Original Research Article

Independent Semitotal Domination in the Join of Graphs

Abstract

A subset $W \subseteq V(G)$ of a graph G is an independent semitotal dominating set of G, abbreviated $ISTd$-set of G, if W is an independent dominating set of G and every element of W is exactly of distance 2 from at least one other element of W. The independent semitotal domination number of G, denoted by $\gamma_{it2}(G)$, is the minimum cardinality of an $ISTd$-set of G. In this paper, we study the concept of independent semitotal domination in graphs and investigate the conditions for graphs on which the $ISTd$-sets exist. Further, the $ISTd$-sets of the join of any two graphs are examined. Consequently, the corresponding independent semitotal domination number of these graphs are obtained.

Keywords: independent semitotal domination, join of graphs, nonsingleton independent set

2020 Mathematics Subject Classification: 05C12, 05C69

1 Introduction

Let $G = (V(G), E(G))$ be a graph. A *dominating set* of G is a set D of vertices of G such that every vertex in $V(G) \setminus D$ is adjacent to at least one vertex in D [1]. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G [1].

The notion of semitotal domination in a graph was introduced by Goddard, Henning and McPillan [2]. It is a strengthening of domination but a relaxation of both total domination and weakly connected domination [3]. A set S of vertices in a graph G with no isolated vertices is a *semitotal* *dominating set* of G, abbreviated *semi-TD-set* of G, if it is a dominating set of G and every vertex in S is within distance 2 of another vertex in S [2]. The semitotal domination number of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a *semi-TD-set* of G, which itself is squeezed between two important domination parameters, namely, domination and total domination numbers of G [3].

On the other hand, the notion of independent domination is also one of the most explored areas of domination (see for instance [4],[5]). Accordingly, a set I of vertices of a graph G is independent if no two vertices in I are adjacent in G [6]. Thus, a set I of vertices in graph G is called an *independent dominating set* if I is both an independent set and a dominating set of G [4]. Along this line we can prescribe that an independent semitotal dominating set of a graph G is a subset $W \subseteq V(G)$ such that W is independent, dominating, and that every element of W is of distance 2 to another element of W.

To illustrate, the placing of checkpoints in imposing community quarantines can be viewed logically in the lens of independent semitotal domination. For instance, police checkpoints cannot be adjacent to each other to maximize use of personnel but cannot be far that when insurgencies arise no one can rescue them.

In this paper, we study the concept of *independent semitotal domination* of a graph and investigate the conditions for graphs on which the $ISTd$-sets exist. In addition, the independent semitotal dominating sets of the join between any connected graph and a connected noncomplete graph shall be examined using some properties posessed by the operation [7],[8] and some inherent properties posessed by the contituents or the individual graphs involved in the operation [9]. This is similar to the motivation in [10],[11].

In this work, all graphs shall be understood in the context of being finite, undirected, connected and simple. To avoid trivial inconsistency, only noncomplete graphs with order at least 3 will be considered. For basic graph theoretic terminologies not specifically described in this paper, please refer to [12].

2 Preliminary Notes

Four of the main concepts in this paper are formally defined below for emphasis.

Definition 2.1. [1] Let $G = (V(G), E(G))$ be a simple, undirected and connected graph and let $v \in V(G)$. The *open neighborhood* $N_G(v)$ of v refers to the set of all vertices of G that are adjacent to v. Moreover, the *closed neighborhood* $N_G[v]$ of v is the union $N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, $N_G(S) = \bigcup_{v\in S} N_G(v)$ and $N_G[S] = \bigcup_{v\in S} N_G[v] = N_G(S) \cup S$.

Definition 2.2. [2] Let $G = (V(G), E(G))$. A subset D of $V(G)$ is a *dominating set* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is, $N_G[D] = V(G)$. The *domination number* of G, denoted by $\gamma(G)$, is the smallest cardinality of a dominating set of G. A dominating set of G with cardinality $\gamma(G)$ is called a γ-set of G or a minimum dominating set of G.

Example 2.1. In Figure 1(a) below, set $S = \{a, b, c\}$ is a dominating set of G of minimum cardinality; thus, $\gamma(G) = 3$. In Figure 1(b), graph H has 8 distinct γ-sets of the form $\{x_i, y_j\}$, $i = 1, 2, 3, 4$ and $j = 1, 2$, with $\{y_1, y_2\}$ as another γ-set of H.

