

Fixed point theorems using α –admissible mappings in metric spaces

Abstract: In this paper, we shall prove the fixed point theorems in metric space using α –admissible mapping. Some existing results of literature shall be deduced from the main results. In the end, we shall provide an example to support our result.

Keywords: α –admissible mappings, complete metric space and fixed point.

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1. Introduction: In 1922, Banach gave a principle to obtain the fixed point in the complete metric space. Since then, many researchers have worked on the Banach fixed point theorem(see, for example, [2-6], [11-13], [17-39]) and tried to generalize this principle. In 2012, Samet *et al.*[33] introduced the new concepts of mappings called α –admissible mappings in metric space. Recently, in 2013 Farhan *et al.* [2] gave new contractions using α –admissible mapping in metric spaces.

In this paper, we shall generalize Farhan’s *et al.*[2] contractions and give fixed point theorems for such contractions.

2. Preliminaries: To prove our main results we need some basic definitions from literature as follows:

Definition([10]): “Let X be a set. A metric space is an ordered pair (X, d) where d is a function $d: X \times X \rightarrow \mathbb{R}$ such that

- (1) $d(x, y) \geq 0$. (non-negativity)
- (2) $d(x, y) = 0$ iff $x = y$. (non-degeneracy)
- (3) $d(x, y) = d(y, x)$ (symmetry)
- (4) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)”

Definition([15]) : “A sequence $\{x_n\}$ in a metric space (X, d) is said to converge if there is a point $x \in X$ and for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for every $n > N$ ”.

Definition: “A sequence $\{x_n\}$ in a metric space (X, d) is Cauchy if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$ for every $n, m > N$ ”.

Definition([15]): “A metric space (X, d) is said to be complete if every Cauchy sequence is convergent”.

Definition([33]) : “Let $f: X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, \infty)$. We say that f is an α –admissible mapping if $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$, for all $x, y \in X$ ”.

3. Main Results:

In this section, we shall prove fixed point theorems.

Theorem 3.1: Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an α – admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$(d(Tx, Ty) + l)^{\alpha(x, Tx)\alpha(y, Ty)} \leq \beta(M(x, y))M(x, y) + l, \text{ for all } x, y \in X \text{ and } l \geq 1. \quad (1)$$

Where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$

Suppose that either

(3.1) T is continuous, or

(3.2) If $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$, for all n , then $(\alpha, Tx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.

If $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, then $Tx_n = x_n$ and we are done.

So, we suppose that $d(x_n, x_{n+1}) > 0$, for all $n \in \mathbb{N}$.

Since T is α – admissible, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ which implies $\alpha(x_0, x_1) \geq 1$.

Similarly, we can say that $\alpha(x_1, x_2) = \alpha(Tx_0, T^2x_0) \geq 1$.

By continuing this process, we get

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \quad (2)$$

By using equation (2), we have

$$d(x_n, x_{n+1}) + l = d(Tx_{n-1}, Tx_n) + l \leq (d(Tx_{n-1}, Tx_n) + l)^{\alpha(x_{n-1}, Tx_{n-1})\alpha(x_n, Tx_n)}.$$

Now using equation (1), we get

$$d(x_n, x_{n+1}) + l \leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n) + l, \quad (3)$$

$$M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\},$$

$$= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\},$$

Assume that if possible $d(x_n, x_{n+1}) > d(x_{n-1}, x_n)$.

Then, $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$.

Using this in equation (3), we get

$$d(x_n, x_{n+1}) < \beta(d(x_n, x_{n+1}))d(x_n, x_{n+1}) \quad (4)$$

$\Rightarrow d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.

So $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, for all n .

It follows that the sequence $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive real numbers. So, it is convergent and suppose that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = d$. Clearly, $d \geq 0$.

Claim: $d = 0$.

Equation (4) implies that

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n)) \leq 1,$$

Which implies that $\lim_{n \rightarrow \infty} \beta(d(x_{n-1}, x_n)) = 1$.

