# On the Pendant Number of Certain Graphs

**Original Research Article**


## Abstract

The present study investigates the pendant number of certain graph classes; complement, line graphs, and total graphs. The pendant number is the minimum number of end vertices of paths in a path decomposition of a graph. A path decomposition of a graph is a decomposition of it into subgraphs; i.e., a sequence of a subset of vertices of the graph such that the endpoints of each edge appear in one of the subsets and each vertex appears in an adjacent sub-sequence of the subsets.



Keywords: Complement; Decomposition; Line Graph; Pendant Number; Total Graph

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## 1 Introduction

The motivation for the present work comes from the study by Sebastian et al. [1], where they introduced the pendant number for certain fundamental graph classes. Our research is carried out specifically to determine this parameter for derived graphs such as complement, line graphs, and total graphs. The pendant number $\Pi_p$ is the minimum number of end vertices of paths in a path decomposition of $G$ [2]. Graph decomposition problems rank among the most prominent areas of research in graph theory which is a collection of edge-disjoint sub-graphs such that every edge belongs to exactly one sub-graph. All graphs that are considered in this paper are undirected, simple, finite, and connected. The graph theoretic terminologies used in this paper are as appeared in [3] and [4].

## 2 Materials and Methods

In this section, terminologies of the types of graphs that we consider are presented as appeared in [1].

### 2.1 Complement Graphs

**Definition 2.1.** If $G$ is a simple graph with vertex set $V(G)$, its complement $\overline{G}$ is the simple graph with vertex set $V(G)$ in which two vertices are adjacent if and only if they are not adjacent in $G$.

1
2
3
4
5 1
2
3
4
5

Figure 1: A Graph and its complement

For example, Figure 1 shows a graph and its complement. Note that the complement of a complete graph is a null graph and that the complement of a complete bipartite graph is the union of two complete graphs.

### 2.2 Line Graphs

1
2
3
4
a
b

c
e
d
a
b
c
e d

Figure 2: A Graph $G$ and the corresponding line graph $L(G)$

Given a graph $G$ with at least one edge, the line graph $L(G)$ is that graph whose vertices are the edges of $G$, with two of these vertices being adjacent if the corresponding edges are adjacent in $G$ [5]. The operation of going from adjacent edges in $G$ to adjacent vertices in $L(G)$ is indicated in Figure 2.

## 2.3 Total Graph

**Definition 2.2.** The total graph $T(G)$ of a graph $G$ is the graph that has as its vertex set $V(G) \cup E(G)$ with a pair of these elements are adjacent if in the graph $G$ they are either

1. adjacent vertices,
2. adjacent edges, or
3. an incident vertex and edge.

This is illustrated schematically in Figure 3 below. It follows from the definition that the total graph of graph $G$ is spanned by the sum (disjoint union) $G + L(G)$ of $G$ and its line graph $L(G)$ [6].

Figure 3: $C_5$ graph and its total graph $T(C_5)$

## 2.4 Pendent Number

**Definition 2.3.** The pendant number of a graph $G$, denoted by $\Pi_p(G)$, is the least number of vertices in a graph such that they are the end vertices of a path in a given path decomposition of a graph.

**Theorem 2.1.** Let $G$ be a connected graph with $n$ vertices. If $G$ has $l$ number of odd degree vertices, then $l \leq \Pi_p(G) \leq n$.

## 2.5 Pendant Number of Acyclic Graphs

An acyclic graph is a graph having no graph cycles. An acyclic-connected graph is called a tree. Since every path $P_n$ has exactly two pendant number of vertices, we have $\Pi_p(G) = 2$ from the following proposition.

**Theorem 2.2.** Let $T$ be a tree with $n \geq 2$ vertices. Then $\Pi_p(T) = 2$ if and only if $T$ is a path $P_n$.

## 2.6 Pendant Number of Cyclic Graphs

Now, we shall state some properties of cyclic graphs which are essential in our study.

**Proposition 2.1.** If $G$ is the cycle $C_n$ on $n \geq 3$ number of vertices, then $\Pi_p(G) = 2$.

**Theorem 2.3.** If $G$ has a Hamilton decomposition of the first kind, then $\Pi_p(G) = 2$.

**Proposition 2.2.** For a complete graph $K_n$ on $n$ vertices, where $n$ is odd, there will be $\frac{n-1}{2}$ edgedisjoint cycles of length $n$.

The following theorem discusses the pendant number of a complete graph of odd order.

**Theorem 2.4.** For $n \geq 3$, $\Pi_p(K_n) = 2$ if and only if $n$ is odd.

# 3 Results and Discussion

Now, we will provide a detailed study on the pendant number of line graphs, total graphs, and complement graphs as suggested in [1].

### 3.1 Pendant Number of Complement Graphs

In graph theory, a cyclic graph $C_n$, sometimes simply known as an $n$−cycle, is a graph on $n$ nodes containing a single cycle through all nodes. When we consider the complement graphs of cycle graphs, those can be obtained by removing a cycle from the complete graph $K_n$ (top) and in the ″standard″ circulant graph from the bottom. By observing the pattern, the following table can be consturcted.