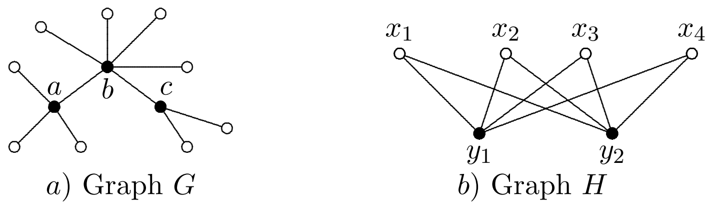


Figure 1: Two graphs G and H, where $\gamma(G) = 3$ and $\gamma(H) = 2$

Definition 2.3. [4] Let $G = (V(G), E(G))$ be a graph. A set $I \subseteq V(G)$ is an *independent dominating set* of G, abbreviated i-set of G, if every vertex in $V(G) \setminus I$ is adjacent to an element in I and that I is independent, i.e., no two elements in I are adjacent. The *independent domination number* of G, denoted by $\gamma_i(G)$, is the minimum cardinality of an independent dominating set of G. An independent dominating set of G with cardinality $\gamma_i(G)$ is called an γ_i-set of G or a minimum independent dominating set of G.

Example 2.2. Consider the graphs G and H as given in Figure 1. The dominating set $S = \{a, b, c\}$ of G is not an independent dominating set of G, while the dominating set $\{y_1, y_2\}$ of H is an independent dominating set of H.

Example 2.2 emphasizes the fact that not all dominating sets are independent dominating sets.

Definition 2.4. [3] Let $G = (V(G), E(G))$ be a graph. A set $S \subseteq V(G)$ is a *semitotal dominating set* of G, abbreviated *semi-TD-set* of G, if every vertex of $V(G) \setminus S$ is adjacent to an element in S and every vertex in S is within distance 2 from another vertex in S. The *semitotal domination number* of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semitotal dominating set of G. A semitotal dominating set of G with cardinality $\gamma_{t2}(G)$ is called a γ_{t2}-set of G or a minimum semitotal dominating set of G.

Example 2.3. Consider the graphs G and H as given in Figure 1. Both the dominating set $S = \{a, b, c\}$ of G and the dominating set $\{y_1, y_2\}$ of H are semitotal dominating sets of G and H, respectively.

With the definitions above, we are now ready to introduce a new variant in the area of domination.

Definition 2.5. Let $G = (V(G), E(G))$ be a connected noncomplete graph with order at least 3. An independent dominating set $W \subseteq V(G)$ is an *independent semitotal dominating set* of G, abbreviated $ISTd$-set of G, if every vertex in W is exactly of distance two from another vertex in W. The *independent semitotal domination number* of G, denoted by $\gamma_{it2}(G)$, is the minimum cardinality of an independent semitotal dominating set of G. An independent semitotal dominating set of G with cardinality $\gamma_{it2}(G)$ is called a γ_{it2}-set of G or a minimum independent semitotal dominating set of G.

Please be it noted that an independent semitotal dominating set is both an independent dominating set and a semitotal dominating set. Additionally, every independent dominating set is an independent set. Moreover, the largest independent set of a graph G is called a maximum independent set and its cardinality is denoted by $\alpha(G)$. Thus, we have this remark below on some natural bounds of the independent semitotal domination number.

Remark 2.6. Let G be a connected noncomplete graph with order at least 3. Then we have $max\{\gamma_{t2}(G), \gamma_i(G)\} \leq \gamma_{it2}(G) \leq \alpha(G)$ as bounds of $\gamma_{it2}(G)$. When G is a noncomplete graph where $\gamma_i(G) = 1$, and from the fact that $\gamma_{t2}(G) \geq 2$ [13], it must necessarily follow that $\gamma_{it2}(G) \geq 2$.

Remarks 2.7. For any connected noncomplete graph G of order n and a maximum degree Δ, we have

$$\gamma_{it2}(G) \geq \frac{2n}{2\Delta+1}.$$

This is anchored on the result of Goddard, Henning, and McPillan [2] where they showed that $\gamma_{t2}(G) \geq \frac{2n}{2\Delta+1}$. By transitive property on the bounds of Remarks 2.6, we have $\gamma_{it2}(G) \geq \frac{2n}{2\Delta+1}$.

