplicative error term as the best model for growth curves data set.

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Appendix

Table 4a: Simulated Errors Term

e_1	e_2	e_3	e_4	e_5
0.847276	0.495763	0.650785	0.872395	-0.55198
0.698757	-0.62224	-0.7613	0.900038	-1.52616
0.997878	1.008155	0.09775	0.272219	-1.12346
-0.25918	-0.23274	0.137352	-0.34672	-0.67755
0.51054	0.566225	-0.82296	-1.22335	2.663928
-1.97856	1.764648	-0.32629	-0.794	0.046252
-0.0602	-1.65686	-0.22397	-0.53609	-0.02824
1.672038	0.079683	1.762497	0.192621	0.08028
-0.5104	-1.02779	0.37025	0.200283	1.334554
0.712781	-0.43665	0.255344	1.519442	0.984084
0.704329	-0.20545	-0.75906	-1.02374	0.051838
-0.17939	0.466009	1.069325	-1.06453	-0.15612
-1.44874	-0.19686	-1.31962	-0.98314	-0.67001
0.431402	1.058873	-0.55061	-0.22671	0.019303
1.209005	-0.29072	0.18121	2.299471	0.520114
0.275059	-0.52165	1.1246	-0.6493	1.248896
-0.07646	-1.25745	-0.65518	0.574409	0.4796
0.523129	1.006714	-0.02769	0.371711	1.364475
0.761608	0.4716	-1.19015	1.319245	-0.62649
-0.43752	0.58176	-1.04554	0.144649	0.793973
0.761562	0.494257	1.54538	1.344123	-0.22049
-0.67741	0.146816	-1.77234	0.851084	1.795567
0.031558	0.011355	-0.5297	-1.51604	-0.12429
0.897098	-2.50639	0.125298	-1.57392	-0.56492
-2.05162	-0.75268	-0.39243	2.162523	2.234036
-1.64978	1.412875	0.025088	1.670954	-0.24801
1.618654	-0.49508	-1.16437	-1.20486	2.402503
1.064672	-1.97689	-0.3184	1.217244	-2.19285
1.713226	-0.8315	2.852005	1.263277	1.098741
0.58589	-0.87901	0.795093	1.675004	0.407639
-0.89895	1.993206	1.266897	0.244445	0.058815
0.020523	0.375971	-2.40115	-0.84789	0.15082
1.095384	-0.04033	-0.4178	-1.61789	0.31802
-0.05753	-0.53919	-0.19073	1.749468	0.454856
0.670082	0.291636	1.630896	1.234524	0.435161
0.400027	-1.34421	-1.64413	-0.7252	0.378692

1.354661	-0.38034	-0.40996	0.360336	-0.5717
-0.56972	1.205344	0.747228	-1.3293	0.959837
1.329592	1.692769	1.089712	-0.31876	-0.88549
-0.72719	0.257685	1.993809	-0.87442	-2.12595
0.118955	-0.1973	0.232341	-0.02553	0.733191
0.630216	0.286721	-0.60528	1.255986	-0.58985
-0.92544	1.042088	-0.98264	0.488205	0.576015
-0.60892	0.994333	-0.29531	0.153313	0.584382
-1.69894	-0.48445	0.464928	-0.60737	0.472417
-1.30637	-0.03443	0.835175	-0.48884	-0.13897
-0.59974	0.728499	-1.75799	0.401702	0.882815
1.070025	1.567425	-0.43176	-0.40031	-0.96336
0.453138	-0.948	1.627992	2.44068	1.252309
0.424652	-0.55112	-0.40236	0.352373	-0.64571
0.115585	1.320929	0.333827	-1.55717	0.293874
0.261082	-0.31921	1.467268	-0.93954	0.346925
-0.34984	0.479973	-0.31376	-0.79368	1.917206
-0.96535	0.368394	0.51028	-0.53947	0.77714
-0.34163	-0.77376	-0.6327	-1.31046	-0.85706
-2.4361	0.492433	-0.57716	-1.51715	-1.13225
-0.89258	-1.96207	1.145799	-1.23018	-1.69365
0.418226	1.655532	-0.2169	0.172094	-0.41094