Table 1: Complement of the cycle graph for $n \leq 10$

| n | V(G) | E(G) | deg(G) | Hamiltonian Yes/No | Number of cycles |
|---|---|---|---|---|---|
| 5 | 5 | 5 | 2 | ✓ | 1 |
| 6 | 6 | 9 | 3 | - | - |
| 7 | 7 | 14 | 4 | ✓ | 2 |
| 8 | 8 | 20 | 5 | - | - |
| 9 | 9 | 27 | 6 | ✓ | 3 |
| 10 | 10 | 33 | 7 | - | - |

**Theorem 3.1.** Let $G$ be the Cyclic Graph $C_n$. For $n \geq 5$ number of vertices, the following holds:

$$\Pi_p(\overline{G}) = 2 \text{ if and only if } n \text{ is odd.} \quad (3.1)$$

Proof. For $n \geq 1$, consider the complement graph of $C_{2n+1}$. Clearly, it has Hamilton decomposition of the first kind, with $n$−Hamiltonian cycles. The degree of the complement graph of a cyclic graph is $2n$. Hence, by Theorem 2.3, $\Pi_p(\overline{G}) = 2$.

Conversely, take a complete graph $C_{2n+1}$ on $n \geq 1$ vertices such that $\Pi_p(\overline{G}) = 2$. Assume, if possible, that $n$ is even. Then, all its vertices are of odd degree $n - 1$, a contradiction. Hence, n cannot be even, completing the proof.

**Theorem 3.2.** For the complement graph of cyclic graph $C_{2n}$ on $n \geq 3$ number of vertices,

$$\Pi_p(\overline{G}) = 2n. \quad (3.2)$$

Proof. Consider the complement graph of $C_{2n}$ for $n \geq 3$. Note that, $C_{2n}$ is a regular graph with $2n+1$ vertices. Hence $C_{2n}$ has odd degree vertices by Theorem 2.1. Therefore the pendant number is $2n$.

### 3.2 Pendant Number of Total Graphs

In this subsection, the pendent number of the total graphs of $P_n$, $K_n$, $C_n$, and $K_{1,n}$ is investigated. Note that the path graph $P_n$ is a tree with two nodes of vertex degree one.

The path graphs and corresponding total graphs of path graphs can be visualized as follows.

$e_{k-1}$ $e_k$

Figure 4: $P_n$ graph and the corresponding total graph $T(P_n)$

**Theorem 3.3.** Let $G$ be a tree with $n \geq 3$ number of vertices, then

$$\Pi_p(T(G)) = 2 \text{ if and only if } G \text{ is the path } P_n. \quad (3.3)$$

Proof. Note that, the total graph of $P_n$ has $2n - 1$ number of vertices and it is a connected graph. Also, $T(P_n)$ on $n \geq 3$ has 2 degree 3 vertices. Hence by Theorem 2.1, the pendant number is 2.

Let $K_n$ be a complete graph with $n$ vertices and $T(K_n)$ be the total graph of a complete graph $K_n$. Note that $T(K_n)$ is a strongly regular graph [7].

**Theorem 3.4.** For the complete graph $K_n$ on $n \geq 3$ vertices,

$\Pi_p(T(K_n)) = 2$. (3.4)

Proof. Note that, the total graph of the complete graph $K_n$ is $2(n-1)$ regular graph. Hence it is clearly a graph with Hamilton decomposition of the first kind. Hence by Theorem 2.3, $\Pi_p(T(K_n)) = 2$.

**Theorem 3.5.** Let $G$ is a star graph $K_{1,n}$ on $n \geq 2$ vertices. If $n$ is even, then

$\Pi_p(T(G)) = n$. (3.5)

Proof. Note that, the total graph of a star graph $K_{1,n}$ on $n \geq 2$. When $n$ is even $K_{1,n}$ has $n$ number of odd vertices. Because $K_{1,n}$ is a connected graph by Theorem 2.1. Therefore the pendant number is $n$.

Let's look at the total graphs of $C_n$. When $n = 5$, the total graph of $C_5$ can be drawn as Figure 3. For the general case, the total graph of $C_n$ can be visualized as Figure 5 [8].

**Theorem 3.6.** If $G$ is the cyclic graph $C_n$, then for $n \geq 3$ number of vertices, the following holds:

$\Pi_p(T(G)) = 2$. (3.6)

Proof. Note that, the total graph of a cycle $C_n$ is a 4-regular graph. Thus it is clear that $C_n$ is a graph with Hamilton decomposition of the first kind. Hence by Theorem 2.3, the pendant number is 2.

Figure 5: $C_n$ graph and the corresponding total graph $T(C_n)$

## 3.3 Pendant Number of Line Graphs

In this section, we present our result on the pendant number of line graphs.

