An investigation into The Properties of Functions Defining Distinguished Varieties

Original Research Paper

Abstract

An inner toral polynomial is a polynomial in $\mathbb{C}[z, w]$ such that its zero set is contained in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$, where $\mathbb{D}$ is the open unit disc, $\mathbb{T}$ is the unit circle and $\mathbb{E}$ is the exterior of the closed unit disc in $\mathbb{C}$([7], [2], [9]). Given such a polynomial $\mathfrak{p}$, it's zero set that lies inside $\mathbb{D}^2$, i.e $V = Z(\mathfrak{p}) \cap \mathbb{D}^2$ is called a distinguished variety, and $\mathfrak{p}$ is called a polynomial defining the distinguished variety V. An inner toral polynomial always gives a distinguished variety and vice versa. Finite Blaschke products generate inner toral polynomials such a way that, given a finite Blaschke product $B(z)$, the numerator of $w^m - B(z)$ is an inner toral polynomial. In this paper, we investigate the conditions that make the sum and the composition of inner toral polynomials generated by finite Blaschke products, inner toral. In fact, the sum of two inner toral polynomials is inner toral if the corresponding finite Blaschke products are of same degree with with same reflective coefficient and differ only by one linear factor. Further, a composition of an inner toral polynomial generated by a finite Blaschke product with an automorphism on the unit bidisk is always inner toral. Likewise, a composition with an automorphism on $\mathbb{C}^2$ is inner toral if the automorphism has the form (az, cw) with $\mid a \mid = \mid c \mid = 1$.

Keywords: Automorphisms, Blaschke products, Distinguished varieties, Inner toral polynomials

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1 Introduction

An algebraic variety is a subset of $\mathbb{C}^n$ that can be written as a set of common zeroes to a collection of polynomials in $\mathbb{C}[z_1, z_2, z_3, ..., z_n]$ ([11]). A distinguished variety is a special variety generated from a single two complex variables polynomial such that it lies in $\mathbb{D}^2$ and extends to $\mathbb{C}^2$ through the distinguished boundary $\mathbb{T}^2$, where $\mathbb{D}$ is the unit disc and $\mathbb{T}$ is the unit circle in the complex plane. Jim Agler and John E. McCarthy recently initiated the work on distinguished varieties ([8]), though the history goes back to Rudin's time. Study on distinguished varieties and function spaces

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on distinguished varieties as well as operators on such spaces later became interesting topics among researchers ([1], [4], [14], [5], [3]). In 2010, Greg Knese ([5]) gave a new proof of a representation for distinguished variety by using the sum of squares formula for two variables polynomials with no zeros on the bidisk. He also discussed the symmetricity properties of polynomials defining distinguished varieties: Polynomials defining distinguished varieties are symmetric with respect to the distinguished boundary $\mathbb{T}^2$. In this work, we focus on exploring the algebraic properties of polynomials defining distinguished varieties. The set of all polynomials defining distinguished varieties becomes closed under scalar multiplication and product naturally. However, as it turned out, the set of all polynomials defining distinguished varieties is not closed under addition and under composition. In ([5]), it was presented that given a finite Blaschke product $B(z)$, the zero set of $w^m - B(z)$ is a distinguished variety. In other words, the numerator of $w^m - B(z)$ is a polynomial defining distinguished variety. In this paper, we worked on polynomials generated by finite Blaschke products to investigate the summation and composition of polynomials defining distinguished varieties. In section 3 and 4, we present the results we obtained for the summation and the composition of such polynomials respectively.