Table 4b: Standardized Simulated Errors Term

Std e_1	Std e_2	Std e_3	Std e_4	Std e_5
3.35403	3.09576	3.15079	2.57239	1.64802
2.94504	1.97776	1.7387	2.60004	0.67384
3.29858	3.60816	2.59775	1.97222	1.07654
1.91315	2.36726	2.63735	1.35328	1.52245
3.09289	3.16623	1.67704	0.47665	4.86393
0.29485	4.36465	2.17371	0.906	2.24625
3.03639	0.94314	2.27603	1.16391	2.17176
4.22629	2.67968	4.2625	1.89262	2.28028
1.44668	1.57221	2.87025	1.90028	3.53455
3.37846	2.16335	2.75534	3.21944	3.18408
2.99228	2.39455	1.74094	0.67626	2.25184
2.08582	3.06601	3.56933	0.63547	2.04388
1.05279	2.40314	1.18038	0.71686	1.52999
3.37858	3.65887	1.94939	1.47329	2.2193
3.59827	2.30928	2.68121	3.99947	2.72011
2.39756	2.07835	3.6246	1.0507	3.4489
2.32418	1.34255	1.84482	2.27441	2.6796
3.04945	3.60671	2.47231	2.07171	3.56448
3.10973	3.0716	1.30985	3.01924	1.57351
1.80261	3.18176	1.45446	1.84465	2.99397
3.40615	3.09426	4.04538	3.04412	1.97951
1.55585	2.74682	0.72766	2.55108	3.99557
2.72924	2.61135	1.9703	0.18396	2.07571
3.40081	0.09361	2.6253	0.12608	1.63508
0.1004	1.84732	2.10757	3.86252	4.43404
1.42376	4.01287	2.52509	3.37095	1.95199
4.66193	2.10492	1.33563	0.49514	4.6025
3.08341	0.62311	2.1816	2.91724	0.00715
3.92152	1.7685	5.352	2.96328	3.29874
2.5617	1.72099	3.29509	3.375	2.60764
1.38218	4.59321	3.7669	1.94445	2.25882
2.78625	2.97597	0.09885	0.85211	2.35082
3.60819	2.55967	2.0822	0.08211	2.51802
2.0905	2.06081	2.30927	3.44947	2.65486
3.19477	2.89164	4.1309	2.93452	2.63516
2.69243	1.25579	0.85587	0.9748	2.57869
3.75779	2.21966	2.09004	2.06034	1.6283
1.4836	3.80534	3.24723	0.3707	3.15984
4.03127	4.29277	3.58971	1.38124	1.31451
1.32935	2.85769	4.49381	0.82558	0.07405
2.8345	2.4027	2.73234	1.67447	2.93319
3.0993	2.88672	1.89472	2.95599	1.61015
1.34125	3.64209	1.51736	2.18821	2.77601

2.14693	3.59433	2.20469	1.85331	2.78438
0.92797	2.11555	2.96493	1.09263	2.67242
1.66818	2.56557	3.33517	1.21116	2.06103
2.27393	3.3285	0.74201	2.1017	3.08282
3.77352	4.16742	2.06824	1.29969	1.23664
2.62364	1.652	4.12799	4.14068	3.45231
2.78471	2.04888	2.09764	2.05237	1.55429
2.47557	3.92093	2.83383	0.14283	2.49387
2.72062	2.28079	3.96727	0.76046	2.54693
2.04728	3.07997	2.18624	0.90632	4.11721
1.60264	2.96839	3.01028	1.16053	2.97714
2.4342	1.82624	1.8673	0.38954	1.34294
0.08717	3.09243	1.92284	0.18285	1.06775
2.32133	0.63793	3.6458	0.46982	0.50635
3.1934	4.25553	2.2831	1.87209	1.78906