Example 2.4. The subset $\{y_1, y_2\}$ of $V(H)$ as shown in Figure 1(b) is clearly an independent semitotal dominating set of H of minimum cardinality. Thus, $\gamma_{it2}(H) = 2$.

3 Main Results

It is necessary to set first the conditions for which type of graphs where $ISTd$-sets exist. We state these as our first and second theorems. In our first theorem, we will utilize the concepts of nonsingleton independent domination and open hop neighborhood in the characterization, so we have to introduce them first. Note that subset a $S \subseteq V(G)$ is called a maximal independent set of a graph G if for any $y \in V(G) \setminus S$, the resulting set $S \cup \{y\}$ loses the property of being independent. Any maximal independent set of G of highest cardinality possible is usually called a maximum independent set of G.

Lemma 3.1. If G is a connected noncomplete graph with order at least 3, then a maximum independent set of G is nonsingleton.

Proof. Any set $\{u, v\}$ containing two non-adjacent vertices u and v of G can always be extended, if necessary, to a maximal independent subset $S \subseteq V(G)$ containing u and v. Clearly, $|S| \geq 2$. Thus, every maximum independent set of G is nonsingleton. □

Definition 3.1. [14] Let $G = (V(G), E(G))$ be a connected graph. The *open hop neighborhood* of a point $u \in V(G)$ is the set $N^2_G(u) = \{v \in V(G) : d_G(u, v) = 2\}$. The *closed hop neighborhood* of u is $N^2_G[u] = N^2_G(u) \cup \{u\}$. For any $S \subseteq V(G)$, $N^2_G(S) = \bigcup_{v \in S} N^2_G(v)$ is called the *open hop neighborhood of S* and $N^2_G[S] = N^2_G(S) \cup S$ is called the *closed hop neighborhood of S.*

Theorem 3.2. Let $G = (V(G), E(G))$ be a connected noncomplete graph with order at least 3. Then G has an independent semitotal dominating set if and only if there exists a nonsingleton independent dominating set W of G satisfying any of the following conditions:

i. $N^2_G(W) \subseteq W$, or

ii. For every $v_0 \notin W$ where $v_0 \in N^2_G(u_i)$ for some $u_i \in W$, there exists $u_j \in W$ such that $u_j \in N^2_G(u_i)$.

Proof. Assume that graph G is connected and noncomplete with order at least 3. Suppose G has an independent semitotal dominating set W. For one thing this means that for every $u_i \in W$ there exists $u_j \in W$ such that $d_G(u_i, u_j) = 2$. Let $W = \{u_0, u_1, u_2, u_3, ..., u_n\}$. Then we consider two general cases for $N^2_G(W)$:

i. $N^2_G(W) \cap (V(G) \setminus W) = \emptyset$. If $N^2_G(W) \cap (V(G) \setminus W) = \emptyset$, then this implies that for every $u_i \in W$ we have $N^2_G(u_i) \subseteq W$. Thus, in this case, we have $N^2_G(W) \subseteq W$.

ii. $N^2_G(W) \cap (V(G) \setminus W) \neq \emptyset$. If $N^2_G(W) \cap (V(G) \setminus W) \neq \emptyset$, then this implies that there exists $v_0 \in V(G) \setminus W$ where $v_0 \in N^2_G(u_i)$ for some $u_i \in W$. Clearly, $d_G(u_i, v_0) = 2$. Since W is an independent semitotal dominating set of G, this implies that there exists $u_j \in W$ such that $u_j \in N^2_G(u_i)$ or equivalently, $d_G(u_i, u_j) = 2$. Thus, in this case, we have $u_j \in W$ whenever there exists $v_0 \notin W$ where $v_0 \in N^2_G(u_i)$ for some $u_i \in W$ such that $u_j \in N^2_G(u_i)$.