2 Preliminaries

2.1 Polynomials Defining Distinguished Varieties

In this section we present the necessary preliminary materials so that the reader can smoothly move to the results section. Through out this article, we consider $\mathbb{D}$ to be the open unit disc, $\mathbb{T}$ to be the unit circle and $\mathbb{E}$ to be the exterior of the closed unit disc in the complex plane. Among the couple of versions of the definition of distinguished varieties available in the literature ([5],[8],[14]), we will stay with the following version:

Definition 2.1. [8] A non-empty set V in $\mathbb{D}^2$ is called a distinguished variety if there is a polynomial $\mathfrak{p} \in \mathbb{C}[z,w]$ such that

$$V = \{(z,w) \in \mathbb{D}^2 : \mathfrak{p}(z,w) = 0\}$$

and the extension of V to $\mathbb{C}^2$ is a subset of $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$.

In this case, we say $\mathfrak{p}$ defines the distinguished variety V and we call $\mathfrak{p}$ an *inner toral polynomial*. For example, $z^n - w^m$ for $n, m \in \mathbb{N}$ is an inner toral polynomial.

A polynomial in two complex variables z and w, $\mathfrak{p}(z,w) \in \mathbb{C}[z,w]$, is said to have *bidegree* (n,m) if $\mathfrak{p}$ has degree n in z and degree m in w ([1]). As it proved in ([5]), polynomials defining distinguished varieties are symmetric with respect to the distinguished boundary $\mathbb{T}^2$ in the following way:

Definition 2.2. [5] Let $\mathfrak{p} \in \mathbb{C}[z,w]$ be a polynomial of bidegree (n,m). The reflection polynomial $\widetilde{\mathfrak{p}}$ of $\mathfrak{p}$ at the degree (n,m) given by,

$$\widetilde{\mathfrak{p}}(z,w) = z^n w^m \overline{\mathfrak{p}\left(\frac{1}{\bar{z}}, \frac{1}{\bar{w}}\right)}.$$

Definition 2.3. [5] A polynomial $\mathfrak{p} \in \mathbb{C}[z,w]$ is said to be essentially $\mathbb{T}^2$-symmetric if there exists a unimodular constant c such that

$$\widetilde{\mathfrak{p}}(z,w) = c\, \mathfrak{p}(z,w).$$

We call this coefficient c *the reflective coefficient of* $\mathfrak{p}$.

Proposition 2.1. *[5] A Polynomial defining a distinguished variety is essentially* $\mathbb{T}^2$*-symmetric.*

For an example, the polynomial $\mathfrak{p}_1\left(z,w\right)=4w^2+izw^2-4z+i$ is an inner toral polynomial and defines a distinguished variety. Observe that $\widetilde{\mathfrak{p}_1}\left(z,w\right)=zw^2\overline{\left(\frac{4}{w^2}+\frac{i}{zw^2}-\frac{4}{z}+i\right)}=-1\left(4w^2+izw^2-4z+i\right)=-\mathfrak{p}_1\left(z,w\right)$. Therefore $\mathfrak{p}_1$ is essentially $\mathbb{T}^2$-symmetric with the reflective coefficient -1.

The rest of this chapter is devoted to discuss a special type of polynomials defining distinguished varieties: polynomials generated by Blaschke products.

Definition 2.4. [13] The rational function $B(z)=\zeta\prod_{i=1}^{k}\left(\frac{z-a_i}{1-\overline{a}_i z}\right)^{m_i}$ is called a finite Blaschke product of degree n, where $\zeta\in\mathbb{T}$, $a_i\in\mathbb{D}$, $n=\sum_{i=1}^{k}m_i$ and m_i is the multiplicity of the zero a_i for $i=1,2,3,...,k$.

Proposition 2.2. *[5] Let $B(z)$ be a finite Blaschke product of degree n. The set $V=\{(z,w)\in\mathbb{D}^2\mid w^m-B(z)=0\}$ is a distinguished variety.*

In other words, the numerator of $w^m-B(z)$ is a polynomial defining a distinguished variety (or an inner toral polynomial). Proposition 2.2 was given as just a statement in page 10 in ([5]). We present a quick proof for the claim for the sake of the completeness.

