Original Research Article

WEIBULL-INVERSE EXPONENTIAL [LOGLOGISTIC] A NEW DISTRIBUTION

ABSTRACT

The Weibull-inverse exponential-loglogistic distribution which is abbreviated as (Weibull-IE-loglogistc) is a member of the neotric *T*- inverse exponential family introduced previously by the authors. Properties of this distribution such as (mode, quantile function, median, hazard function, survival function, moments, order statistics and Shannon's entropy) are derived, and maximum likelihood estimates of its parameters are obtained. The usefulness of this neoteric distribution in analyzing data is illustrated. A simulation study is conducted to evaluate the performance of this distribution.

Keywords: Quantile function, Shannon's entropy, T-IE family, T-X[Y], Weibull-inverse exponential [loglogistic] distribution.

1. INTRODUCTION

Processes generating data are becoming complex and complicated resulting in data that are diversified in all aspects and shapes. This diversification in data requires new statistical distribution to accommodate this continuous diversification in data. The number of new families of distributions introduced in the literature since the 1980's is over whelming. However, it does not meet all the requirements for this invasion of data types. To get a better fit for data analysis, many researchers have recently expressed an interest in expanding the generating family. Some of the well-known generating families are;

- beta-G, [12] used the beta distribution as a generator function. The cumulative distribution function cdf of the beta generated distribution is defined as;

$$G(x)=\int_0^{F(x)} b(\tau)\,d\tau, \qquad (1)$$

where, *F* is the cdf of any random variable, say *X*, and $b(\tau)$ is the pdf of beta distirbution. The pdf of beta generated distribution is given by;

$$g(x) = \frac{f(x)}{B(\alpha, \beta)} F^{\alpha-1}(x)\,(1 - F(x))^{\beta-1},\ \alpha;\ \beta > 0, \tag{2}$$

where, $B(\alpha, \beta)$ is the beta function. To produce beta distributions, several researchers used various *F* in (2).

- Kumaraswamy-G [13] and [10] used Kumaraswamy distribution as a generator function instead of beta distribution.
- The transformed transformers family (*T-X* family) [6], which enables the use of any continuous pdf as a generator instead of beta or Kumaraswamy distribution, was proposed as a general technique for producing families of distributions. This technique is based on three functions $(R,\ F,\ \text{and}\ W)$, with *R* and *F* serving as the cdfs of two random variables $(T\ \text{and}\ X)$. $W(\cdot)$ is a real value function from $[0, 1]$ into the support of *T*. The cdf and pdf of *T-X* family of distributions is given as, respectively;

$$G(x) = \int_c^{W(F(x))} r(t)\,dt = R(W(F(x))), \tag{3}$$

where, *R* is the cdf of the generated random variable *T* and *r* is the pdf of *T*.

$$g(x) = \left[\frac{d}{dx} W(F(x))\right][r(W(F(x)))]. \tag{4}$$

- $T\text{-}X[Y]$ family of distributions [4] have been proposed. Substituting the quantile function of a random variable *Y* for $W(\cdot)$ in the *T-X* family. The T-X[Y] approach is based on 3 functions $F_T(x)$, $F_X(x)$ and Q(Y), with $F_T(x)$ and $F_X(x)$ serving as the cdfs of two random variables T and X, Q(Y) is the quantile function of some variable Y . The cdf and pdf of T-X[Y] family of distributions is provided respectively as;

$$G(x) = \int_a^{Q_Y(F_X(x))} f_T(t)\,dt = F_T\left(Q_Y\left(F_X(x)\right)\right), \tag{5}$$

and

$$g(x) = f_X(x).\frac{f_T\left(Q_Y\left(F_X(x)\right)\right)}{f_Y\left(Q_Y\left(F_X(x)\right)\right)}. \tag{6}$$

Several new distributions have been suggested by many researchers and statisticians; among them the beta-Gumbel [18], the beta- generalized Pareto [17], the gamma-Pareto distribution [5], the exponentiated generalized class of distributions [11], the exponentiated Kumaraswamy distribution [14], the Pareto-Weibull [generalized lambda] distribution [2], the Lomax-Gumbel Fréchet distribution [15], the Weibull-Lomax [log-logistic] distribution [7], the inverse power logistic exponential [1], the logistic-exponential [3], the Weibull-exponential [9], have been proposed.