Using the property of the function β , we conclude that $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0$. (5)

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k , we have

$$n(k) > m(k) > k, d(x_{n(k)}, x_{m(k)}) \geq \epsilon \text{ and } d(x_{n(k)}, x_{m(k)-1}) < \epsilon.$$

By the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) \\ &< \epsilon + d(x_{m(k)-1}, x_{m(k)}), \text{ for all } k \in \mathbb{N}. \end{aligned}$$

Taking the limit as $k \rightarrow +\infty$ in the above inequality and using equation (5), we get

$$\lim_{k \rightarrow +\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon. \quad (6)$$

Again by triangle inequality, we have

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}).$$

Taking the limit as $k \rightarrow +\infty$, together with (5) and (6), we deduce that

$$\lim_{k \rightarrow +\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon. \quad (7)$$

From equations (1),(2),(6) and (7), we get

$$\begin{aligned} d(x_{n(k)+1}, x_{m(k)+1}) + l &\leq (d(x_{n(k)+1}, x_{m(k)+1}) + l)^{\alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})}, \\ &= (d(Tx_{n(k)}, Tx_{m(k)}) + l)^{\alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})} \\ &\leq \beta(M(x_{n(k)}, x_{m(k)}))M(x_{n(k)}, x_{m(k)}) + l \end{aligned} \quad (8)$$

$$M(x_{n(k)}, x_{m(k)}) = \max \{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1})\}$$

So, equation (8) implies that

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq \beta(M(x_{n(k)}, x_{m(k)}))M(x_{n(k)}, x_{m(k)}) \leq 1,$$

Letting $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} \beta(d(x_{n(k)}, x_{m(k)})) = 1.$$

By using definition of β function, we get

$$\Rightarrow \lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)}) = 0 < \epsilon, \text{ which is a contradiction.}$$

Hence, $\{x_n\}$ is a Cauchy sequence.

Since (X, d) is a complete space, so $\{x_n\}$ is convergent and assume that $x_n \rightarrow x$ as $n \rightarrow \infty$.

Since T is continuous, then we have

$$Tx = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = x.$$

So, x is a fixed point of T .

Now, suppose that (3.2) holds, then $\alpha(x, Tx) \geq 1$ and by using equations (1) and (2) we get

$$\begin{aligned} d(Tx, x_{n+1}) + l &\leq (d(Tx, Tx_n) + l)^{\alpha(x, Tx)\alpha(x_n, Tx_n)} \\ &\leq \beta(M(x, x_n))M(x, x_n) + l. \end{aligned} \quad (9)$$

Where $M(x, x_n) = \max\{d(x, Tx), d(x, x_n), d(x, Tx_n)\}$.

Clearly, from equation (9) and using triangle inequality, we get

$$\begin{aligned} d(Tx, x) &\leq d(Tx, x_{n+1}) + d(x_{n+1}, x) \\ &\leq \beta(M(x, x_n))M(x, x_n) + d(x_{n+1}, x) \end{aligned}$$

Letting $n \rightarrow \infty$, we get

$$d(Tx, x) = 0 \text{ which implies } Tx = x.$$

Theorem 3.2: Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an α -admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ which implies that $t_n \rightarrow 0$ and

$$(\alpha(x, Tx) \cdot \alpha(y, Ty) + 1)^{d(Tx, Ty)} \leq 2^{\beta(M(x, y))M(x, y)}, \text{ for all } x, y \in X. \quad (10)$$

Suppose that either

(3.3) T is continuous or

(3.4) If $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n , then $\alpha(x, Tx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$.

Define a sequence $\{x_n\}$ in X as $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

If $x_{n+1} = x_n$, for some $n \in \mathbb{N}$, then $Tx_n = x_n$ and we are done. Hence, we suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. As in Theorem (3.1), we conclude that $\alpha(x_n, Tx_n) \geq 1$ for all $n \in \mathbb{N}$.

From equation (10), we get

$$\begin{aligned} 2^{d(Tx_{n-1}, Tx_n)} &\leq (\alpha(x_{n-1}, Tx_{n-1})\alpha(x_n, Tx_n) + 1)^{d(Tx_{n-1}, Tx_n)} \\ &\leq 2^{\beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n)} \end{aligned}$$

Which yields that

$$d(x_n, x_{n+1}) \leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n). \quad (11)$$

$$\begin{aligned} M(x_{n-1}, x_n) &= \max \{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}, \\ &= \max \{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}. \end{aligned}$$

If possible suppose that

$$d(x_n, x_{n+1}) > d(x_{n-1}, x_n).$$

$$\text{Then } M(x_{n-1}, x_n) = d(x_n, x_{n+1})$$

Using this, equation (11) implies that $d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.

So, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$.

So, $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive reals. So, there exists $d \in R^+ \cup \{0\}$ such that

$$d(x_n, x_{n+1}) \rightarrow d \text{ as } n \rightarrow \infty$$

Claim: $d = 0$.