Proof. Note that

$$\mid B(z)\mid \text{ is } \begin{cases} =1 & \text{iff} \quad \mid z\mid=1 \\ <1 & \text{iff} \quad \mid z\mid<1 \\ >1 & \text{iff} \quad \mid z\mid>1 \end{cases}.$$

If (z_0,w_0) is a zero of $w^m-B(z)$, then $\mid w_0\mid^m=\mid B(z_0)\mid$. Observe that $\mid w_0\mid=1$ iff $\mid B(z_0)\mid=1$ iff $\mid z_0\mid=1$. Likewise, $\mid w_0\mid<1$ iff $\mid z_0\mid<1$ and $\mid w_0\mid>1$ iff $\mid z_0\mid>1$. Therefore, $(z_0,w_0)\in\mathbb{D}^2\cup\mathbb{T}^2\cup\mathbb{E}^2$. Therefore $\{(z,w)\in\mathbb{C}^2 : w^m-B(z)=0\}$ is a distinguished variety. Therefore, the numerator of $w^m-B(z)$ is inner toral. ☐

By Proposition 2.2, The numerator of $w^m-B(z)$, is an inner toral polynomial with the bidegree (n,m). For the rest of the article, we focus on such inner toral polynomials.

3 On the Summation of Inner Toral Polynomials Generated by Finite Blaschke Products

It is obvious that the scalar multiplication and products of inner toral polynomials are also inner toral. However, the summation of two inner toral polynomials is not always inner toral. For example, $z-w$ and z^2-w are two inner toral polynomials, however the sum, z^2+z-2w, is not inner toral, because $(-1,0)$ is a zero of z^2+z-2w, but it is not in $\mathbb{D}^2\cup\mathbb{T}^2\cup\mathbb{E}^2$. In this section, we investigate the factors that make a sum of two inner toral polynomials generated by Blaschke products inner toral, and we present the following results.

Proposition 3.1. *The reflective coefficient of an inner toral polynomial generated by the finite Blaschke product $B(z)=\zeta\prod_{i=1}^{k}\left(\frac{z-m_i}{1-\overline{a_i}z}\right)^{m_i}$ is $-\overline{\zeta}$.*

Proof. Let $\mathfrak{p}(z,w)$ be the numerator $w^m-B(z)$ and let $n=m_1+m_2+....+m_k$. Note that $\mathfrak{p}(z,w)=\prod_{i=1}^{k}\left(1-\overline{a_i}z\right)^{m_i}w^m-\zeta\prod_{i=1}^{k}\left(z-a_i\right)^{m_i}$ and by Proposition 2.2, $\mathfrak{p}$ is an inner toral polynomial of bidegree (n,m). Therefore, by Proposition 2.1, $\mathfrak{p}(z,w)$ is essentially $\mathbb{T}^2$-symmetric. To find the reflective coefficient, consider $\widetilde{\mathfrak{p}}(z,w)$.

$$\begin{aligned}
\widetilde{\mathfrak{p}}(z,w) &= w^m z^n \overline{\left\{\prod_{i=1}^{k}\left(1-\frac{\overline{a_i}}{\overline{z}}\right)^{m_i}\frac{1}{\overline{w}^m}-\zeta\prod_{i=1}^{k}\left(\frac{1}{\overline{z}}-a_i\right)^{m_i}\right\}}\\
&= w^m z^n\left\{\prod_{i=1}^{k}\left(1-\frac{a_i}{z}\right)^{m_i}\frac{1}{w^m}-\overline{\zeta}\prod_{i=1}^{k}\left(\frac{1}{\overline{z}}-\overline{a_i}\right)^{m_i}\right\}\\
&= \prod_{i=1}^{k}\left(\frac{z-a_i}{z}\right)^{m_i}z^n-\overline{\zeta}w^m z^n\prod_{i=1}^{k}\left(\frac{1-\overline{a_i}z}{z}\right)^{m_i}\\
&= \overline{\zeta}\left\{\prod_{i=1}^{k}\left(1-\overline{a_i}z\right)^{m_i}w^m-\zeta\prod_{i=1}^{k}\left(z-a_i\right)^{m_i}\right\}\\
&= -\overline{\zeta}\mathfrak{p}(z,w).
\end{aligned}$$

