Original Research Article

On Rings Domination of Total Graph of Some Graph Families

Abstract

For a nontrivial connected graph G with no isolated vertex, a nonempty subset $D \subseteq V(G)$ is a rings dominating set if D is a dominating set and for each vertex $v \in V \setminus D$ is adjacent to at least two vertices in $V \setminus D$. Thus, the dominating set D of $V(G)$ is a rings dominating set if for all $v \in V \setminus D$, $|N(v) \cap (V \setminus D)| \geq 2$. Moreover, D is called a minimum rings dominating set if D is a rings dominating set of smallest size in a given graph. The cardinality of minimum rings dominating set of G is the rings domination number of G, denoted by $\gamma_{ri}(G)$. Here, we determine how the minimum rings dominating set is constructed in the total graph of some graph families with the inclusion of generated conditions for this type of domination and provide their respective rings domination number.

Keywords: total graph, rings dominating set, minimum rings dominating set, rings domination number.

2020 Mathematics Subject Classification: 05C35

1 Introduction

Rings domination is a newly presented parameter in graph theory that resembles the usual domination almost exactly. Given its uses in computer networks, electronics, and other fields, it has been a great area of study in graph theory, as mentioned in [10], [15], and [17], which motivates the writing of this paper. It plans to introduce the notion of rings dominating sets given over the total graph $T(G)$ of some graph G.

Additionally, rings domination put a condition on the $V \setminus D$ set in the graph. Hence, we will be studying the behavior of rings dominating set in the total graph of some graphs focusing on its $V \setminus D$ set. At the end of this paper, fundamental results are presented showcasing the conditions of minimum rings dominating set and rings domination number for the above-mentioned graphs.

2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [1], [5], [6], [11], [12].

Definition 2.1. [6] **(Total Graph)** The total graph of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G. The total graph of G is denoted by $T(G)$.

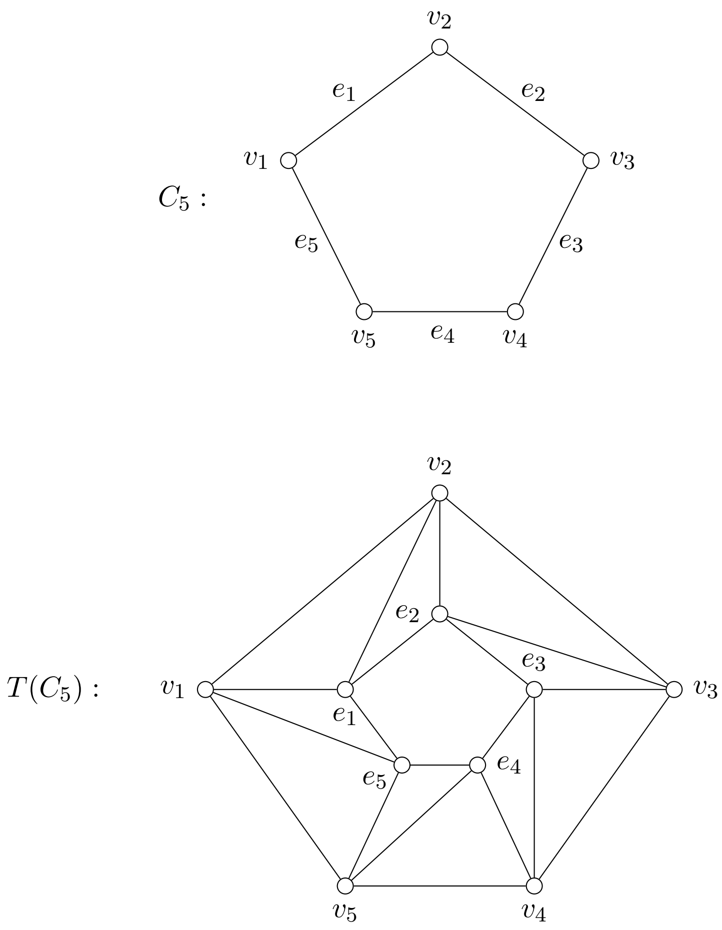

Figure 1: The total graph of C_5, $T(C_5)$

Definition 2.2. [1] **(Rings Dominating Set, Minimum Rings Dominating set, Rings Domination Number)** A dominating set $D \subseteq V(G)$ is a rings dominating set if each vertex $v \in V(G) \setminus D$ is adjacent to at least two vertices in $V(G) \setminus D$. Whereas, the rings dominating set of smallest size in a given graph is referred as minimum rings dominating set. The cardinality of the minimum rings dominating sets of G is called the rings domination number of G and is denoted by $\gamma_{ri}(G)$.

Example 2.1. *Refer on the graph of $T(C_5)$ shown in Figure 1. Suppose we take the vertices v_1 and e_3, then we are left with the subgraph shown in Figure 2. Clearly, the set $K = \{v_1, e_3\}$ is* *a dominating set. Note that each vertex in Figure 2 has degree 3. Hence, the set K is a rings dominating set. Moreover, K is rings dominating set of smallest size. Therefore, $\gamma_{ri}[T(C_5)] = 2$.*

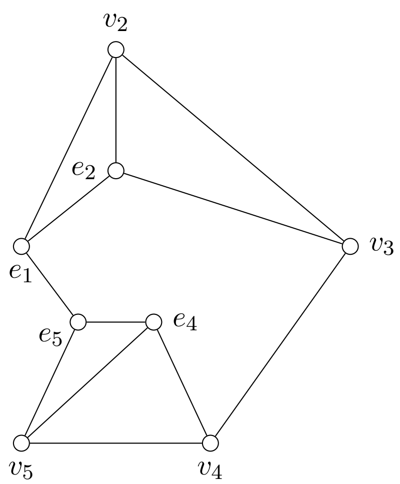

Figure 2: The induced subgraph $\langle V[T(C_5)] - K\rangle$ of $T(C_5)$

3 Main Results

In this section, the characteristics of minimum rings dominating set in the total graph of some graph families are presented. We also determine the rings domination number for each of the graphs being classified in this paper.