Equation (11) implies that

$$\frac{d(x_n, x_{n+1})}{M(x_{n-1}, x_n)} \leq M(d(x_{n-1}, x_n)) \leq 1, \text{ which implies}$$

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n)) \leq 1.$$

Taking $n \rightarrow \infty$, we get $\lim_{n \rightarrow \infty} \beta(d(x_{n-1}, x_n)) = 1$. Using definition of β function, we get

$$\Rightarrow \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (12)$$

We prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k ,

$$n(k) > m(k) > k, d(x_{n(k)}, x_{m(k)}) \geq \epsilon \text{ and } d(x_{n(k)}, x_{m(k)-1}) < \epsilon.$$

Following the related lines in the proof of Theorem (3.1), we get

$$\lim_{k \rightarrow +\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon \quad (13)$$

$$\text{and } \lim_{k \rightarrow +\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon. \quad (14)$$

Using equations (10), (13) and (14), we get

$$\begin{aligned} 2^{d(x_{n(k)+1}, x_{m(k)+1})} &\leq (\alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)}) + 1)^{d(x_{n(k)+1}, x_{m(k)+1})} \\ &= (\alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)}) + 1)^{d(Tx_{n(k)}, Tx_{m(k)})} \\ &\leq 2^{\beta(M(x_{n(k)}, x_{m(k)}))M(x_{n(k)}, x_{m(k)})} \end{aligned}$$

Where $M(x_{n(k)}, x_{m(k)}) = \max\{d(x_{n(k)}, x_{m(k)}), d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1})\}$.

$$\text{Hence, } \frac{d(x_{n(k)+1}, x_{m(k)+1})}{M(x_{n(k)}, x_{m(k)})} \leq \beta(M(x_{n(k)}, x_{m(k)})) \leq 1.$$

By taking $k \rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} \beta(d(x_{n(k)}, x_{m(k)})) = 1.$$

$$\Rightarrow \lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)}) = 0 < \epsilon, \text{ which is a contradiction.}$$

So, $\{x_n\}$ is a Cauchy sequence and as X is complete, so $\{x_n\} \rightarrow x$.

Now suppose the T is continuous.

$$Tx = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = x.$$

$$\Rightarrow Tx = x.$$

$$\Rightarrow x \text{ is fixed point of } T.$$

Next, we suppose that the condition (3.4) holds, then $\alpha(x, Tx) \geq 1$.

Now by equation (10), we get

$$\begin{aligned} 2^{d(Tx, x_{n+1})} &\leq (\alpha(x, Tx)\alpha(x_n, Tx_n) + 1)^{d(Tx, Tx_n)}, \\ &\leq 2^{\beta(M(x, x_n))M(x, x_n)}, \end{aligned}$$

Where

$$\begin{aligned}
M(x, x_n) &= \max\{d(x, x_n), d(x, Tx), d(x_n, x_{n+1})\} \\
\Rightarrow d(Tx, x_{n+1}) &\leq \beta(M(x, x_n))M(x, x_n)
\end{aligned} \tag{15}$$

Using triangle inequality,

$$d(Tx, x) \leq d(Tx, x_{n+1}) + d(x_{n+1}, x).$$

Letting $n \rightarrow \infty$ and using (15), we get

$$d(Tx, x) = 0.$$

$$\Rightarrow Tx = x.$$

Theorem 3.3: Let (X, d) be a complete metric space and $T: X \rightarrow X$ be an α -admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$\alpha(x, Tx)\alpha(y, Ty)d(Tx, Ty) \leq \beta(M(x, y))M(x, y), \forall x, y \in X. \tag{16}$$

Suppose that either

(3.5) T is continuous or

(3.6) If $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n , then $\alpha(x, Tx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Proof: Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Construct a sequence $\{x_n\}$ in X such that $x_{n+1} = Tx_n \forall n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then $Tx_n = x_n$ and we are done.

So, assume $x_n \neq x_{n+1}$ for all $n \geq 1$.

$$\text{As in Theorem (3.1), we conclude that } \alpha(x_n, x_{n+1}) \geq 1 \text{ for all } n. \tag{17}$$

Now by equation (16), we get

$$d(x_n, x_{n+1}) \leq \alpha(x_n, Tx_{n-1})\alpha(x_n, Tx_n)d(Tx_{n-1}, Tx_n) \leq \beta(M(x_{n-1}, x_n))M(x_{n-1}, x_n). \tag{18}$$

Where $M(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}$.

$$= \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})\}$$

If possible assume that $d(x_{n-1}, x_n) < d(x_n, x_{n+1})$, then $M(x_{n-1}, x_n) = d(x_n, x_{n+1})$.