Therefore the reflective coefficient of $\mathfrak{p}$ is $-\overline{\zeta}$. □

Proposition 3.2. *Let $B_t\,(z)$ is the Blacshke product of degree n given by $B_t\,(z)=\zeta_t\prod_{i=1}^{k_t}\left(\frac{z-a_{ti}}{1-\overline{a_{ti}}z}\right)^{m_{ti}}$ where $t=1,2$, $i=1,2,...,k_t$ and $n=\sum_{i=1}^{n_t}m_{ti}$. Let $\mathfrak{p}_t\,(z,w)$ be the inner toral polynomial generated by $B_t\,(z)$. If $\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_2$ is inner toral, then $\zeta_1=\zeta_2$.*

Proof. Note that $\mathfrak{p}_t\,(z,w)=\prod_{i=1}^{k_t}(1-\overline{a_{ti}}z)^{m_{ti}}\,w^m-\zeta_t\prod_{i=1}^{n_t}(z-a_{ti})^{m_{ti}}$ for $t=1,2$. By Proposition 2.2 , both $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are inner toral polynomials with bidegree (n,m). Note that $\mathfrak{p}\,(z,w)=\mathfrak{p}_1\,(z,w)+\mathfrak{p}_2\,(z,w)$ has bidegree (n,m) if $\zeta_1\neq-\zeta_2$. Now, assuming $\zeta_1\neq-\zeta_2$, we have

$$\widetilde{\mathfrak{p}}=\widetilde{\mathfrak{p}_1+\mathfrak{p}_2}=w^m z^n\overline{(\mathfrak{p}_1+\mathfrak{p}_1)\left(\frac{1}{\overline{z}},\frac{1}{\overline{w}}\right)}=w^m z^n\overline{\mathfrak{p}_1\left(\frac{1}{\overline{z}},\frac{1}{\overline{w}}\right)}+w^m z^n\overline{\mathfrak{p}_2\left(\frac{1}{\overline{z}},\frac{1}{\overline{w}}\right)}=\widetilde{\mathfrak{p}_1}+\widetilde{\mathfrak{p}_2}.$$

By Proposition 3.1, $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are essentially $\mathbb{T}^2$-symmetric with reflective coefficients $-\overline{\zeta_1}$ and $-\overline{\zeta_2}$ respectively. Therefore $\widetilde{\mathfrak{p}}=-\overline{\zeta_1}\mathfrak{p}_1\,(z,w)-\overline{\zeta_2}\mathfrak{p}_2\,(z,w)$. If $\mathfrak{p}$ is inner toral, then by Proposition 2.1, $\mathfrak{p}$ is essentially $\mathbb{T}^2$-symmetric with reflective coefficient $-\overline{\zeta}$(say). That is $\widetilde{\mathfrak{p}}=-\overline{\zeta}\mathfrak{p}$. Therefore, $\zeta_1=\zeta_2=\zeta$. □

A sort of a converse to the above result can be given as follows:

Proposition 3.3. *Let $B\,(z)=\prod_{i=1}^{k-1}\left(\frac{z-a_i}{1-\overline{a_i}z}\right)^{m_i}$ where $m_i\in\mathbb{N}$ and $i=1,2,..(k-1)$. For $t=1,2$, let $B_t\,(z)=\zeta_t\;B\,(z)\left(\frac{z-a_{k_t}}{1-\overline{a_{k_t}}z}\right)$ and $\mathfrak{p}_t(z,w)$ be the inner toral polynomial generated by $B_t(z)$. If $\zeta_1=\zeta_2$, then $\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_2$ is an inner toral polynomial.*