3.1 Rings Domination Number of the Total Graph of Path P_n, $T(P_n)$

Theorem 3.1. *Let $T(P_n)$ be the total graph of path of order n. Then*

$$\gamma_{ri}[T(P_n)] = \begin{cases} \frac{2n+5}{5}, & \text{if } n \equiv 0 (\text{mod } 5) \\ \frac{2n+3}{5}, & \text{if } n \equiv 1 (\text{mod } 5) \\ \frac{2n+6}{5}, & \text{if } n \equiv 2 (\text{mod } 5) \\ \frac{2n+4}{5}, & \text{if } n \equiv 3 (\text{mod } 5) \\ \frac{2n+7}{5}, & \text{if } n \equiv 4 (\text{mod } 5) \end{cases}$$

.

Proof. Let P_n be a path of order n and $T(P_n)$ be the total graph of P_n. For convenience, let $V(P_n) = \{v_1, v_2, v_3, ..., v_{n-1}, v_n\}$, $E(P_n) = \{e_1, e_2, e_3, ..., e_{n-2}, e_{n-1}\}$ where e_1 is the edge incident with v_1 and v_2, e_2 is the edge incident with v_2 and v_3, and so on and so forth. By definition of total graph, $V[T(P_n)] = V(P_n) \cup E(P_n)$. Suppose S is a rings dominating set of minimum cardinality, that is, S is a $\gamma_{ri} - set$, then we consider the following cases.

case 1: $n \equiv 0 \pmod 5$.

Let $K \subseteq V[T(P_n)]$ with $K = \{v_1, v_6, v_{11}, ..., v_{n-4}, v_{n-2}, v_n\} \cup \{e_3, e_8, e_{13}, ..., e_{n-17}, e_{n-12}, e_{n-7}\}$. Observe that K is a dominating set and for each vertex $v \in V[T(P_n)] \setminus K$, $deg(v)$ is either 2 or 3. Hence, K is a rings dominating set. To this end, notice that $|K| = \frac{2n+5}{5}$. Since S is a $\gamma_{ri} - set$, $|K| \geq |S|$. So, $|S| \leq |K| = \frac{2n+5}{5}$. On the other hand, since S is a $\gamma_{ri} - set$ of $T(P_n)$, then S must have at least $\frac{2n+5}{5}$ vertices in $T(P_n)$. Hence, $|S| \geq \frac{2n+5}{5}$. Therefore, $|S| = \frac{2n+5}{5}$.

case 2: $n \equiv 1 \pmod 5$.

Let $K \subseteq V[T(P_n)]$ with $K = \{v_1, v_6, v_{11}, ..., v_{n-10}, v_{n-5}, v_n\} \cup \{e_3, e_8, e_{13}, ..., e_{n-13}, e_{n-8}, e_{n-3}\}$. Observe that K is a dominating set and for each vertex $v \in V[T(P_n)] \setminus K$, $deg(v)$ is either 2 or 3. Hence, K is a rings dominating set. To this end, notice that $|K| = \frac{2n+3}{5}$. Since S is a $\gamma_{ri} - set$, $|K| \geq |S|$. So, $|S| \leq |K| = \frac{2n+3}{5}$. On the other hand, since S is a $\gamma_{ri} - set$ of $T(P_n)$, then S must have at least $\frac{2n+3}{5}$ vertices in $T(P_n)$. Hence, $|S| \geq \frac{2n+3}{5}$. Therefore, $|S| = \frac{2n+3}{5}$.

case 3: $n \equiv 2 \pmod 5$.

Let $K \subseteq V[T(P_n)]$ with $K = \{v_1, v_6, v_{11}, ..., v_{n-6}, v_{n-2}, v_n\} \cup \{e_3, e_8, e_{13}, ..., e_{n-14}, e_{n-9}, e_{n-4}\}$. Observe that K is a dominating set and for each vertex $v \in V[T(P_n)] \setminus K$, $deg(v)$ is either 2 or 3. Hence, K is a rings dominating set. To this end, notice that $|K| = \frac{2n+6}{5}$. Since S is a $\gamma_{ri} - set$, $|K| \geq |S|$. So, $|S| \leq |K| = \frac{2n+6}{5}$. On the other hand, since S is a $\gamma_{ri} - set$ of $T(P_n)$, then S must have at least $\frac{2n+6}{5}$ vertices in $T(P_n)$. Hence, $|S| \geq \frac{2n+6}{5}$. Therefore, $|S| = \frac{2n+6}{5}$.

case 4: $n \equiv 3 \pmod 5$.

