Homotopy Continuation Method of Arbitrary Order of Convergence for Solving Differenced Kepler's Equation

Abstract: This paper deals with an efficient iterative method of arbitrary integer order of convergent ≥ 2 which will be established for the solution of differenced Kepler's equation. This method by the way of dynamical aspect, where going from one iterative model to another subsequent using additional instruction . Where, the most important hint that, the method does not need any prior knowledge of the initial guess. A property avoids the critical situations between divergent to very slow convergent solutios, that may exist in other numberical methods that depend on initial guess. Finally, computeational algorithm and numerical example for the method is given.

Key Words: Initial value problem, differenced Kepler's Equation, Homotopy continuation method, space dynamics.

1.Introduction

Mostly , equations resulting in much problems of applied mathematics , are highly transcendental and could be solved by iterative methods which in turn need: (a) initial guess, (b) an iterative scheme. Realy , these two points are not separated from each other, but there is a full agreement that, even accurate iterative schemes are extremely sensitive to initial guess. Moreover, in many cases the initial guess may lead to drastic situation between divergent and very slow convergent solutions.

In the field of numerical analysis, powerful techniques have been devoted [4] to solve transcendental equations without any priori knowledge of the initial guess. These techniques are known as homotopy continuation methods. The method was first applied to the universal initial value problem of space dynamics [5] , in stellar statistics [6] and for non linear algebraic equation as many papers as [7].

In the present work , we will establish an efficient iterative method of arbitrary integer order of convergent ≥ 2 for the solution of differenced kepler's equation. This method is of dynamic nature , where, on going from one iterative scheme to the subsequent one, only additional instruction is needed. Moreover, the most important of this method that, it does not need any priori knowledge of the initial guess. A property which avoids the critical situations between divergent to very slow convergent solutios, that may exist in other numberical methods which depend on initial guess. Finally, computeational package for digital implementation of the method is given

2-One-Point Iteration Formulae for Solving Y(x) = 0

Let Y(x) = 0 such that Y: **R** $\rightarrow$ **R** smooth map and has a solution $x = \xi(\text{say})$. To construct iterative schemes for solving this equation, some basic definitions are to be recalled as follows:

1-The error in the k^{th} iterate is defined as

$$\varepsilon_k = \xi - x_k .$$

2- If the sequence $\{x_k\}$ converges to $x = \xi,$ then

$$\lim_{k\to\infty} x_k = \xi$$

3 -If there exists a real number $p \geq 1$ such that

$$\lim_{i\to\infty} \frac{|x_{i+1} - \xi|}{|x_i - \xi|^p} = \lim_{i\to\infty} \frac{|\varepsilon_{i+1}|}{|\varepsilon_i|^p} = K \neq 0$$

we say that, the iterative scheme is of order p at ξ. The constant K is called the asymptotic error constant. For $p=1$, the convergence is linear; for $p=2$, the convergence is quadratic; for $p=3,4,5$ the convergence is cubic, quartic and quintic, respectively.

4- One- point iteration formulae are those which use information at only one point. Here, we shall consider only stationary one-point iteration formulae which have the form

$$x_{i+1} = R(x_i), i = 0,1,\cdots \tag{1}$$

5- The order of one point iteration formulae could be determine either from: (a) The Taylor series of the iteration function $R(x_n)$ about ξ e.g [1]. or from , (b) The Taylor series of the function $Y(x_{k+1})$ about x_k [2].

On the bases of the second approach mentioned above [point (b)] it is easy to form a class of iterative formulae containing members of all integral orders [3] to solve Equation (1) as

$$x_{i+1} = x_i + \delta_{i,m+2};\ i=0,1,2,\ldots;\ m=0,1,2,... \tag{2}$$

where

$$\delta_{i,m+2} = \frac{-Y_i}{\sum_{j=1}^{m+1} (\delta_{i,m+1})^{j-1} Y_i^{(j)} / j!}; \qquad \delta_{i,1} = 1; \forall i \ge 0. \tag{3}$$

$$Y_i^{(j)} \equiv \left.\frac{d^j Y(x)}{dx^j}\right|_{x=x_i}; \qquad Y_i \equiv Y_i^{(0)}. \tag{4}$$

The convergence order is $m+2$ and is given as

$$\varepsilon_{i+1} = -\frac{1}{(m+2)!} \; \frac{Y(\xi)^{(m+2)}}{Y_i^{(1)}(\xi_1)} \varepsilon_i^{m+2}, \tag{5}$$

where ξ between x_{i+1} and x_i and ξ_1 between x_{i+1} and ξ.