Proof. Note that for $t=1,2$, $\mathfrak{p}_t\,(z,w)=\left[\prod_{i=1}^{k-1}(1-\overline{a_i}z)^{m_i}\right](1-\overline{a_{k_t}}z)\,w^m-\zeta_t\left[\prod_{i=1}^{k-1}(z-a_i)^{m_i}\right](z-a_{k_t})$. Suppose that $\zeta_1=\zeta_2=\zeta$ (say). A simple calculation shows that,

$$\mathfrak{p}\,(z,w)=\left[\prod_{i=1}^{k-1}(1-\overline{a_i}z)^{m_i}\right]\left[2-\overline{(a_{k_1}+a_{k_2})}z\right]w^m-\zeta\left[\prod_{i=1}^{k-1}(z-a_i)^{m_i}\right]\left[2z-(a_{k_1}+a_{k_2})\right].$$

Let $a = \left(\frac{a_{k_1}+a_{k_2}}{2}\right)$. Since $a_{k_1}, a_{k_2} \in \mathbb{D}$, a is also in $\mathbb{D}$. If $(z_0, w_0) \in Z(P)$, then $w_0^m = \zeta B\left(z_0\right)\left(\frac{z_0-a}{1-\bar{a}z_0}\right)$. It easily follows that $(z_0, w_0) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$. Therefore, $\mathfrak{p}$ is an inner toral polynomial. □

4 On the Composition of Inner Toral Polynomials Generated by Finite Blaschke Products

An automorphism is simply an isomorphism from one set to itself. Every automorphism on the unit disc $\mathbb{D}$, is of the form $\dfrac{az+b}{\bar{b}z+\bar{a}}$ where $a, b \in \mathbb{C}$ and $\mid a \mid^2 - \mid b \mid^2 = 1$. Alternatively, an automorphism on $\mathbb{D}$ has the form $e^{i\theta}\dfrac{z-c}{1-\bar{c}z}$ where $c \in \mathbb{D}$ and $\theta \in (0, 2\pi]$. Every automorphism on the bidisc $\mathbb{D}^2$, is of the form $f = (f_1, f_2)$, where f_1 and f_2 are automorphisms on the unit disc ([12],[10]). Every automorphism, f, on the complex plane is of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$. An automorphism on the two dimensional complex plane is of the form $f = (f_1, f_2)$ where f_k for $k = 1, 2$ are automorphism on the complex plane; that is an a automorphism $f : \mathbb{C}^2 \to \mathbb{C}^2$ can be given by $f(z_1, z_2) = (f_1(z_1), f_2(z_2))$ for all $(z_1, z_2) \in \mathbb{C}^2$ ([10]).

Definition 4.1. [6]Let $f = f(x_1, x_2, ..., x_n)$ be a function of n variables and let $g = (g_1, g_2, ..., g_n)$ be an n-tuple of functions. The composition f with g is defined as $f \circ g := f(g_1, g_2, ..., g_n)$.

Proposition 4.1. *Let $B(z)$ be a finite Blaschke product of degree n and let $\mathfrak{p}(z, w)$ be the inner toral polynomial generated by $B(z)$. If f is an automorphism on the bidisc, then the composition of $\mathfrak{p}$ with f, $\mathfrak{p} \circ f$ also defines an inner toral polynomial.*

Proof. Let $B(z)$= $\zeta \prod_{i=1}^{k}\left(\dfrac{z-a_i}{1-\overline{a_i}z}\right)^{m_i}$ where $\zeta \in \mathbb{T}$, $a_i \in \mathbb{D}$, $n = \sum_{i=1}^{k} m_i$ and m_i is the multiplicity of the zero a_i for $i = 1, 2, 3, ..., k$. Recall that polynomial $\mathfrak{p}$ is of the form $\mathfrak{p}(z, w) = w^m \prod_{i=1}^{k}(1 - \overline{a_i}z)^{m_i} - \zeta\prod_{i=1}^{k}(z - a_i)^{m_i}$. Let f be an automorphism on the bidisc of the form $f = (f_1, f_2) = \left(e^{i\theta_1}\dfrac{z-a_1}{1-\overline{a_1}z}, e^{i\theta_2}\dfrac{w-a_2}{1-\overline{a_2}w}\right)$ where $a_1, a_2 \in \mathbb{D}$ and $\theta_1, \theta_2 \in (0, 2\pi]$. Observe that