Let $K \subseteq V[T(P_n)]$ with $K = \{v_1, v_6, v_{11}, ..., v_{n-7}, v_{n-2}, v_n\} \cup \{e_3, e_8, e_{13}, ..., e_{n-15}, e_{n-10}, e_{n-5}\}$. Observe that K is a dominating set and for each vertex $v \in V[T(P_n)] \setminus K$, $deg(v)$ is either 2 or 3. Hence, K is a rings dominating set. To this end, notice that $|K| = \frac{2n+4}{5}$. Since S is a $\gamma_{ri} - set$, $|K| \geq |S|$. So, $|S| \leq |K| = \frac{2n+4}{5}$. On the other hand, since S is a $\gamma_{ri} - set$ of $T(P_n)$, then S must have at least $\frac{2n+4}{5}$ vertices in $T(P_n)$. Hence, $|S| \geq \frac{2n+4}{5}$. Therefore, $|S| = \frac{2n+4}{5}$.

case 5: $n \equiv 4 \pmod 5$.

Let $K \subseteq V[T(P_n)]$ with $K = \{v_1, v_6, v_{11}, ..., v_{n-8}, v_{n-3}, v_n\} \cup \{e_3, e_8, e_{13}, ..., e_{n-11}, e_{n-6}, e_{n-1}\}$. Observe that K is a dominating set and for each vertex $v \in V[T(P_n)] \setminus K$, $deg(v)$ is either 2 or 3. Hence, K is a rings dominating set. To this end, notice that $|K| = \frac{2n+7}{5}$. Since S is a $\gamma_{ri} - set$, $|K| \geq |S|$. So, $|S| \leq |K| = \frac{2n+7}{5}$. On the other hand, since S is a $\gamma_{ri} - set$ of $T(P_n)$, then S must have at least $\frac{2n+7}{5}$ vertices in $T(P_n)$. Hence, $|S| \geq \frac{2n+7}{5}$. Therefore, $|S| = \frac{2n+7}{5}$. □

Example 3.2. *Refer on Figure 3. Clearly,* $K = \{v_1, v_3, v_5\}$ *is dominating set. Observe that, for every vertex* $v \in V[T(P_n)] \setminus K$, $deg\ v = 2$ *or* $deg\ v = 3$. *Thus,* K *is a rings dominating set. Here,* K *is a rings dominating set of smallest size, so* $\gamma_{ri}[T(P_5)] = 3$. *Using Theorem 3.1, for* $n = 5 \equiv 0\ (mod\ 5)$,

$$\gamma_{ri}[T(P_5)] = \frac{2n+5}{5} = \frac{2(5)+5}{5} = \frac{15}{5} = 3.$$

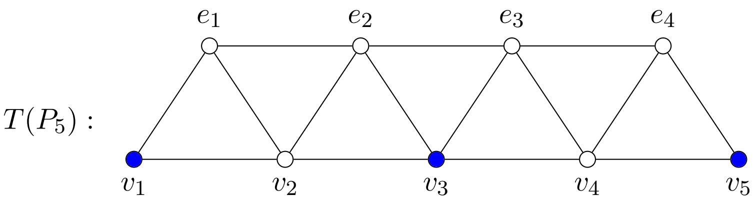

Figure 3: The minimum rings dominating set of $T(P_5)$

3.2 Rings Domination Number of the Total Graph of Cycle C_n, $T(C_n)$

A cycle C_n has n vertices and n edges. From the definition of total graph, the order of $T(C_n)$ is the sum of the order and size of C_n. So, there are $2n$ vertices in $T(C_n)$. Also, observe that $v \in V(C_n)$ is adjacent to two vertices in the same set and is incident to two edges in $E(C_n)$. Moreover, $e \in E(C_n)$ is adjacent to two other edges in the same set and incident to two vertices in $V(C_n)$. Hence, $deg\, v = deg\, e = 4$, that is, $T(C_n)$ is 4-regular. We will use this fact in the proof of some results below. Now, for convenience, let $V[T(C_n)] = V(C_n) \cup E(C_n) = \{v_1, v_2, ..., v_n\} \cup \{v'_1, v'_2, ..., v'_n\}$ and the following shows few outputs on the total graph $T(C_n)$.

Lemma 3.3. *For $n = 5k$ and $k \geq 1$, $S = \{v_1, v_6..., v_{n-4}\} \cup \{v'_3, v'_8, ...v'_{n-2}\}$ is a minimum rings dominating set of $T(C_n)$ if and only if for every $u, v \in S$, $N[u] \cap N[v] = \emptyset$ with $u \neq v$.*

Proof. For $n = 5k$, suppose $S = \{v_1, v_6..., v_{n-4}\} \cup \{v'3, v'8, ...v'_{n-2}\}$ is a minimum rings dominating set and there exists $u, v \in S$ such that $N[u] \cap N[v] \neq \emptyset$, then

$$|N[S]| = |N[v_1] \cup N[v_6] \cup ... \cup N[v_{n-4}] \cup N[v'_3] \cup N[v'_8] \cup ... \cup N[v'_{n-2}]| \tag{3.1}$$
$$< |N[v_1]| + |N[v_6]| + ... + |N[v_{n-4}]| + |N[v'_3]| + |N[v'_8]| + ... + |N[v'_{n-2}]| \tag{3.2}$$
$$= 5(2k) \tag{3.3}$$
$$= 2(5k) \tag{3.4}$$
$$= 2n \tag{3.5}$$

which implies that S is not a dominating set.