Mahmoud et al. [16] used the $T\text{-}X[Y]$ approach to form the *T*-inverse exponential$[Y]$ $(T\text{-}IE[Y])$ family of distributions, with X following the inverse exponential distribution. The cdf and pdf of T-IE[Y] family of distributions are given by;

$$G_X(z) = \int_a^{Q_Y(e^{-\frac{\vartheta}{z}})} f_T(\tau)\,d\tau = F_T[Q_Y(e^{-\frac{\vartheta}{z}})], \tag{7}$$

and

$$g_X(z) = \frac{\vartheta}{z^2} e^{-\frac{\vartheta}{z}} \frac{f_T[Q_Y(e^{-\frac{\vartheta}{z}})]}{f_Y[Q_Y(e^{-\frac{\vartheta}{z}})]}. \tag{8}$$

A new three parameter distribution based on the $T\text{-}IE[Y]$ family will be studied in this article. The rest of the paper is settled out accordingly; In Section 2 a neoteric distribution is presented. In Section 3 some basic characteristics of Weibull-IE-loglogistc distribution are studied. In Section 4 the estimation of parameters is investigated by maximum likelihood method. In Section 5 Weibull-IE-loglogistic application along with other distributions are fitted to a real data. Simulation study is performed in Section 6. Section 7 ends with some concluding remarks on our study.

2. A Neoteric Distribution

We will display in here the formation of the cdf for Weibull-IE-loglogistic. Also, the pdf, survival function and hazard function are derived. In addition, plots of all of those functions at specific values of the parameters are displayed. The distribution function of Weibull-IE-loglogistic distribution cdf $(\text{for } z > 0)$ is given by;

$$G(z) = 1 - \exp\left[-\left(\frac{e^{-\frac{\vartheta}{z}}}{\eta(1 - e^{-\frac{\vartheta}{z}})}\right)^{\beta}\right], \tag{9}$$

where ϑ is scale parameter and β, η are shape parameters. The associated probability denisty function pdf can be written as follow;

$$g(z) = \frac{\beta\vartheta}{\eta^{\beta} z^2} \frac{e^{-\frac{\beta\vartheta}{z}}}{(1 - e^{-\frac{\vartheta}{z}})^{\beta+1}} \exp\left[-\left(\frac{e^{-\frac{\vartheta}{z}}}{\eta(1 - e^{-\frac{\vartheta}{z}})}\right)^{\beta}\right], \quad z > 0,\ \vartheta,\ \eta,\ \beta > 0. \tag{10}$$

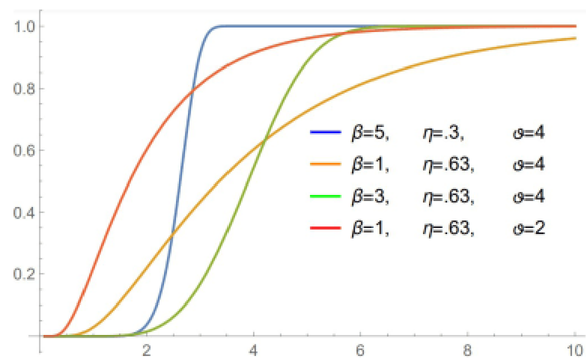

Figure 1. Weibull- IE -loglogistic distribution cdfs for different parameter values.