Using this from equation (16), we get

$$d(x_n, x_{n+1}) \leq \beta(d(x_{n-1}, x_n))d(x_{n-1}, x_n) \tag{19}$$

$d(x_n, x_{n+1}) < d(x_n, x_{n+1})$, which is a contradiction.

So, $d(x_n, x_{n+1}) \leq d(x_n, x_{n-1})$ for all $n \in \mathbb{N}$.

It follows that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive reals. So, there exists $d \geq 0$ such that $d(x_n, x_{n+1}) \rightarrow d$ as $n \rightarrow \infty$.

Therefore, (19) implies that

$$\frac{d(x_n, x_{n+1})}{d(x_{n-1}, x_n)} \leq \beta(d(x_{n-1}, x_n)) \leq 1.$$

Thus we find that $\lim_{n \rightarrow \infty} \beta(d(x_n, x_{n+1})) = 1$.

$$\Rightarrow \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (20)$$

Next, we will prove that the sequence $\{x_n\}$ is Cauchy. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k ,

$$n(k) > m(k) > k, d(x_{n(k)}, x_{m(k)}) \geq \epsilon \text{ and } d(x_{n(k)}, x_{m(k)-1}) < \epsilon.$$

Again, by following the lines of the proof of Theorem (3.1), we derive that

$$\lim_{k \rightarrow +\infty} d(x_{n(k)}, x_{m(k)}) = \epsilon. \quad (21)$$

$$\text{and } \lim_{k \rightarrow +\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon. \quad (22)$$

now, combining (16), (21) and (22), we get

$$\begin{aligned} d(x_{n(k)+1}, x_{m(k)+1}) &\leq \alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})d(x_{n(k)+1}, x_{m(k)+1}) \\ &= \alpha(x_{n(k)}, Tx_{n(k)})\alpha(x_{m(k)}, Tx_{m(k)})d(Tx_{n(k)}, Tx_{m(k)}) \\ &\leq \beta(M(x_{n(k)}, x_{m(k)}))M(x_{n(k)}, x_{m(k)}). \end{aligned} \quad (23)$$

Where $M(x_{n(k)}, x_{m(k)}) = \max \{d(x_{n(k)}, x_{n(k)+1}), d(x_{m(k)}, x_{m(k)+1}), d(x_{n(k)}, x_{m(k)})\}$

Now, equation (23) implies

$$\frac{d(x_{n(k)+1}, x_{m(k)+1})}{M(x_{n(k)}, x_{m(k)})} \leq \beta(M(x_{n(k)}, x_{m(k)})) \leq 1.$$

Taking $\rightarrow \infty$, we get

$$\lim_{k \rightarrow \infty} \beta(d(x_{n(k)}, x_{m(k)})) = 1.$$

$$\Rightarrow \lim_{k \rightarrow \infty} d(x_{n(k)}, x_{m(k)}) = 0 < \epsilon, \text{ which is a contradiction.}$$

So, $\{x_n\}$ is a Cauchy sequence. Since X is complete, so $\{x_n\} \rightarrow x$.

First suppose that T is continuous. So,

$$Tx = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = x.$$

$\Rightarrow x$ is fixed point.

Now suppose that (3.6) holds, then $\alpha(x, Tx) \geq 1$ and using (16), we have

$$\begin{aligned} d(Tx, x_{n+1}) &\leq d(Tx, Tx_n)\alpha(x, Tx)\alpha(x_n, Tx_n) \\ &\leq \beta(M(x, x_n))M(x, x_n) \end{aligned} \quad (24)$$

Where $M(x, x_n) = \max \{d(x, x_n), d(x, Tx), d(x_n, Tx_n)\}$

and $d(Tx, x) \leq d(Tx, x_{n+1}) + d(x_{n+1}, x)$

Using (24), we get

$$d(Tx, x) \leq \beta(M(x, x_n))M(x, x_n) + d(x_{n+1}, x)$$

Taking as $n \rightarrow \infty$, we get

$$d(Tx, x) = 0 \Rightarrow Tx = x.$$

Theorem 3.4: Assume that all the hypothesis of theorem (3.1), (3.2) and (3.3) hold. Adding the following condition:

(3.7) If $x = Tx$, then $\alpha(x, Tx) \geq 1$.

We obtain the uniqueness of fixed point.

Proof: Let z and z^* be two distinct fixed point of T in the setting of Theorem (3.1) and condition (3.7) holds, then

$\alpha(z, Tz) \geq 1$ and $\alpha(z^*, Tz^*) \geq 1$.