Conversely, if $u, v \in S$ and no pair of vertices share neighbors, then $|N[v_1]| = |N[v_6]| = ... = |N[v_{n-4}]| = |N[v'_3]| = |N[v'_8]| = ... = |N[v'_{n-2}]| = 5$ as $T(C_n)$ is 4-regular. Hence,

$$|N[S]| = |N[v_1] \cup N[v_6] \cup ... \cup N[v_{n-4}] \cup N[v'_3] \cup N[v'_8] \cup ... \cup N[v'_{n-2}]| \tag{3.6}$$
$$= |N[v_1]| + |N[v_6]| + ... + |N[v_{n-4}]| + |N[v'_3]| + |N[v'_8]| + ... + |N[v'_{n-2}]| \tag{3.7}$$
$$= 5(2k) \tag{3.8}$$
$$= 2(5k) \tag{3.9}$$
$$= 2n \tag{3.10}$$

Thus, $N[S] = V[T(C_n)]$. Clearly, S is a dominating set. To this end, observe that for every vertex $w \in V[T(C_n)] \setminus S$, $deg\ w = 3$. With this, S is a rings dominating set. It only remains to show that S is a rings dominating set of minimum cardinality. Note that $|S| = 2k$. Now, suppose $|S| = 2k - 1$. Then,

$$|N[S]| = 5(2k - 1) = 10k - 5 < 10k = 2n,$$

a contradiction since this shows that S is not a dominating set. Therefore, S with $|S| = 2k$ is a minimum rings dominating set. □

For an example, refer on Example 2.1. Also, below shows some minimum rings dominating set in $T(C_5)$ which are identified by colors. Thus, vertices of the same color is an example of minimum rings dominating set.

$T(C_5)$:

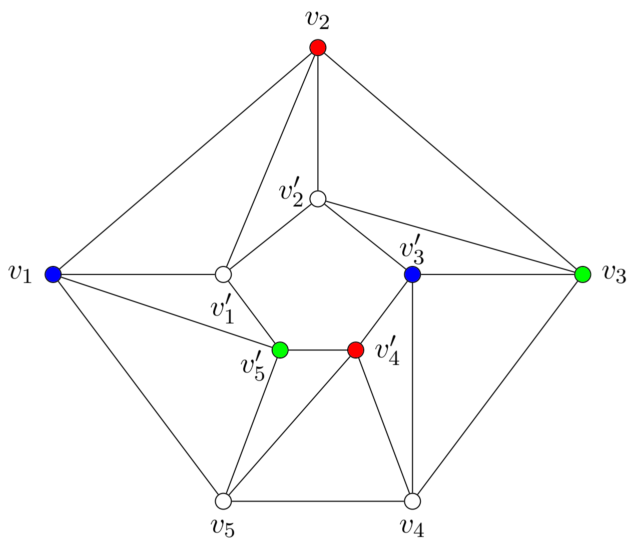

Figure 4: The total graph of C_5, $T(C_5)$

Theorem 3.4. *Let $T(C_n)$ be the total graph of cycle C_n of order $n \geq 3$. Then*

$$\gamma_{ri}[T(C_n)] = \left\lceil \frac{2n}{5} \right\rceil.$$

Proof. Let $T(C_n)$ be the total graph of cycle C_n of order n. Now suppose S is a $\gamma_{ri} - set$ of $T(C_n)$, that is, S is a rings dominating set of minimum cardinality, then we consider following cases.

case 1: $n \equiv 0 \pmod 5$

From Lemma 3.3, for $n = 5k$ and $k \geq 1$, $S = \{v_1, v_6..., v_{n-4}\} \cup \{v_3', v_8', ...v_{n-2}'\}$ is a minimum rings dominating set of $T(C_n)$. Notice that $|S| = 2k$. Clearly,

$$2k = \frac{10k}{5} = \frac{2(5k)}{5} = \frac{2n}{5} = \left\lceil \frac{2n}{5} \right\rceil$$

as $\frac{2n}{5}$ is always an integer.

case 2: $n \equiv 1 \pmod 5$

Since $n \geq 3$, we only observe the total graph $T(C_n)$ with $n = 6, 11, 16, ...$. Choose $R = \{v_1, v_6, ..., v_{n-5}\} \cup \{v_3', v_8', ..., v_{n-3}'\}$. Observe that R with $|R| = \lceil \frac{2n}{5} \rceil - 1$ is not a dominating set as there exists $u \in V[T(C_n)] \setminus R$ that are not dominated by any vertex in R. Notice that $|V[T(C_n)] \setminus N[R]| = 2$ and $\langle V[T(C_n)] \setminus N[R] \rangle \cong P_2$. Now, take a unique vertex $u \in V[T(C_n)] \setminus N[R]$, then $R \cup \{u\}$ is now a dominating set and every vertex $x \in V[T(C_n)] \setminus (R \cup \{u\})$, $deg\, x = 2$ or $deg\, x = 3$. Hence, T is a rings dominating set. For coherence, let $T = R \cup \{u\}$, then

$$|T| = |R \cup \{u\}| = |R| + |\{u\}| = \left(\left\lceil \frac{2n}{5} \right\rceil - 1 \right) + 1 = \left\lceil \frac{2n}{5} \right\rceil.$$

On this account, $|S| \leq |T| = \lceil \frac{2n}{5} \rceil$ as by assumption, S is a $\gamma_{ri} - set$. On the other hand, again since S is a rings dominating set of minimum cardinality of $T(C_n)$, then S must have at least $\lceil \frac{2n}{5} \rceil$ vertices in $T(C_n)$. Hence, $|S| \geq \lceil \frac{2n}{5} \rceil$. Therefore, $|S| = \lceil \frac{2n}{5} \rceil$.