3-Homotopy Continuation Method for solving Y(x) = 0

3-1 Formulations

Suppose one wishes to obtain a solution of a single non-linear equation in one variable x (say)

$$Y(x) = 0, \tag{6}$$

where , $Y : \mathbf{R} \to \mathbf{R}$ is a mapping which , for our application assumed to be smooth that is , a map has as many continuous derivatives as requires . Let us consider the situation in which no priori knowledge concerning the zero point of Y is available. Since we assume that such a priori knowledge is not available, then any of the iterative methods will often fail to calculate the zero $\bar{x}$, because poor starting value is likely to be chosen As a possible remedy, one defines a homotopy or deformation $H : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ such that

$$H(x,1) = Q(x) \quad ; \quad H(x,0) = Y(x),$$

where $Q : \mathbf{R} \to \mathbf{R}$ is a (trivial) smooth map having known zero point and H is also smooth . Typically, one may choose a convex

$$H(x,\lambda) = \lambda \; Q(x) + (1-\lambda)\, Y(x). \tag{7}$$

and attempt to trace an implicitly defined curve $\Phi(z) \in H^{-1}(0)$ from a starting point $(x_1, 1)$ to a solution point $(\bar{x}, 0)$ .If this succeeds, then a zero point $\bar{x}$ of Y is obtained.

3-2 Embedding methods

The basic idea of the embedding methods referred to at the end of Subsection 3-1 is explained in the following algorithm for tracing the curve $\Phi(z) \in H^{-1}(0)$ from, say $\lambda = 1$ to $\lambda = 0.$

Computational Algorithm1

● **Purpose**

To solve Y(x) = 0 by embedding method.

● **Input**

(1) The function Q(x) with defined root x_1 such that H (x_1,1) = 0,

(2) positive integer m.

● **Output**

Solution x of Y(x) = 0.

Computational sequence:

1-Set $x = x_1, \lambda = (m-1)/m, \Delta\lambda = 1/m.$

2-For i = 1 to m do

begin

Solve $H(y, \lambda) = \lambda Q(y) + (1-\lambda)Y(y) = 0$ iteratively for y using x as starting value.

x = y.

$\lambda = \lambda - \Delta\lambda.$

end

4- Application of Homotopy Method for Solving Differenced Kepler's Equation

4-1 Differenced Kepler's equation

Let $(E_n, E_\ell), (M_n, M_\ell)$ be the eccentric and the mean anomalies associated with the position vectors $(\mathbf{r_n}, \mathbf{r}_\ell)$ at the two epochs t_n and t_ℓ $(\ell \succ n)$ of an elliptic orbit. .

The differenced Kepler's equation is given as:

$$M_\ell - M_n = (E_\ell - E_n) + \frac{\sigma n}{\sqrt{a}}\{1 - \cos(E_\ell - E_n)\} - \left(1 - \frac{r_n}{a}\right)\sin(E_\ell - E_n),$$

where

$$\sigma_n = \frac{1}{\sqrt{\mu}} < \mathbf{r_n}, \mathbf{v_n} >$$

The above equation could be written as:

$$Y = G - C_n \sin G - S_n \cos G + S_n - W = 0, \quad (8)$$

where

$$W = M_\ell - M_n \quad , \quad G = E_\ell - E_n \quad ; \quad C_n = 1 - \frac{r_n}{a} \quad ; \quad S_n = \frac{\sigma_n}{\sqrt{a}}. \quad (9)$$

The computed value of G may be checked by the condition that;

$$\text{Check} = G - C_n \sin G - S_n \cos G + S_n - W \approx 0$$

Two notes are to be recorded as follows:

1-From Equation (5) it is clear that, an iterative scheme for solving Equation (8) includes derivatives of Y as much as the order of the scheme. On the other hand, the higher the order of an iterative scheme, the higher its accuracy and rate of convergence will be. Regarding this last fact, the remarkable simplicity of the derivative formulae of Y which are

$$Y^{(1)}(G) = 1 - C_n \cos G + S_n \sin G,$$

$$Y^{(2)}(G) = C_n \sin G - S_n \cos G,$$

$$Y^{(3)}(G) = C_n \cos G - S_n \sin G,$$

$$Y^{(k)}(G) = -Y^{(k-2)}; k \geq 4$$

enables us to find derivatives of $Y(G)$ as many as we need.