$$p \circ f(z,w) = \left(e^{i\theta_2}\frac{w-a_2}{1-\overline{a_2}w}\right)^m \prod_{i=1}^{k}\left\{1-\overline{a_i}\left(e^{i\theta_1}\frac{z-a_1}{1-\overline{a_1}z}\right)\right\}^{m_i} - \zeta\prod_{i=1}^{k}\left\{\left(e^{i\theta_1}\frac{z-a_1}{1-\overline{a_1}z}\right)-a_i\right\}^{m_i}.$$

Let (z_0, w_0) be a zero of $\mathfrak{p} \circ f$, and let $Z_0 = e^{i\theta_1}\dfrac{z_0-a_1}{1-\overline{a_1}z_0}$ and $W_0 = e^{i\theta_2}\dfrac{w_0-a_2}{1-\overline{a_2}w_0}$. Observe that $\mathfrak{p} \circ f(Z_0, W_0) = W_0^m \prod_{i=1}^{k}(1 - \overline{a_i}Z_0)^{m_i} - \zeta\prod_{i=1}^{k}(Z_0 - a_i)^{m_i} = 0$. That is, (Z_0, W_0) is a zero of $\mathfrak{p}$. Consequently, we have $W_0^m \prod_{i=1}^{k}(1 - \overline{a_i}Z_0)^{m_i} = \zeta\prod_{i=1}^{k}(Z_0 - a_i)^{m_i}$. Taking the modulus of both sides, we have $\mid W_0 \mid^m = \prod_{i=1}^{k} \mid \dfrac{Z_0-a_i}{1-\overline{a_i}Z_0} \mid^{m_i}$.

If $W_0 \in \mathbb{D}$, then $\mid \dfrac{Z_0-a_i}{1-\overline{a_i}Z_0} \mid < 1$ for $a_i \in \mathbb{D}$ where $i = 1, 2, 3, .., k$, and hence $Z_0 \in \mathbb{D}$. Likewise, if $Z_0 \in \mathbb{D}$, then $W_0 \in \mathbb{D}$. Similarly, if $Z_0 \in \mathbb{T}$, then $W_0 \in \mathbb{T}$ and vice versa, and if $Z_0 \in \mathbb{E}$, then

$W_0 \in \mathbb{E}$ and vice versa. Therefore $(Z_0, W_0) \in \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$. Solving for z_0 and w_0, we have $Z_0 = e^{-i\theta_1}\dfrac{z_0 - e^{i\theta_1}a_1}{1-\overline{e^{i\theta_1}a_1}z_0}$ and $W_0 = e^{-i\theta_2}\dfrac{w_0 - e^{i\theta_2}a_2}{1-\overline{e^{i\theta_2}a_2}w_0}$. Since $(Z_0, W_0) \in \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$, so is (z_0, w_0). Therefore, $\mathfrak{p} \circ f$ defines an inner toral polynomial. ☐

Proposition 4.2. *Let $B(z) = \zeta z^n$, where $\zeta \in \mathbb{T}$ and let $\mathfrak{p}$ be the inner toral polynomial generated by $B(z)$. Let f be an automorphism on the two dimensional complex plane $\mathbb{C}^2$. The composition $\mathfrak{p} \circ f$ defines an inner toral polynomial iff f has the form $f = (az, cw)$ with $\mid a \mid^n = \mid c \mid^m$.*

Proof. Suppose that f has the form $f = (az, cw)$ with $\mid a \mid^n = \mid c \mid^m$. Let (z_0, w_0) be a zero of $\mathfrak{p} \circ f$. So we have $\mathfrak{p} \circ f(z_0, w_0) = (cw_0)^m - \zeta(az_0)^n = 0$. That is $(cw_0)^m = \zeta(az_0)^n$. Taking the modulus of both sides, since $\mid a \mid^n = \mid c \mid^m$, we have $\mid z_0 \mid^n = \mid w_0 \mid^m$. Therefore $(z_0, w_0) \in \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$ and $\mathfrak{p} \circ f$ defines an inner toral polynomial.