case 3: $n \equiv 2 \pmod 5$

Here, we observe the total graph $T(C_n)$ with $n = 7, 12, 17, ...$. Choose $R = \{v_1, v_6, ..., v_{n-6}\} \cup \{v_3', v_8', ..., v_{n-4}'\}$. Observe that R with $|R| = \lceil \frac{2n}{5} \rceil - 1$ is not a dominating set as there exists $u \in V[T(C_n)] \setminus R$ that are not dominated by any vertex in R. Notice that $|V[T(C_n)] \setminus N[R]| = 4$. Now, take a unique vertex $u \in V[T(C_n)] \setminus N[R]$ such that each $u \neq w \in V[T(C_n)] \setminus N[R]$, $w \in N(u)$. Then $R \cup \{u\}$ is now a dominating set and every vertex $x \in V[T(C_n)] \setminus (R \cup \{u\})$, $deg\, x = 2$ or $deg\, x = 3$. Hence, T is a rings dominating set. This time, let $T = R \cup \{u\}$, then

$$|T| = |R \cup \{u\}| = |R| + |\{u\}| = \left(\left\lceil \frac{2n}{5} \right\rceil - 1 \right) + 1 = \left\lceil \frac{2n}{5} \right\rceil.$$

As a result, $|S| \leq |T| = \lceil \frac{2n}{5} \rceil$ as by assumption, S is a $\gamma_{ri} - set$. On the other hand, again since S is a rings dominating set of minimum cardinality of $T(C_n)$, then S must have at least $\lceil \frac{2n}{5} \rceil$ vertices in $T(C_n)$. Hence, $|S| \geq \lceil \frac{2n}{5} \rceil$. Therefore, $|S| = \lceil \frac{2n}{5} \rceil$.

case 4: $n \equiv 3 \pmod 5$

In similar manner, choose $R = \{v_1, v_6, ..., v_{n-2}\} \cup \{v_3', v_8', ..., v_{n-5}'\}$. Observe that R with $|R| = \lceil \frac{2n}{5} \rceil - 1$ is not a dominating set as there exists a unique vertex in $V[T(C_n)] \setminus R$ which is v_{n-1}' that is not dominated by any vertex in R. Taking this vertex or any of its neighbors would result to a dominating set $R \cup \{v_{n-1}'\}$. Further, observe that every vertex $x \in V[T(C_n)] \setminus (R \cup \{v_{n-1}'\})$, $deg\, x = 2$ or $deg\, x = 3$. Hence, T is a rings dominating set. To this end, let $T = R \cup \{v_{n-1}'\}$, then

$$|T| = |R \cup \{v_{n-1}'\}| = |R| + |\{v_{n-1}'\}| \tag{3.11}$$

$$= \left(\left\lceil \frac{2n}{5} \right\rceil - 1 \right) + 1 = \left\lceil \frac{2n}{5} \right\rceil. \tag{3.12}$$

Consequently, $|S| \leq |T| = \lceil \frac{2n}{5} \rceil$ as by assumption, S is a $\gamma_{ri} - set$. On the other hand, again since S is a rings dominating set of minimum cardinality of $T(C_n)$, then S must have at least $\lceil \frac{2n}{5} \rceil$ vertices in $T(C_n)$. Hence, $|S| \geq \lceil \frac{2n}{5} \rceil$. Therefore, $|S| = \lceil \frac{2n}{5} \rceil$.

case 5: $n \equiv 4 \pmod 5$

In this case, choose $R = \{v_1, v_6, ..., v_{n-3}\} \cup \{v'_3, v'_8, ..., v'_{n-6}\}$. Observe that R with $|R| = \lceil \frac{2n}{5} \rceil - 1$ is not a dominating set as there exists $u \in V[T(C_n)] \setminus R$ that are not dominated by any vertex in R. Notice that $|V[T(C_n)] \setminus N[R]| = 3$ and $\langle V[T(C_n)] \setminus N[R] \rangle \cong C_3$. Now, take a unique vertex $u \in V[T(C_n)] \setminus N[R]$, then $R \cup \{u\}$ is now a dominating set and every vertex $x \in V[T(C_n)] \setminus (R \cup \{u\})$, $deg\, x = 2$ or $deg\, x = 3$. Hence, T is a rings dominating set. Now, let $T = R \cup \{v'_{n-1}\}$, then

$$|T| = |R \cup \{v'_{n-1}\}| = |R| + |\{v'_{n-1}\}| \tag{3.13}$$

$$= \left(\left\lceil \frac{2n}{5} \right\rceil - 1 \right) + 1 = \left\lceil \frac{2n}{5} \right\rceil . \tag{3.14}$$

Consequently, $|S| \leq |T| = \lceil \frac{2n}{5} \rceil$ as by assumption, S is a $\gamma_{ri} - set$. On the other hand, again since S is a rings dominating set of minimum cardinality of $T(C_n)$, then S must have at least $\lceil \frac{2n}{5} \rceil$ vertices in $T(C_n)$. Hence, $|S| \geq \lceil \frac{2n}{5} \rceil$. Therefore, $|S| = \lceil \frac{2n}{5} \rceil$.