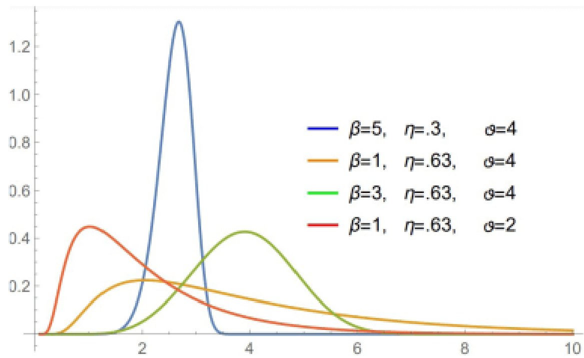

Figure 2. Weibull- IE -loglogistic distribution pdfs for different parameter values.

The survival function and hazard function are provided as;

$$R(z) = \exp\left[-\left(\frac{e^{-\frac{\vartheta}{z}}}{\eta(1-e^{-\frac{\vartheta}{z}})}\right)^{\beta}\right], \quad (11)$$

and

$$h(z) = \frac{\beta\vartheta}{\eta^{\beta} z^{2}} \frac{e^{-\frac{\beta\vartheta}{z}}}{(1-e^{-\frac{\vartheta}{z}})^{\beta+1}}. \quad (12)$$

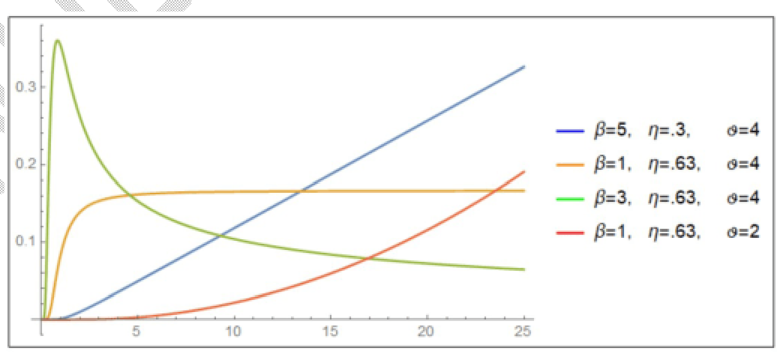

Figure 3. Weibull- IE -loglogistic distribution hazard function for different parameter values.

Plots of the cdf, pdf and hazard function for some values of β, η and ϑ are given in Figures 1-3 respectivly. The hazard function can be monotonically decreasing, increasing or an upside-down bathtub depending on the values of its parameters.

3. Basic Statistical Characteristics

Several general properties are found in this section concerning Weibull-IE-loglogistc distribution, including quantile function, median, skewness, kurtosis, mode, Shannon entropy, moments and order statistics.

$$Q_z(u) = \frac{-\vartheta}{\ln\left(\frac{-\eta[\ln(1-u)]^{1/\beta}}{1-\eta[\ln(1-u)]^{1/\beta}}\right)}. \tag{13}$$

3.1. Median

The median of the Weibull-IE-loglogistic distribution computation can be made by puting $u = 0.5$ in $Q_z(u)$ (Equation (13)) as follow:

$$median = \frac{-\vartheta}{\ln\left(\frac{-\eta[\ln(1-0.5)]^{1/\beta}}{1-\eta[\ln(1-0.5)]^{1/\beta}}\right)}. \tag{14}$$

3.2. Skewness and Kurtosis

Quartile function can be used as an alternative to moments if one does not have enough information about the mean, mode, and standard deviation to compute skweness and kurtosis (see [19]). The Bowley skewness (S_B) and Moors kurtosis (K_M) definitions are given as;

$$S_B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)},$$

and

$$K_M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(3/4) - Q(1/4)},$$

where, $Q(\cdot)$ denotes the quantile function. Skweness and kurtosis for Weibull-IE-loglogistic distribution are calculated for $\beta = 1$, $\eta = 1$, and $\vartheta = 2$, the result of skweness and kurtosis are; $S_B = 0.229207$ and $K_M = 1.30617$.