Conversely, suppose that polynomial $\mathfrak{p} \circ f$ defines an inner toral polynomial. By Proposition 2.1, $\mathfrak{p} \circ f$ is essentially $\mathbb{T}^2$ symmetric. That is, $(cw+d)^m - \delta(az+b)^n = \gamma\{z^n(\bar{c} + \bar{d}w)^m - \bar{\delta}(\bar{a} + \bar{b}z)^n\}$ with $\mid \gamma \mid = 1$. By comparing coefficient, $\mid a \mid^n = \mid c \mid^m$ and $b = d = 0$. Therefore, f has the form $f = (az, cw)$ and $\mid a \mid^n = \mid c \mid^m$. ☐

We further generalize this result in the following way. Proposition 4.3 provides the necessary and sufficient condition for a composition of an inner toral polynomial generated by a finite Blaschke product with an automorphism on $\mathbb{C}^2$ to be inner toral.

Proposition 4.3. *Let $B(z)$ be a finite Blaschke product of degree n and $\mathfrak{p}$ be the inner toral polynomial generated by $B(z)$. If f is an automorphism on $\mathbb{C}^2$, then $\mathfrak{p} \circ f$ defines an inner toral polynomial iff $f = (az, cw)$ with $\mid a \mid = \mid c \mid = 1$.*

Proof. Let $B(z) = \zeta \prod_{i=1}^{k}\left(\dfrac{z-a_i}{1-\overline{a_i}z}\right)^{m_i}$, where $\zeta \in \mathbb{T}$, $a_i \in \mathbb{D}$, $n = \sum_{i=1}^{k} m_i$ and m_i is the zero of a_i for $i = 1, 2, 3, ..., k$. The polynomial $\mathfrak{p}$ has the form $\mathfrak{p}(z, w) = w^m \prod_{i=1}^{k}(1-\overline{a_i}z)^{m_i} - \zeta\prod_{i=1}^{k}(z-a_i)^{m_i}$. Suppose that $f \in$ Aut($\mathbb{C}^2$) has the form $f = (az, cw)$ with $\mid a \mid = \mid c \mid = 1$. For a zero (z_0, w_0) of $\mathfrak{p} \circ f$, we have $\mathfrak{p} \circ f(z_0, w_0) = (cw_0)^m \prod_{i=1}^{k}(1-\overline{a_i}az_0)^{m_i} - \zeta\prod_{i=1}^{k}(a_0z_0 - a_i)^{m_i} = 0$. That is, $(cw_0)^m \prod_{i=1}^{k}(1-\overline{a_i}az_0)^{m_i} = \zeta\prod_{i=1}^{k}(az_0 - a_i)^{m_i}$. Taking the modulus of both sides, since $\mid a \mid = \mid c \mid = 1$, we have $\mid w_0 \mid^m = \prod_{i=1}^{k} \mid \dfrac{z_0 - (\frac{a_i}{a})}{1-\overline{(\frac{a_i}{a})}z_0} \mid^{m_i}$. Therefore $(z_0, w_0) \in \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$ and $\mathfrak{p} \circ f$ defines an inner toral polynomial.

Conversely, let f be an automophism on $\mathbb{C}^2$ and suppose that $f \neq (az, cw)$, or $\mid a \mid \neq 1$ or $\mid c \mid \neq 1$. If $f \neq (az, cw)$, then f has the form $f = (az+b, cw)$ or $f = (az, cw+d)$ or $f = (az+b, cw+d)$ where $b, d \neq 0$. Now we consider the two cases separately.