□

Example 3.5. *The shaded nodes of the same color in $T(C_8)$ is an example of minimum rings dominating set.*

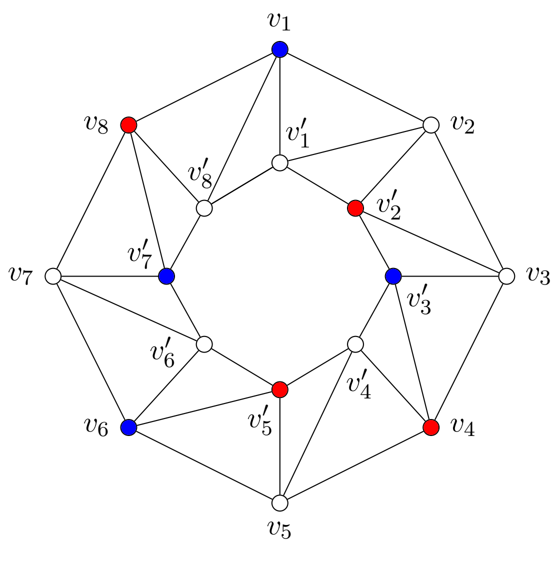

Figure 5: The minimum rings dominating sets in $T(C_8)$

3.3 Rings Domination Number of the Total Graph of Complete graph K_n, $T(K_n)$

We know that a complete graph denoted by K_n is a graph where every pair of distinct vertices is connected by a unique edge. The graph considered here is simple (no loops and multiple edges) and is undirected graph. Now, the following is a result in the total graph of complete graph.

Theorem 3.6. *Let $T(K_n)$ be the total graph of complete graph K_n of order $n \geq 3$. Then*

$$\gamma_{ri}[T(K_n)] = \left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil .$$

Proof. From [12], the order of $T(K_n)$ is $\frac{n(n+1)}{2}$ and $T(k_n)$ is $2(n-1)$ - regular. Choose $R = \{u_i \in V[T(K_n)] : N[u_i] \cap N[u_j] = \emptyset, i \neq j\}$. Note that we can always choose consecutive disjoint neighborhoods. Now, observe that R with $|R| = \left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil - 1$ is not a dominating set since there exists vertex $v \in V[T(K_n)] \setminus R$ that are not dominated by any vertex in R. Suppose set M contains these vertices, $v \in V[T(K_n)] \setminus N[R]$ and $N[u_1] \cup N[u_2] \cup ... \cup N[u_{\lceil \frac{n(n+1)}{4(n-1)} \rceil - 1}] \cup M = V[T(K_n)]$, then we are forced to choose another unique vertex $v \in M$ to be in R such that for every $v \neq w \in M$, $w \in N(v)$ so that $R = \{u_1, u_2, ..., u_{\lceil \frac{n(n+1)}{4(n-1)} \rceil - 1}, v\}$ now is a dominating set. At this time, for every vertex $x \in V[T(K_n)] \setminus R$, $deg\, x = 2n-3$ or $deg\, x = 2n-4$, both of which is greater than or equal to 2 as $n \geq 3$. Hence, R is a rings dominating set. Notice that,

$$|R| = \left| \{u_1, u_2, ..., u_{\lceil \frac{n(n+1)}{4(n-1)} \rceil - 1}\} \right| + |\{v\}| \tag{3.15}$$

$$= \left(\left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil - 1 \right) + 1 = \left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil . \tag{3.16}$$

To this end, assume S to be a γ_{ri}-set, that is, S is a rings dominating set of minimum cardinality. This implies that

$$|S| \leq |R| = \left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil . \tag{3.17}$$

On the other hand, again since S is a γ_{ri}-set of $T(C_n)$, then S must have at least $\left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil$ vertices in $T(C_n)$. Hence,

$$|S| \geq \left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil . \tag{3.18}$$

Therefore, from equations 3.17 and 3.18,

$$|S| = \left\lceil \frac{n(n+1)}{4(n-1)} \right\rceil .$$

□

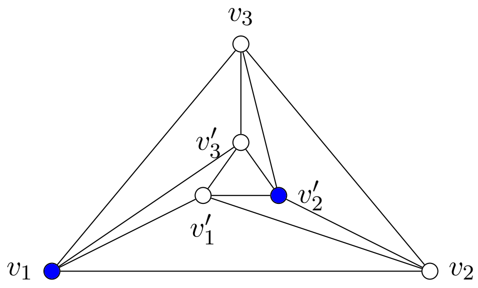

Figure 6: The minimum rings dominating sets in $T(K_3)$

3.4 Characteristics of Minimum Rings Dominating Set and Rings Domination Number of the Total Graph of Star graph $K_{1,n}$, $T(K_{1,n})$

Theorem 3.7. *Let S be a nonempty subset of $V[T(K_{1,n})]$, then S is a minimum rings dominating set of $T(K_{1,n})$ if and only if S contains exactly the end-vertices of $K_{1,n}$.*

Proof. Let $K_{1,n}$ be a star with vertex set $V(K_{1,n}) = \{v_1, v_2, ..., v_{n+1}\}$ where the vertices $v_1, v_2, ..., v_n$ are the end-vertices. Observe that in $T(K_{1,n})$, v_{n+1} is adjacent to all other vertices and $deg\, v_i = 2$ for $i = 1, 2, ..., n$. Now, suppose S is a minimum rings dominating set of $T(K_{1,n})$ and S contains a non-end-vertex of $K_{1,n}$, that is, either $v_{n+1} \in S$ or $e_i \in S$ for some i.

case 1: $v_{n+1} \in S$

Here, since $v_{n+1} \in S$, then S is a dominating set. However, this time $deg\, v_i = 1$ in $\langle V[T(K_{1,n})] \setminus S\rangle$ and so we are forced to include these vertices v_i to be in S so that S is a rings dominating set. Thus, we now have $S = \{v_1, v_2, ..., v_{n+1}\}$. Observe that, $\langle V[T(K_{1,n})] \setminus S\rangle \cong K_n$. Hence, S is indeed a rings dominating set. To this end, note that $|S| = n + 1$. But, $S_1 = \{v_1, v_2, ..., v_n\}$ is a rings dominating set as S is dominating set with $\langle V[T(K_{1,n})] \setminus S\rangle \cong K_{n+1}$. This time, we have

$$|S| = n + 1 < n = |S_1|.$$

Therefore, S is not a minimum rings dominating set, a contradiction.

case 2: $e_i \in S$ for some i.