2-Homotopy continuation method is powerful technique for solving $Y(G) = 0$ without priori knowledge of the initial guess.

From these two notes, we can now establish for the solution of Kepler's Equation (8), an iterative algorithm of any positive integer order $l \geq 2$. Moreover, the algorithm does not need priori knowledge of the initial guess. According to Equation (5), the algorithm is of dynamic nature in the sense that, it includes iterative schemes up to the l^{th} order such that, in going from one scheme to the subsequent one, only additional instruction is needed.

This algorithm is illustrated in what follows with algorithm 1 augmented to it, together with the Q function of the homotopy H [Equation (7)] as Q(x) = x-1, so that

$H(x_1, 1) = 0$, where $x_1 = 1$.

4.2 Computational Algorithm

Purpose : To solve kepler's equation by iterative schemes of quadratic up to l^{th} convergence orders without priori knowledge of the initial guess using homotopy continuation method with $Q(G) = G - 1$

Input :m (positive integer $3 \leq m \leq 20$), $W, Cn(\equiv C_n), Sn(\equiv S_n), n(\equiv \ell)$ ε (specified tolerance $\approx 10^{-6}$) ,

Computational Sequence

1- Set $G = 1$; $\Delta\lambda = 1/m$; $\lambda = 1 - \Delta\lambda$

2- For i : =1 to m do

begin{i}

$$Q = 1 - \lambda$$

$$Y = \lambda(G - 1) + Q\{G - Cn \ \ \sin G - Sn \cos G + Sn - W\}$$

$$Y^{(1)} = \lambda + Q\{1 - Cn \cos G + Sn Sin G\}$$

$$\Delta G = -Y / Y^{(1)}$$

If $[\ell = 2, \text{If}[|\Delta G| \leq \in, \text{ go to step 4}]$

$$Y^{(2)} = Q(Cn \sin G + Sn \cos G)$$

$$H = Y^{(1)} + DE * Y^{(2)} / 2$$

$$\Delta G = -Y / H$$

If $[\ell = 3, \text{If}[|\Delta G| \leq \in, \text{ go to step 4}]$

$$Y^{(3)} = Q(Cn \cos G - Sn \sin G)$$

$$H = Y^{(1)} + DE * Y^{(2)} / 2 + (DE)^2 * Y^{(3)} / 6$$

$$\Delta G = -Y / H$$

If $[\ell = 4, \text{If}[|\Delta G| \leq \in, \text{ go to step 4}]$

$$L = \ell - 1$$

For k: = 4 to L do

begin {k}

set $Y^{(k)} = -Y^{(k-2)}; \quad n = k - 1; \quad H = Y^{(1)}; B = 1$

For j: = 1 to n do

begin {j}

$B = \Delta G * B/(j+1)$

$H = H + B * Y^{(j+1)}$

end {j}

$\Delta G = -Y/H$

end {k}

$G = G + \Delta G$

$\lambda = \lambda - \Delta\lambda$

end {i}

4- End

4.3 Numerical example

Consider the values $W = 6.30025$; $C_n = -0.324852$; $S_n = 0.41876$ with

$m = 10, \ell = 15, \varepsilon = 10.^{-6}$ .The result is: $G = 6.29604$, $\text{Check} = -8.88178 \times 10^{-16}$.

5- Conclusion

In concluding the present paper we stress that, an efficient iterative method of arbitrary integer order of convergent ≥ 2 will be established for the solution of differenced Kepler's equation. The method is of dynamic nature in the sense that, on going from one iterative scheme to the subsequent one, only additional instruction is needed. Moreover, which is the most important, the method does not need any priori knowledge of the initial guess. A property which avoids the critical situations between divergent to very slow convergent solutios, that may exist in other numberical methods which depend on initial guess.

Computeational package for digital implementation of the method is given

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