Table 4c: Additive errors of Data 1

Errors	TIME(min)	VOLTAGE(volts)	LN.VOLTAGE(volts)	ln(TIME.min)
0.29485	0.7949	1.3	0.262364	-0.2296
1.05279	2.5528	1.3	0.262364	0.937187
1.44668	3.9467	1.9	0.641854	1.372875
1.91315	5.4132	3.4	1.223775	1.688831
2.08582	6.5858	5.3	1.667707	1.884919
2.94504	8.445	7.1	1.960095	2.133579
2.99228	9.4923	10.6	2.360854	2.250479
3.03639	10.536	16	2.772589	2.354835
3.09289	11.593	16.4	2.797281	2.450392
3.29858	12.799	18.3	2.906901	2.549334
3.35403	13.854	20.9	3.039749	2.628576
3.37846	14.878	20.5	3.020425	2.699915
3.37858	15.879	21.3	3.058707	2.764971
3.59827	17.098	21.2	3.054001	2.838977
4.22629	18.726	20	2.995732	2.929928

Table 4d: Multiplicative errors of Data 1

Errors	TIME(min)	VOLTAGE(volts)	LN.VOLTAGE(volts)	ln(TIME.min)
0.29485	0.147425	1.3	0.262364	-1.91444
1.05279	1.579185	1.3	0.262364	0.456909
1.44668	3.6167	1.9	0.641854	1.285562
1.91315	6.696025	3.4	1.223775	1.901514
2.08582	9.38619	5.3	1.667707	2.239239
2.94504	16.19772	7.1	1.960095	2.78487
2.99228	19.44982	10.6	2.360854	2.967838
3.03639	22.77293	16	2.772589	3.125572
3.09289	26.28957	16.4	2.797281	3.269172
3.29858	31.33651	18.3	2.906901	3.444784
3.35403	35.21732	20.9	3.039749	3.561538
3.37846	38.85229	20.5	3.020425	3.659767
3.37858	42.23225	21.3	3.058707	3.743184
3.59827	48.57665	21.2	3.054001	3.883143
4.22629	61.28121	20	2.995732	4.115473

Table 4e: Additive errors of Data 2

ei	Time(days after sowing)	Experimental(g/m2)	Ln(Time(days after sowing))	Ln(Experimental(g/m2))
0.29485	21.29485	1.49	0.398776	3.058465
1.44668	29.44668	5.56	1.715598	3.382581
1.91315	36.91315	20.69	3.02965	3.608568
2.94504	44.94504	60.34	4.099995	3.80544
3.03639	52.03639	137.41	4.922969	3.951943
3.09289	59.09289	261.38	5.565975	4.079111
3.29858	66.29858	373.52	5.922972	4.194168
3.35403	73.35403	462.66	6.136992	4.295297
3.37846	80.37846	505.23	6.225014	4.386746
4.22629	88.22629	485.6	6.185385	4.479905

Table 4f: Multiplicative errors of Data 2

ei		Time(days after sowing)	Experimental(g/m2)	Ln(Time(days after sowing))
0.29485	6.19185	1.49	0.398776	1.823234
1.44668	40.50704	5.56	1.715598	3.701476
1.91315	66.96025	20.69	3.02965	4.204099
2.94504	123.6917	60.34	4.099995	4.817792
3.03639	148.7831	137.41	4.922969	5.00249
3.09289	173.2018	261.38	5.565975	5.154458
3.29858	207.8105	373.52	5.922972	5.336627
3.35403	234.7821	462.66	6.136992	5.458658
3.37846	260.1414	505.23	6.225014	5.561225
4.22629	355.0084	485.6	6.185385	5.872141