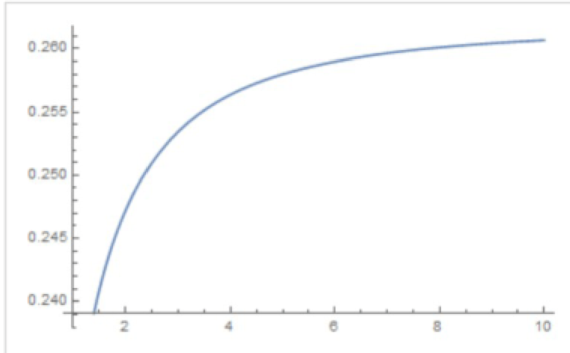

Figure 4. Weibull-IE-loglogistic distribution skewness.

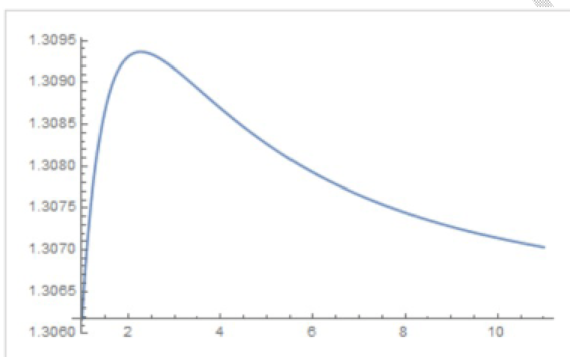

Figure 5. Weibull-IE-loglogistic distribution kurtosis.

3.3. Shannon's entropy

Entropy is a widely used term as a measure of uncertainty in social science. Thus, in this section, Shannon's entropy (ξ_x) for a random variable *X* with PDF $f(x)$ will be formed as $\xi_x = E[-\ln[f(x)]]$. Weibull-IE-loglogistic distribution Shannon's entropy is obtained as;

$$\xi_z = \beta \ln[\eta] - \ln[\beta\vartheta] + 2E(\ln[z]) + \beta\vartheta E\left(\frac{1}{z}\right) + (\beta+1)E(\ln[1-e^{-\frac{\vartheta}{z}}]) + \frac{\beta}{\eta}E\left(\frac{e^{-\frac{\vartheta}{z}}}{1-e^{-\frac{\vartheta}{z}}}\right). \tag{15}$$

3.4. Mode

Mahmoud et al. [16] obtained an equation to get the mode of the *T-IE* [log-logistic] sub family, which is;

$$z = \frac{f_T\left(\dfrac{e^{-\frac{\vartheta}{z}}}{1-e^{-\frac{\vartheta}{z}}}\right)}{f_T'\left(\dfrac{e^{-\frac{\vartheta}{z}}}{1-e^{-\frac{\vartheta}{z}}}\right)}\left[2-\frac{\vartheta}{z}\left(1+2\frac{e^{-\frac{\vartheta}{z}}}{1-e^{-\frac{\vartheta}{z}}}\right)\right],$$

where $f(T)$ is the pdf of a random variable *T*, and in this paper the variable *T* follows Weibull distribution. Therefore, the mode of Weibull-IE-loglogistic distribution is the solution of this Equation;

$$z = \frac{\vartheta}{z^2}\frac{e^{-\frac{\vartheta}{z}}}{1-e^{-\frac{\vartheta}{z}}}\cdot\left[\frac{\beta}{\eta(1-e^{-\frac{\vartheta}{z}})}+\frac{\beta-1}{e^{-\frac{\vartheta}{z}}}\right]\cdot\left[2-\frac{\vartheta}{z}\left(1+2\frac{e^{-\frac{\vartheta}{z}}}{1-e^{-\frac{\vartheta}{z}}}\right)\right]. \tag{16}$$

3.5. Order statistics

If $Z_{(1)} < Z_{(2)} < \cdots < Z_{(n)}$ denote the ordered observations in a data set from Weibull-IE-loglogistic distribution given by equation (8) and equation (9), then the PDF $g_{v_{r:n}}(z)$ of the *i*th order statistic $z_{(i)}$ is;

$$g_{z_r}(z) = \frac{1}{\beta(r,\, n-r+1)}\, g(z)[G(z)]^{r-1}[1-G(z)]^{n-r}. \tag{17}$$