**Theorem 3.7.** If $G$ is the cycle $C_n$ on $n \geq 3$ vertices, then

$\Pi_p(L(G)) = 2$. (3.7)

Proof. Note that, the line graph of the cycle graph $C_n$ is isomorphic to $C_n$ itself. From Proposition 2.1, if $G$ is the cycle $C_n$ on $n \geq 3$ vertices, then $\Pi_p(G) = 2$. Since $L(C_n) = n$, the pendent number of the line graph of $C_n$ is 2.

**Theorem 3.8.** Let $T$ be a tree with $n \geq 3$ number of vertices. If $T$ is a path $P_n$, then

$\Pi_p(L(T)) = 2$. (3.8)

Proof. Since it has two pendent vertices, the path graph $P_n$ is a tree with two nodes of vertex degree one. Note that the line graph of $P_n$ is isomorphic to $P_{n-1}$. Hence by Theorem 2.5, the pendant number is 2.

**Theorem 3.9.** If $G$ is the star graph $S_n$ on $n \geq 3$ number of vertices, then

$$\Pi_p(L(G)) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases} \quad (3.9)$$

Proof. Note that, the line graph of the star graph $S_n$ is isomorphic to the complete graph $K_{n-1}$. From Proposition 2.2 and Theorem 2.4, the pendant number of a complete graph $K_n$ is equal to $n$ when it has odd order. Furthermore, for $n \geq 3$, $\Pi_p(K_n) = 2$ if and only if $n$ is even. Therefore,

$$\Pi_p(L(G)) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ n & \text{if } n \text{ is odd.} \end{cases} \quad (3.10)$$

Our results can be extended to 3-dimensional graphs as well.

**Theorem 3.10.** If $G$ is a tetrahedral graph, then

$$\Pi_p(L(G)) = 2. \quad (3.11)$$

Proof. Note that, the line graph of the tetrahedral graph is the octahedral graph. The Hamiltonian cycle covers the entire vertex set of $G$ and there are three cycles. Since the pendant number of a cyclic graph $C_n$ is 2 and each $C_n$ has common vertices, the pendant number of $L(G)$ is equal to 2.

Figure 6 shows the degree of the line graph of the bipartite graph for fixed $n = 2$.

Figure 6: Bipartite graph and corresponding line graph for fix $n = 2$

By observing the graphs and line graphs the following theorem can be obtained.

**Theorem 3.11.** If $G$ is a complete bipartite graph $K_{m,n}$ with $m \leq n$, then the degree of $L(G)$ can be generalized as follows:

$$\deg(L(G)) = \begin{cases} m + n - 2 & \text{if } m < n, \\ 2(n-1) & \text{if } m = n. \end{cases} \quad (3.12)$$

**Theorem 3.12.** If $G$ is the complete bipartite graph $K_{m,n}$ with $m \leq n$, then

$$\Pi_p(L(G)) = \begin{cases} mn & \text{if } \deg(L(G)) \text{ is odd}, \\ n^2 & \text{if } \deg(L(G)) \text{ is even}. \end{cases} \quad (3.13)$$

Proof.

Case 1: $m < n$, if $m$ is odd and $n$ is even or $m$ is even and $n$ is odd

Now $L(G)$ is a connected graph with odd-degree vertices. Hence all vertices have odd-degree by Theorem 2.1, and $\Pi_p(L(G)) = x$, where $x$ is the number of vertices in $L(G)$ and it is $mn$.

Case 2: $m < n$, if both $m$ and $n$ are both odd or both are even

Now $L(G)$ is a connected graph with even degree vertices and there are no odd degree vertices.

Case 3: $m = n$

Here $L(G)$ is a connected graph with $n^2$ number of even degree vertices.

By considering the above three cases we get, the pendant number as required.

**Theorem 3.13.** If $G$ is a cubic graph, then

$$\Pi_p(L(G)) = 2. \quad (3.14)$$

Proof. Note that, the line graph of the cubic graph is the cuboctahedral graph. It is a Hamiltonian, and a regular graph. Hamiltonian cycle covers the entire vertex set of $G$ and there are $x$ cycles that have $x-$fold symmetry. Since the pendant number of the cyclic graph $C_n$ is 2 and each $C_n$ has common vertices, the pendant number of $L(G)$ is equal to 2.

# 4 CONCLUSIONS

Th present study aimed to determine the pendant number of certain derived graph classes, including the complement, total, and line graphs. The study can be extended to other general graph classes. Although there is no direct relationship between the pendant number and other well-known graph parameters, it was observed that the pendant number highly depends on the number of odd-degree vertices in a graph. Further studies could compare the pendant number with other graph parameters, such as the domination number and graph diameter. The concept of pendant number can be applied in combination with path decomposition, which has applications in Computer Science and Engineering, including network design [9] and graph algorithms [10]. One of the recent studies along these lines is the study of the pendant number of corona products and rooted products of paths and cycles [11]. Also, in [12], the pendant number of regular graphs, complete r-partite graphs, line graphs, total graphs, and line graphs of total graphs are studied. Our approach is different than to [12] in that we used Hamiltonian cycles to establish our results.

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