Table 4g: Additive errors of Data 3

ei	X	Nigeria Population growth	Ln(x)	Ln(Nigeria Population growth)
0.08717	1.08717	41547.48	0.083578	10.63459
0.1004	2.1004	42771.55	0.742128	10.66363
0.29485	3.29485	44033.21	1.192361	10.6927
0.92797	4.92797	45333.22	1.594927	10.7218
1.05279	6.05279	46675.41	1.800519	10.75097
1.32935	7.32935	48064.04	1.991887	10.78029
1.34125	8.34125	49491.14	2.121213	10.80955
1.38218	9.38218	50948.61	2.238812	10.83857
1.42376	10.42376	52445.06	2.344088	10.86752
1.44668	11.44668	53990.04	2.4377	10.89655
1.4836	12.4836	55585.64	2.524416	10.92568
1.55585	13.55585	57234.22	2.606818	10.95491
1.60264	14.60264	58941.18	2.681202	10.9843
1.66818	15.66818	60702.38	2.751632	11.01374
1.80261	16.80261	62523.71	2.821534	11.0433
1.91315	17.91315	64421.5	2.885535	11.0732
2.04728	19.04728	66392.9	2.946924	11.10335
2.08582	20.08582	68421.17	3.000014	11.13344

2.0905	21.0905	70483.45	3.048823	11.16313
2.14693	22.14693	72588.34	3.097699	11.19256
2.27393	23.27393	74821.27	3.147334	11.22286
2.32133	24.32133	77149.58	3.191354	11.2535
2.32418	25.32418	79510.98	3.23176	11.28365
2.39756	26.39756	81163.74	3.273272	11.30422
2.4342	27.4342	82724.29	3.31179	11.32327
2.47557	28.47557	84889.38	3.349047	11.3491
2.5617	29.5617	87092.24	3.38648	11.37472
2.62364	30.62364	89405.02	3.421772	11.40093
2.69243	31.69243	91760.42	3.456078	11.42694
2.72062	32.72062	94159.65	3.488005	11.45275
2.72924	33.72924	96603.76	3.518365	11.47837
2.78471	34.78471	99092.5	3.549178	11.50381
2.78625	35.78625	101625.1	3.577564	11.52905
2.8345	36.8345	104199.7	3.606435	11.55406
2.94504	37.94504	106814	3.636139	11.57884
2.99228	38.99228	109465.1	3.663364	11.60336
3.03639	40.03639	112149	3.689789	11.62758
3.04945	41.04945	114862	3.714777	11.65149
3.08341	42.08341	117599.6	3.739654	11.67504
3.09289	43.09289	120369.2	3.763358	11.69832
3.0993	44.0993	123178.8	3.786444	11.72139
3.10973	45.10973	126014.3	3.809098	11.74415
3.1934	46.1934	128864.3	3.832837	11.76652
3.19477	47.19477	131727.5	3.854283	11.78849
3.29858	48.29858	134604.5	3.877402	11.8101
3.35403	49.35403	137495.1	3.899019	11.83134
3.37846	50.37846	142615	3.919564	11.8679
3.37858	51.37858	146417	3.939221	11.89421
3.40081	52.40081	150347	3.958922	11.9207
3.40615	53.40615	154420	3.977926	11.94743
3.59827	54.59827	158578	4.000002	11.974
3.60819	55.60819	162877	4.01833	12.00075
3.75779	56.75779	167297	4.038793	12.02753
3.77352	57.77352	171829	4.056531	12.05426
3.92152	58.92152	176461	4.076206	12.08086
4.03127	60.03127	181181	4.094866	12.10725
4.22629	61.22629	185990	4.114577	12.13345
4.66193	62.66193	190886	4.137754	12.15943

Table 4h: Summary of the Identified models

	Agricultural Data 1			Engineering Data 2			Population Growth Data 3		
Model	Error Terms $\hat{\varepsilon}_1$	Error Terms $\hat{\varepsilon}_2$	Error Terms $\hat{\varepsilon}_3$	Error Terms $\hat{\varepsilon}_1$	Error Terms $\hat{\varepsilon}_2$	Error Terms $\hat{\varepsilon}_3$	Error Terms $\hat{\varepsilon}_1$	Error Terms $\hat{\varepsilon}_2$	Error Terms $\hat{\varepsilon}_3$
Weibull	**MET**	**MET**	AET	AET	AET	**MET**	AET	AET	AET
Logistic	AET	**MET**	**MET**	AET	AET	**MET**	AET	AET	AET
Gompertz	**MET**	**MET**	AET	AET	AET	AET	AET	AET	**MET**
Hill	**MET**	**MET**	AET	AET	AET	AET	AET	AET	AET
Richard	**MET**	**MET**	**MET**	**MET**	AET	**MET**	AET	AET	**MET**