Applying Equation 9, and 10 in Equation 17, then we have the pdf of Weibull-IE-loglogistic distribution order statistics;

$$g_{z_r}(z) = \frac{1}{\beta(r,\, n-r+1)}\frac{\beta\vartheta}{\eta^{\beta} z^2}\frac{e^{-\frac{\beta\vartheta}{z}}}{(1-e^{-\frac{\vartheta}{z}})^{\beta+1}}\left[1-\exp\left[-\left(\frac{e^{-\frac{\vartheta}{z}}}{\eta(1-e^{-\frac{\vartheta}{z}})}\right)^{\beta}\right]\right]^{r-1}$$
$$\times\left[\exp\left[-\left(\frac{e^{-\frac{\vartheta}{z}}}{\eta(1-e^{-\frac{\vartheta}{z}})}\right)^{\beta}\right]\right]^{n-r+1}. \tag{18}$$

3.6. Moments

Moments are important to know the characteristics of a distribution. We derived the moments of a random variable *z* which has the Weibull-IE-loglogistic distribution.

Applying [16] moments formula for T-IE[log-logistic] subfamily, the Weibull-IE-loglogistic distribution moments can be formed as follow;

$$E(z^r) = \vartheta^r r \sum_{i=0}^{\infty} \binom{i+r}{i} \sum_{j=0}^{i} \frac{(-1)^{i+j}}{-r-j} \binom{i}{j} P_{j,i}, \tag{19}$$

where $P_{j,i}$ is a constant and can be computed like that;

$$P_{j,i} = \frac{1}{i} \sum_{m=1}^{i} \frac{(jm - i + m)(-1)^m}{m+1} P_{j,i-m}, \quad \text{for } i = 1, 2, 3, ..., \text{ and } P_{j,0} = 1.$$

4. Estimation of Weibull-IE-LogLogistic Parameters

In this section the maximum likelihood method is used to obtain the unknown parameters of Weibull-IE-loglogistic distribution based on complete samples. Let $z_1, z_2, ..., z_n$ be a random sample from pdf (10) with set of parameters $\Theta = (\vartheta, \beta, \eta)$. The likelihood function, denoted by $L(z; \Theta)$, is given by;

$$L(z, \Theta) = L(z; \vartheta, \beta, \eta)$$

$$= \frac{\beta^n \vartheta^n}{\eta^{\beta n} \prod_{i=1}^{n} z_i} \frac{e^{\frac{-\beta\vartheta}{\sum_{i=1}^{n} z_i}}}{\left(1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right)^{\beta+1}} \exp\left[-\frac{e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}}{\eta\left(1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right)}\right]. \tag{20}$$

The natural logarithm of the likelihood function denoted by $\ln L(z; \vartheta, \beta, \eta)$ is given by;

$$\ln L(z;\,\theta,\,\beta,\,\eta) = n\ln[\beta\vartheta] - \beta n\ln[\eta] - \sum_{i=0}^{n}\ln[z_i^2] - \frac{\beta\vartheta}{\sum_{i=0}^{n} z_i^2} - \beta\ln\left[1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right]$$

$$-\ln\left[1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right] - \left(\frac{e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}}{\eta\left(1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right)}\right)^{\beta}. \quad (21)$$