For the converse, suppose that there exists a nonsingleton independent dominating set W satisfying either conditions (i) or (ii). If $N_G^2(W) \subseteq W$, then this implies that for every $u_i \in W$ we have $u_j \in W$ for all $u_j \in V(G)$ for which $d_G(u_i, u_j) = 2$. On the other hand, let $u_i \in W$ where there exists $v_0 \notin W$ for which $v_0 \in N_G^2(u_i)$, i.e., $d_G(u_i, v_0) = 2$. By condition (ii), there exists $u_j \in W$ such that $d_G(u_i, u_j) = 2$. In either cases, we have W as an independent semitotal dominating set of G. □

Theorem 3.3. Let $G = (V(G), E(G))$ be a connected noncomplete graph with order at least 3. Then G has an independent semitotal dominating set if and only if $\mathcal{C}_1(G) \cap \mathcal{C}_2(G) \neq \emptyset$ where $\mathcal{C}_1(G)$ is the collection of all semitotal dominating sets of G and $\mathcal{C}_2(G)$ is the collection of all independent sets of G.

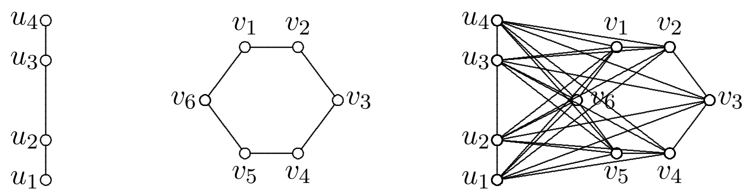
Proof. This is a straightforward characterization of graphs having at least one independent semitotal dominating set. □

Corollary 3.4. Let $G = (V(G), E(G))$ be a connected noncomplete graph with order at least 3. Then $\gamma_{it2}(G) = 2$ if and only if there exists a dominating set $W = \{u, v\}$ in G such that $d_G(u, v) = 2$.

Proof. Assume that graph G is connected and noncomplete with order at least 3. Suppose $\gamma_{it2}(G) = 2$. Let $W = \{u, v\}$ be an γ_{it2}-set of G. Then W is an independent dominating set in G such that $d_G(u, v) = 2$.

For the converse, let $W = \{u, v\}$ be a dominating set in G such that $d_G(u, v) = 2$. Clearly W is an independent semitotal dominating set of G. As a consequence, $\gamma_{it2}(G) = 2$. □

On the Join of Graphs. Recall that the *join* or *complete product* of two graphs G and H is the graph $G+H$ with vertex set $V(G+H) = V(G) \dot{\cup} V(H)$ and edge set $E(G+H) = E(G) \dot{\cup} E(H) \dot{\cup} \{uv : u \in V(G), v \in V(H)\}$, where the symbol $\dot{\cup}$ denotes disjoint union. The figure below provides an illustration of this binary operation.



(a) Path P_4 (b) Cycle C_6 (c) $P_4 + C_6$

Figure 2: The graphs P_4 and C_6, and the join $P_4 + C_6$

Theorem 3.5. Let G and H be graphs where at least one of them is noncomplete. A subset $W \subseteq V(G+H)$ is an independent semitotal dominating set of $G+H$ if and only if either $W \subseteq V(G)$ or $W \subseteq V(H)$, where W is a nonsingleton independent dominating set of either G or H, respectively.

Proof. From the general assumption and the definition of the sum of graphs, we have $G + H$ connected, noncomplete, and $|V(G + H)| \geq 3$. Now, assume first that $W \subseteq V(G + H)$ is an independent semitotal dominating set of $G + H$. Clearly, W contains two or more elements. Moreover, since W is an independent semitotal dominating set of $G + H$, it follows from the adjacency of the vertices in $G + H$ that either $W \subseteq V(G)$, where W is an independent dominating set in G, or $W \subseteq V(H)$ where W is an independent dominating set in H. From either cases, W comes out to be a nonsingleton independent dominating set in G or in H, respectively.