Case 1: $f \neq (az, cw)$

Suppose $f = (az+b, cw)$ for $b \neq 0$. Let $B_1(z) = \dfrac{z-(1/2)}{1-(1/2)z}$ and $\mathfrak{p}_1(z, w) = w\{1-(1/2)z\} - \{z-(1/2)\}$ be the inner toral polynomial generated by $B_1(z)$. Also let $f_1 = (z+1, w)$. Note that, $\mathfrak{p}_1 \circ f_1(z, w) = w\{1-(1/2)(z+1)\} - \{(z+1)-(1/2)\}$ and $(1/2, 4)$ is a zero of $\mathfrak{p} \circ f$. However, $(1/2, 4) \notin \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$. Therefore $\mathfrak{p}_{1_\circ} f_1$ does not define an inner toral polynomial. Similarly, if $f = (az, cw+d)$ with $d \neq 0$, we can make choices for f and $\mathfrak{p}$ such that $\mathfrak{p} \circ f$ does not define an inner toral polynomial. Now suppose Suppose $f = (az+b, cw+d)$ for $b, d \neq 0$. Let $f_2 = (z+2, w+1)$, $B_2(z) = \dfrac{z-(1/2)}{1-(1/2)z}$, and $\mathfrak{p}_2(z,w) = w\{1-(1/2)z\} - \{z-(1/2)\}$ be the inner toral polynomial generated by $B_2(z)$. Observe that $(0,-6)$ is a zero of $\mathfrak{p}_2 \circ f_2$. However, $(0,-6) \notin \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$. Therefore $\mathfrak{p}_{2\circ} f_2$ does not define an inner toral polynomial.

Case 2: $| a |\neq 1$ **or** $| c |\neq 1$

Without loss of generality, assume $| a |\neq 1$. Let $f_3 = (2z, w)$, $B_3(z) = \dfrac{z-(1/2)}{1-(1/2)z}$, and $\mathfrak{p}_3(z,w) = w\{1-(1/2)z\} - \{z-(1/2)\}$ be the inner toral polynomial generated by $B_3(z)$. Observe that $(1/2, 1)$ is a zero of $\mathfrak{p}_3 \circ f_3$ that does not lie in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$, and hence $\mathfrak{p}_3 \circ f_3$ does not define an inner toral polynomial.

Therefore, if $f \neq (az, cw)$, or $| a |\neq 1$ or $| c |\neq 1$, then $\mathfrak{p} \circ f$ does not define an inner toral polynomial. By taking the contrapositive, if $\mathfrak{p} \circ f$ defines an inner toral polynomial, then $f = (az, cw)$ with $| a |=| c |= 1$. $\square$

5 Conclusions

In this study we have proved that the reflective coefficient of a polynomial that define a distinguished variety, generated by a finite Blaschke product with Blaschke coefficient is $-\overline{\zeta}$ and if the sum of two polynomials that define distinguished variety generated by Blaschke products gives distinguished variety, then the Blaschke coefficients are the same. In addition, we proved that the sum of two polynomials that define distinguished variety generated by Blaschke products which differ only by one factor with multiplicity one, is defining distinguished variety iff their Blaschke coefficients are the same. Moreover, we prove that when we take the composition of a polynomial that defines a distinguished variety and that generated by Blaschke product, with an automorphism on the bidisc we get back a polynomial defining a distinguished variety. Further, we have generalized this result by considering the composition of such polynomials with automorphisms on the complex plane and obtained similar results.

6 Completing Interest

Authors have declared that no competing interests exist.

7 Authors Contribution

All authors contributed to the study conception and design. The first draft of the manuscript was written by Y. G. D. M. Dharmasiri and R. D. P. M. Wijerathne and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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