The maximum likelihood estimate $\hat{\Theta}$ of Θ is obtained by solving the system $\frac{\partial \ln L(z;\,\vartheta,\,\beta,\,\eta)}{\partial\vartheta} = 0,$ as follows:

$$\frac{n}{\beta} - n\ln[\eta] - \frac{\vartheta}{\sum_{i=1}^{n} z_i} - \ln\left[1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right]\frac{n}{\beta} - n\ln[\eta] - \frac{\vartheta}{\sum_{i=1}^{n} z_i} - \ln\left[1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right] = 0 \quad (22)$$

$$\frac{\beta}{\eta}\left[-n + \left(\frac{e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}}{\eta\left(1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right)}\right)^{\beta}\right] = 0, \quad (23)$$

$$\frac{n}{\vartheta} - \frac{\beta}{\sum_{i=1}^{n} z_i} - \frac{e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}}{\left(1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}\right)\sum_{i=1}^{n} z_i} \cdot \left[\beta + 1 - \frac{\beta}{\eta^{\beta}}\frac{1}{1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}}\left(\frac{e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}}{1 - e^{\frac{-\vartheta}{\sum_{i=1}^{n} z_i}}}\right)^{\beta-1}\right] = 0. \quad (24)$$

The resulting equations above can not be solved analytically, so we usually use some software's like Mathematica to solve them numerically.

The information matrix is given by

$$I(\Theta) = -E\left(\frac{\partial^2 \log L(\Theta;\, z)}{\partial \Theta^2}\right). \tag{25}$$

The common used formula for Fisher information matrix is what is usually referred to as the observed Fisher information given by

$$\hat{I}(\Theta) = -\left(\frac{\partial^2 \log L(\Theta;\, z)}{\partial \Theta^2}\right)\Bigg|_{\Theta=\hat{\Theta}} \tag{26}$$

we can use equation (26) to make interval estimation of the distribution parameters.

5. An Application

In this section, the usefulness of Weibull-IE-loglogistic distribution for modelling reliability data is illustrated.

The flexibility of Weibull-IE-loglogistic is clarified by the use of a real data set. 34 observations of vinyl chloride data in mg/L obtained from clean up gradient ground-water monitoring wells provided by [8] is used. A differentiation is made between Weibull-IE-loglogistic distribution and a number of other distributions such as (inverse power logistic exponential [1], Weibull-exponential [9], logistic-exponential [3], Weibull-Lomaxloglogistic) [7], and Fréchet) using Akaike information criterion (AIC), Hannan-Quinn information criterion (HQIC), Bayesian information criterion (BIC), and loglikelihood value. The better model, on the other hand, has the lowest loglikelihood, AIC, HQIC and BIC values. Distribution parameters are estimated using the maximum likelihood estimation method.

Table 1 includes the values of AIC, HQIC, BIC and log-likelihood. The figures in Table 1 show that among the listed models, the Weibull-IE-loglogistic distribution is the most closely matches data. The results in this section are obtained using the Mathematica 12 program.

Table 1. AIC, HQC, BIC and Log-likelihood measures for the data

Distribution	AIC	HQC	BIC	Log-likelihood
Weibull-IE-LogLogistic	128.629	126.410	133.208	–61.314
Fréchet	137.240	135.021	141.819	–65.619
Inverse power logistic exponential	137.240	135.021	141.819	–65.619
Weibull-Lomax-loglogistic	133.800	131.581	138.379	–63.900
Weibull-exponential	135.800	132.841	141.905	–63.900
Logistic-exponential	131.800	130.321	134.853	–63.900

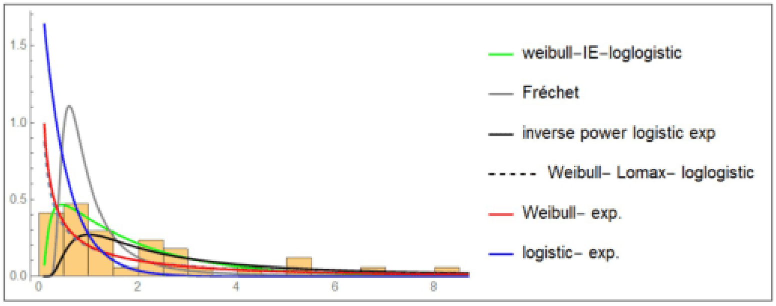

Figure 6. The dataset's histogram and fitted PDFs.