For the converse, suppose that W is a nonsingleton independent dominating set of G. Then by the adjacency of the vertices in $G+H$, we can see that every vertex of W is of distance 2 from any other vertex of W and that W dominates the entire $G+H$. Thus, W is an independent semitotal dominating set of $G+H$. The case where W is a nonsingleton independent dominating set of H also leads to a similar result. $\square$

Minimum Nonsingleton Independent Dominating Set. In this part, we introduce the concept of *minimum nonsingleton independent dominating set* of a noncomplete graph G, denoted by γ_{i^*}-set of G, as any nonsingleton independent dominating set whose cardinality is equal to $\gamma_i^*(G) = min\{|W| : W \ is \ a \ nonsingleton \ independent \ dominating \ set \ of \ G\}$, where $\gamma_i^*(G) = \gamma_i(G)$ whenever $\gamma_i(G) \neq 1$. In Figure 3, the γ_i-set of G is $\{u_1\}$, hence $\gamma_i(G) = 1$. But the sets $\{u_4, u_5\}$, $\{u_4, u_6\}$, $\{u_2, u_3, u_5\}$, and $\{u_2, u_3, u_6\}$ are the nonsingleton independent dominating sets of G. Hence, $\gamma_{i^*}(G) = 2$.

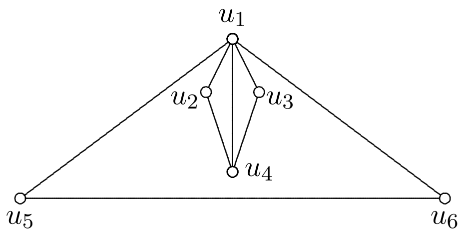


Figure 3: A graph G where $\gamma_{i^*}(G) = 2$

Corollary 3.6. If G and H are any graphs where at least one of them is noncomplete, then

$$\gamma_{it2}(G+H) = \begin{cases} \gamma_{i^*}(G) & if \ G \ is \ noncomplete \ and \ H \ is \ complete, \\ \gamma_{i^*}(H) & if \ G \ is \ complete \ and \ H \ is \ noncomplete, \\ min \ \{\gamma_{i^*}(G), \gamma_{i^*}(H)\} & if \ both \ G \ and \ H \ are \ noncomplete. \end{cases}$$

Proof. Whenever only one of G and H is noncomplete, Theorem 3.5 immediately implies that the γ_{it2}-sets of $G+H$ are the nonsingleton independent dominating sets of minimum cardinality of that noncomplete graph. On the other hand, if both G and H are noncomplete, then the minimum between $\gamma_{i^*}(G)$ and $\gamma_{i^*}(H)$ is the $\gamma_{it2}(G+H)$. $\square$

Example 3.7. In Figure 2, $\{u_1, u_3\}$, $\{u_2, u_4\}$, and $\{u_1, u_4\}$ are the γ_{i^*}-sets of P_4, so that $\gamma_{i^*}(P_4) = 2$. On the other hand, $\{v_1, v_4\}$, $\{v_2, v_5\}$ and $\{v_3, v_6\}$ are the γ_{i^*}-sets of C_6, so that $\gamma_{i^*}(C_6) = 2$. Hence, $\gamma_{it2}(P_4 + C_6) = 2$.

Theorem 3.5 provides an assurance that as long as H is noncomplete, the join $G' + H$ will always have $ISTd$-sets for any graph G'. Our last result in this paper, which provides a large family of graphs having $ISTd$-sets, is an application of Theorem 3.5.

Corollary 3.8. Every noncomplete graph G of order $n \geq 3$ having a spanning star has an $ISTd$-set.

Proof. Let $V(G) = \{u_1, u_2, ..., u_n\}$ with $deg_G(u_1) = n-1$, and let $G = G_1 + H$ with $V(G_1) = \{u_1\}$. Since G is noncomplete, it follows that H is noncomplete. As a consequence, H contains a nonsingleton independent subset which can be extended, if necessary, to a nonsingleton independent dominating set W of H. Clearly, by Theorem 3.5 W is an $ISTd$-set of $G = \{u_1\} + H$. This completes the proof. $\square$

4 Conclusion

Independent semitotal domination is squeezed between independent domination and semitotal domination. As introduced here, a subset $W \subseteq V(G)$ of a graph G is an independent semitotal dominating set of G, abbreviated $ISTd$-set of G, if W is an independent dominating set of G and every element of W is exactly of distance 2 from at least one other element of W. The independent semitotal domination number of G, denoted by $\gamma_{it2}(G)$, is the minimum cardinality of an $ISTd$-set of G. In this paper, we set the criteria of graphs for which independent semitotal domination exists and characterized the independent semitotal dominating set of join of these kinds of graphs. Finally, the corresponding γ_{it2} of this operation was obtained.

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