Without loss of generality, suppose $e_1 \in S$, then observe that in $K_{1,n}$, e_1 is incident to an end-vertex, say v_1 and to v_{n+1}. Hence, $deg\, v_1 = 1$ in $\langle V[T(K_{1,n})] \setminus S\rangle$ and this forces us to include v_1 to be in S. Clearly, this is not a dominating set. On this reason, we have to observe the rest of the vertices. From case 1, we are given the idea that $v_{n+1} \notin S$. However, note that $\bigcap_i N[v_i] = \{v_{n+1}\}$ and no two end-vertex in $K_{1,n}$ share neighbors in $T(K_{1,n})$ other than $\{v_{n+1}\}$. To this end, we consider two subcases. (1) Suppose $e_i \in S$ for all i, then v_i must also be in S. But this is not a rings dominating set. Hence in this subcase, $S = V[T(K_{1,n})]$ so that S is a rings dominating set. (2) Assume $e_1, v_i \in S$. Here, clearly S is a rings dominating set since $\langle V[T(K_{1,n})] \setminus S\rangle \cong K_n$. Obviously,

$$|S| = |v_i| + |\{e_1\}| = n + 1 < n = |S_1|.$$

Both of which subcases contradict the assumption that S is a minimum rings dominating set.That being the case, we can say that if S is a minimum rings dominating set in $T(K_{1,n})$ then S must contain only the end-vertices in $K_{1,n}$.

Conversely, choose $S_1 = \{v_1, v_2, ..., v_n\}$. With this, we can observe that S_1 is rings dominating set with $\langle V[T(K_{1,n})] \setminus S_1 \rangle \cong K_{n+1}$. It only means to show that S_1 is a rings dominating set of minimum cardinality. Clearly, $|S_1| = n$. Now, suppose on the contrary that $|S_2| = n-1$ is a minimum rings dominating set. This implies that $S_2 = \{v_1, v_2, ..., v_{n-1}\}$. At this point,

$$|N[S_2]| = |N[v_1] \cup N[v_2] \cup ... \cup N[v_{n-1}]| = 2n - 1 < 2n + 1$$

.

It is conclusive that S_2 is not a dominating set. A contadiction. Therefore, $S = \{v_1, v_2, ..., v_n\}$ is the minimum rings dominating set of $T(K_{1,n})$.

□

Corollary 3.8. *Let $T(K_{1,n})$ be the total graph of star K_n of order $n \geq 2$. Then*

$$\gamma_{ri}[T(K_{1,n})] = n$$

.

Proof. This is an immediate consequence of Theorem 3.8. □

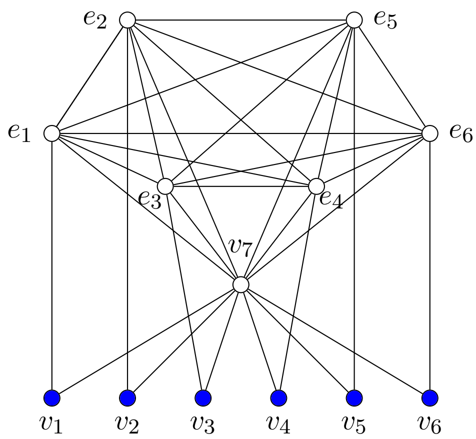

Figure 7: The minimum rings dominating set in $T(K_{1,6})$

Remark 3.1. The minimum rings dominating set in $T(K_{1,n})$ is unique, that is, $S = \{v_1, v_2, ..., v_n\}$ is the only minimum rings dominating set in $T(K_{1,n})$.

3.5 Rings Domination Number of the Total Graph of Fan graph F_n, $T(F_n)$

For $n \geq 2$, the fan graph $F_n = K_1 \vee P_n$ of order $n+1$ is a graph produced from the complete product of an isolated vertex and a path P_n. For convenience, we let the following $V(P_n) = \{v_1, v_2, ..., v_n\}$;

$V(K_1) = \{v_{n+1}\}$; and $E(P_n) = \{e_1, e_2, ..., e_{n-1}\}$. Also, we denote the edges joining v_{n+1} and every vertex in $V(P_n)$ as e_i, $n \leq i \leq 2n-1$.

Theorem 3.9. *Let $T(F_n)$ be the total graph of fan graph F_n of order $n+1$. Then*

$$\gamma_{ri}[T(F_n)] = \gamma(P_{n-1}) + 1$$

.