6. A Simulation Study

A simulation study is carried out to assess the performance of the MLEs of the Weibull-IE-loglogistic distribution. The process is carried out as follow:

• The process is replicated one hundered times each with sample size $n = 50, 70$ and 100 from equation (9).

• Initial values for the parameters are selected as shown in Table 2.

• Compute the MLEs for the one hundered samples, say $(\hat{\beta}, \hat{\eta}, \hat{\vartheta})$ for $i = 1, 2, ..., 100.$

The figures in Table 2 shows that, the absolute value of relative bias (RBias) and the root of mean square error (RMSE) decreases as the sample size increases. The actual values for β, η and $\vartheta (80, 10, 75)$, respectively, has the lowest RBias and RMSE values for sample size 50, 70 and 100. The RBais and RMSE values for sample size 50 are considered suitable to be used.

7. Summary and Conclusion

The three-parameter Weibull-IE-loglogistic distribution is defined in this paper as a member of the T-IE family of distributions. A number of properties are introduced, such as mode, quantile function, median, hazard function, survival function, moments, order statistics, and Shannon's entropy. The parameters of the new distribution were estimated using the maximum likelihood method using a real data set and a numerical simulation study. The Weibull-IE-logistic distribution provides a better fit for the real data used in the study than the inverse power logistic exponential, Weibull-exponential, logistic-exponential, Weibull-Lomax-loglogistic, and Fréchet distributions.

Table 2. Results of the Weibull-IE-loglogistic simulation using MLE for a few values of β, η and ϑ

Acutal value			n	mean			RBias			RMSE		
β	η	ϑ		$\hat{\beta}$	$\hat{\eta}$	$\hat{\vartheta}$	β	η	ϑ	β	η	ϑ
			50	31.880	4.567	31.960	0.601	0.7716	0.7710	0.085	0.10912	0.10913
80	20	140	70	36.539	5.830	40.800	0.543	0.7084	0.7085	0.064	0.08468	0.08469
			100	40.926	7.108	49.751	0.488	0.6445	0.6446	0.048	0.06445	0.06446
			50	31.918	2.288	17.148	0.601	0.771	0.771	0.084	0.10906	0.10908
80	10	75	70	36.610	2.924	21.927	0.542	0.707	0.707	0.064	0.08456	0.08457
			100	41.304	3.611	27.079	0.483	0.6388	0.6389	0.048	0.06388	0.06389
			50	25.059	1.936	11.609	0.642	0.8063	0.8065	0.090	0.11403	0.11405
70	10	60	70	29.069	2.552	15.307	0.584	0.7447	0.7448	0.069	0.08901	0.08902
			100	32.976	3.170	19.015	0.528	0.6829	0.6830	0.052	0.06829	0.06830
			50	25.585	2.036	10.175	0.634	0.7963	0.7964	0.089	0.11262	0.11264
70	10	50	70	28.688	2.494	12.468	0.590	0.7505	0.7506	0.070	0.08970	0.08971
			100	32.856	3.150	15.748	0.530	0.6849	0.6850	0.053	0.06849	0.06850
			50	18.577	1.589	7.940	0.690	0.8410	0.8411	0.097	0.11894	0.11896
60	10	50	70	21.592	2.088	10.439	0.640	0.7911	0.7912	0.076	0.09455	0.09456
			100	24.884	2.651	13.253	0.585	0.7348	0.7349	0.058	0.07348	0.07349
			50	12.661	1.237	6.181	0.746	0.8762	0.8763	0.105	0.12392	0.12393
50	10	50	70	14.857	1.616	8.079	0.702	0.8383	0.8384	0.084	0.10019	0.10020
			100	17.417	2.094	10.467	0.651	0.7905	0.7906	0.065	0.07905	0.07906
			50	6.591	0.700	3.500	0.835	0.9299	0.9299	0.118	0.13151	0.13152
40	10	50	70	8.217	0.988	4.940	0.749	0.9011	0.9012	0.094	0.10770	0.10771
			100	10.284	1.398	6.989	0.742	0.8601	0.8602	0.074	0.08681	0.08602
			50	6.757	2.116	3.526	0.831	0.9294	0.9295	0.117	0.13144	0.13145
40	30	50	70	8.564	3.162	5.269	0.785	0.8945	0.8946	0.093	0.10692	0.10693
			100	10.811	4.534	7.556	0.729	0.8487	0.8488	0.072	0.08487	0.08488