Table 4I: Multiplicative errors of Data 3

ei	x	Nigeria Population growth	Ln(x)	Ln(Nigeria Population growth)
0.08717	0.08717	41547.48	-2.4399	10.63459
0.1004	0.2008	42771.55	-1.60545	10.66363
0.29485	0.88455	44033.21	-0.12268	10.6927
0.92797	3.71188	45333.22	1.311538	10.7218
1.05279	5.26395	46675.41	1.660882	10.75097
1.32935	7.9761	48064.04	2.07645	10.78029
1.34125	9.38875	49491.14	2.239512	10.80955
1.38218	11.05744	50948.61	2.403104	10.83857
1.42376	12.81384	52445.06	2.550526	10.86752
1.44668	14.4668	53990.04	2.671856	10.89655
1.4836	16.3196	55585.64	2.792367	10.92568
1.55585	18.6702	57234.22	2.926929	10.95491
1.60264	20.83432	58941.18	3.036602	10.9843
1.66818	23.35452	60702.38	3.150791	11.01374
1.80261	27.03915	62523.71	3.297286	11.0433
1.91315	30.6104	64421.5	3.42134	11.0732
2.04728	34.80376	66392.9	3.549725	11.10335
2.08582	37.54476	68421.17	3.625534	11.13344
2.0905	39.7195	70483.45	3.681842	11.16313
2.14693	42.9386	72588.34	3.759771	11.19256
2.27393	47.75253	74821.27	3.866032	11.22286
2.32133	51.06926	77149.58	3.933183	11.2535
2.32418	53.45614	79510.98	3.978862	11.28365
2.39756	57.54144	81163.74	4.052505	11.30422
2.4342	60.855	82724.29	4.108494	11.32327
2.47557	64.36482	84889.38	4.164567	11.3491
2.5617	69.1659	87092.24	4.236508	11.37472
2.62364	73.46192	89405.02	4.296767	11.40093
2.69243	78.08047	91760.42	4.35774	11.42694
2.72062	81.6186	94159.65	4.402057	11.45275
2.72924	84.60644	96603.76	4.43801	11.47837
2.78471	89.11072	99092.5	4.48988	11.50381
2.78625	91.94625	101625.1	4.521204	11.52905
2.8345	96.373	104199.7	4.568226	11.55406
2.94504	103.0764	106814	4.63547	11.57884
2.99228	107.7221	109465.1	4.679555	11.60336
3.03639	112.3464	112149	4.721587	11.62758
3.04945	115.8791	114862	4.752547	11.65149
3.08341	120.253	117599.6	4.789598	11.67504
3.09289	123.7156	120369.2	4.817985	11.69832

3.0993	127.0713	123178.8	4.844748	11.72139
3.10973	130.6087	126014.3	4.872206	11.74415
3.1934	137.3162	128864.3	4.922286	11.76652
3.19477	140.5699	131727.5	4.945705	11.78849
3.29858	148.4361	134604.5	5.000155	11.8101
3.35403	154.2854	137495.1	5.038804	11.83134
3.37846	158.7876	142615	5.067568	11.8679
3.37858	162.1718	146417	5.088657	11.89421
3.40081	166.6397	150347	5.115834	11.9207
3.40615	170.3075	154420	5.137606	11.94743
3.59827	183.5118	158578	5.212279	11.974
3.60819	187.6259	162877	5.23445	12.00075
3.75779	199.1629	167297	5.294123	12.02753
3.77352	203.7701	171829	5.316992	12.05426
3.92152	215.6836	176461	5.373813	12.08086
4.03127	225.7511	181181	5.419433	12.10725
4.22629	240.8985	185990	5.484376	12.13345
4.66193	270.3919	190886	5.599873	12.15943