Proof. Let S be a minimum rings dominating set in $T(F_n)$. Note that in F_n, v_{n+1} is adajcent to all vertices in $V(P_n)$ and is incident to e_i for every $n \leq i \leq 2n-1$. Thus, by definition of total graph, $N(v_{n+1}) = V(P_n) \cup \{e_i : n \leq i \leq 2n-1\}$. So, $V[T(F_n)] \setminus N[v_{n+1}] = E(P_n)$ are the vertices not dominated by any vertex in $N[v_{n+1}]$. For this reason, taking the minimum dominating set of $E(P_n)$, say set D, then we now have a dominating set. Furthermore, each vertex $u \in V[T(F_n)] \setminus (D \cup \{v_{n+1}\})$, $deg\, u \geq 2$. Hence, $D \cup \{v_{n+1}\}$ is a rings dominating set. Moreover, in this context, $\langle E(P_n) \rangle \cong P_{n-1}$. Now, suppose $K = D \cup \{v_{n+1}\}$, then

$$|K| = \gamma(P_{n-1}) + 1.$$

With this,

$$|S| \leq |K| = \gamma(P_{n-1}) + 1.$$

On the other hand, since S is a rings dominating set, then S must have at least $\gamma(P_{n-1})+1$ vertices. This being said, we have

$$|S| \geq \gamma(P_{n-1}) + 1.$$

Therefore,

$$|S| = \gamma(P_{n-1}) + 1.$$

□

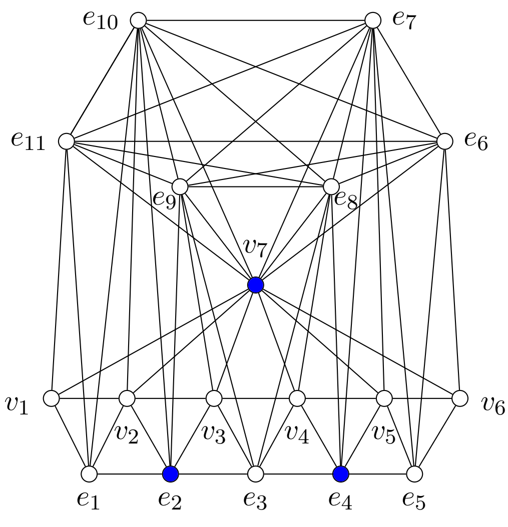

Figure 8: A minimum rings dominating set in $T(F_6)$

3.6 Rings Domination Number of the Total Graph of Wheel graph W_n, $T(W_n)$

The wheel graph W_n on $n+1$ vertices is defined as $W_n = K_1 \vee C_n$ where C_n is a n-cycle. Let $V(W_n) = \{v_i : 1 \leq i \leq n\} \cup \{v_{n+1}\}$ and $E(W_n) = \{e_i = v_iv_{i+1} : 1 \leq i \leq n, \text{subscripts modulo n}\} \cup \{e'_i = v_{n+1}v_i : 1 \leq i \leq n\}$. Now, the following provides a result on the total graph of W_n.

Theorem 3.10. *Let $T(W_n)$ be the total graph of wheel graph W_n of order $n+1$. Then*

$$\gamma_{ri}[T(W_n)] = \gamma(C_n) + 1.$$

Proof. Let S be a minimum rings dominating set in $T(W_n)$. Note that in W_n, v_{n+1} is adajcent to all vertices in $\{v_i : 1 \leq i \leq n\}$ and is incident to every edge in $\{e'_i = v_{n+1}v_i : 1 \leq i \leq n\}$. Thus, by definition of total graph, $N(v_{n+1}) = \{v_i : 1 \leq i \leq n\} \cup \{e'_i = v_{n+1}v_i : 1 \leq i \leq n\}$ in $T(W_n)$. So, $V[T(F_n)] \smallsetminus N[v_{n+1}] = \{e_i = v_iv_{i+1} : 1 \leq i \leq n\}$ are the vertices in $T(W_n)$ that are not dominated by any vertex in $N[v_{n+1}]$. For this reason, taking the minimum dominating set of $\{e_i = v_iv_{i+1} : 1 \leq i \leq n\}$, say set D, then we now have a dominating set. Furthermore, each vertex $u \in V[T(F_n)] \smallsetminus (D \cup \{v_{n+1}\})$, $deg\ u \geq 3$. Hence, $D \cup \{v_{n+1}\}$ is a rings dominating set. Moreover, in this context, $\langle \{e_i = v_iv_{i+1} : 1 \leq i \leq n\} \rangle \cong C_n$. Now, suppose $K = D \cup \{v_{n+1}\}$, then $|K| = \gamma(C_n)+1$. With this, $|S| \leq |K| = \gamma(C_n)+1$. On the other hand, since S is a rings dominating set, then S must have at least $\gamma(C_n)+1$ vertices. Consequently, we have $|S| \geq \gamma(C_n)+1$. Therefore, $|S| = \gamma(C_n) + 1$.

□

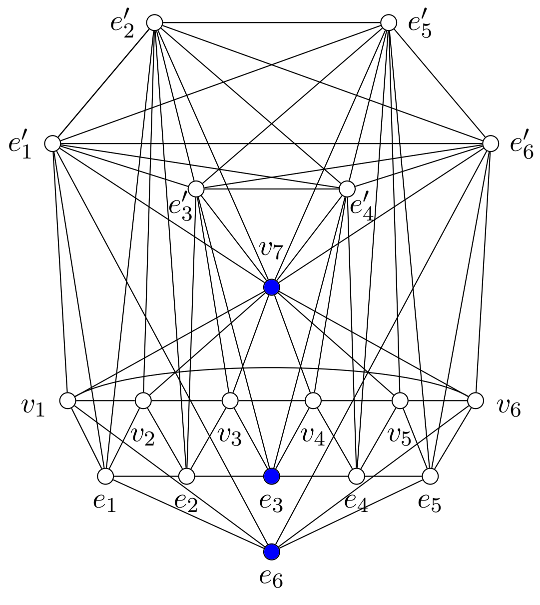

Figure 9: The minimum rings dominating set in $T(W_6)$

4 Conclusion

In this article, rings dominating sets in the total graph of some graph families are studied. Further, the rings domination number is also determined. Lastly, we intend to examine the rings dominating set and rings domination number for few unstudied graph families in the future.

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