So, $d(Tz, Tz^*) + l \leq (d(Tz, Tz^*) + l)\alpha(z, Tz)\alpha(z^*, Tz^*)$

$$\leq \beta(M(z, z^*))M(z, z^*) + l. \quad (25)$$

Where $M(z, z^*) = \max \{d(z, z^*), d(Tz, z), d(Tz^*, z)\}$

$$= d(z, z^*).$$

So, equation (25) implies

$$d(z, z^*) = d(Tz, Tz^*) \leq \beta(d(z, z^*))d(z, z^*)$$

$$\Rightarrow \beta(d(z, z^*)) = 1$$

$$\Rightarrow d(z, z^*) = 0 \Rightarrow z = z^*.$$

Similarly, one can prove for theorem (3.2) and (3.3).

Example 3.5: Let $X = \{0, 1, 2\}$ and $d(x, y) = |x - y|$. Clearly, (X, d) is a complete metric space.

Define $T(0) = 0, T(1) = 0$ and $T(2) = 2$.

All possible pairs of (x, y) are as follows:

(x, y)	$d(Tx, Ty)$	$d(x, Tx)$	$d(y, Ty)$	$d(x, y)$	$M(x, y)$
$(0, 0)$	0	0	0	0	0
$(0, 1)$	0	0	1	1	1
$(0, 2)$	2	0	0	2	2
$(1, 1)$	0	1	1	0	1
$(1, 2)$	2	1	0	1	1
$(2, 2)$	0	0	0	0	0

Let $\alpha = 1$ and $\beta = \frac{1}{2}$.

Putting these values in equation (1), we get

When $(x, y) = (0, 0), (0 + l) \leq \frac{1}{2}(0) + l \Rightarrow l \leq l$.

When $(x, y) = (0, 1), (0 + l) \leq \frac{1}{2}(1) + l \Rightarrow l \leq \frac{1}{2} + l$.

When $(x, y) = (0, 2), (2 + l) \leq \frac{1}{2}(2) + l \Rightarrow 2 + l \leq 1 + l$.

When $(x, y) = (1, 1), (0 + l) \leq \frac{1}{2}(1) + l \Rightarrow l \leq \frac{1}{2} + l$.

When $(x, y) = (1, 2), (2 + l) \leq \frac{1}{2}(1) + l \Rightarrow 2 + l \leq \frac{1}{2} + l$.

When $(x, y) = (2, 2), (0 + l) \leq \frac{1}{2}(0) + l \Rightarrow l \leq l$.

Hence, theorem (3.1) is verified.

4. Consequences:

Some existing results of literature can be deduced from our main results as follows:

Corollary 4.1.(Farhan *et al.*[2]) Let (X, d) be a complete metric space and $f: X \rightarrow X$ be an α -admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$(d(fx, fy) + l)^{\alpha(x, fx)\alpha(y, fy)} \leq \beta(d(x, y))d(x, y) + l$$

for all $x, y \in X$ where $l \geq 1$. Suppose that either

- (a) f is continuous, or
- (b) if $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n , then $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point.

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 1, one can get the proof.

Corollary 4.2. (Farhan *et al.*[2]) Let (X, d) be a complete metric space and $f: X \rightarrow X$ be an α -admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$(\alpha(x, fx)\alpha(y, fy) + 1)^{d(fx, fy)} \leq 2^{\beta(d(x, y))d(x, y)}$$

for all $x, y \in X$. Suppose that either

- (a) f is continuous, or
- (b) If $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n , then $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point.

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 2.

Corollary 4.3. (Farhan *et al.*[2]) Let (X, d) be a complete metric space and $f: X \rightarrow X$ be an α -admissible mapping. Assume that there exists a function $\beta: [0, \infty) \rightarrow [0, 1]$ such that, for any bounded sequence $\{t_n\}$ of positive reals, $\beta(t_n) \rightarrow 1$ implies $t_n \rightarrow 0$ and

$$\alpha(x, fx)\alpha(y, fy)d(fx, fy) \leq \beta(d(x, y))d(x, y)$$

for all $x, y \in X$. Suppose that either

- (a) f is continuous, or
- (b) If $\{x_n\}$ is a sequence in X such that $x_n \rightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$ for all n , then $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$, then f has a fixed point.

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 3, one can get the proof easily.

Corollary 4.4. (Farhan *et al.*[2]) Assume that all the hypotheses of theorem (1), (2) and (3) hold. Adding the following condition:

- (c) If $x = fx$, then $\alpha(x, fx) \geq 1$,

we obtain the uniqueness of the fixed point of f .

Proof: Taking $M(x, y) = d(x, y)$ in Theorem 4.

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