REFERENCES

[1] AL Sobhi, M. M. The inverse-power logistic-exponential distribution: Properties, estimation methods, and application to insurance data. Mathematics. 2020; 8(11): 2060_2075.

[2] Aldeni, M., Lee, C., and Famoye, F. Families of distributions arising from the quantile of generalized lambda distribution. Journal of Statistical Distributions and Applications. 2017; 4(1):1_18.

[3] Ali, S., Dey, S., Tahir, M. H., and Mansoor, M. Two-parameter logistic-exponential distribution: Some new properties and estimation methods. American Journal of Mathematical and Management Sciences. 2020; 39(3):270_298.

[4] Aljarrah, M. A., Lee, C., and Famoye, F. On generating T-X family of distributions using quantile functions. Journal of Statistical Distributions and Applications. 2014; 1(1):2_19.

[5] Alzaatreh, A., Famoye, F., and Lee, C. Gamma-Pareto distribution and its applications. Journal of Modern Applied Statistical Methods. 2012; 11(1):7.

[6] Alzaatreh, A., Lee, C., and Famoye, F. A new method for generating families of continuous distributions. Metron. 2013; 71(1):63_79.

[7] Alzaghall, A. and Hamed, D. New families of generalized Lomax distributions: Properties and applications. International Journal of Statistics and Probability. 2019; 8(6):51_68.

[8] Bhaumik, D. K., Kapur, K., and Gibbons, R. D. Testing parameters of a gamma distribution for small samples. Technometrics. 2009; 51(3):326_334.

[9] Bilal, M., Mohsin, M., and Aslam, M. Weibull-exponential distribution and its application in monitoring industrial process. Mathematical Problems in Engineering. 2021; 2021

[10] Cordeiro, G. M. and de Castro, M. A new family of generalized distributions. Journal of Statistical Computation and Simulation. 2011; 81(7):883_898.

[11] Cordeiro, G. M., Ortega, E. M., and da Cunha, D. C. The exponentiated generalized class of distributions. Journal of Data Science. 2013; 11(1):1_27.

[12] Eugene, N., Lee, C., and Famoye, F. Beta-normal distribution and its applications. Communications in Statistics-Theory and Methods. 2002; 31(4):497_512.

[13] Jones, M. Kumaraswamy's distribution: A beta-type distribution with some tractability advantages. Statistical Methodology.2009; 6(1):70_81.

[14] Lemonte, A. J., Barreto-Souza, W., and Cordeiro, G. M. The exponentiated Kumaraswamy distribution and its log-transform. Brazilian Journal of Probability and Statistics.2013; 27(1):31_53.

[15] Mahmoud, M., Mandouh, R., and Abdelatty R. Lomax-Gumbel {Fréchet}: A new distribution. Journal of Advances in Mathematics and Computer Science. 2019; 31(2): 1-19.

[16] Mahmoud, M. R., Ahmad, M. A. M., and Ismail, A. E. T-inverse exponential family of distributions. Journal of University of Shanghai for Science and Technology. 2021; 23(9):556_572.

[17] Mahmoudi, E. The beta generalized pareto distribution with application to lifetime data. Mathematics and Computers in Simulation. 2011; 81(11):2414_2430.

[18] Nadarajah, S. and Kotz, S. The beta Gumbel distribution. Mathematical Problems in Engineering.2004; 2004(4):323_332.

[19] Zar, J. H. Biostatistical analysis. Prentice-Hall, Inc. 2010.