Original Research Article

DERIVATION OF FIXED-POINT THEOREM USING EXPANSIVE MAPPING APPROACH

ABSTRACT: Application of Fixed-Point Theorem has tremendously increased in different areas of interest and research. Fixed Point Theorem presents that if $T: X \to X$ is a contraction mapping on a complete metric space (X, d) then there exists a unique fixed point in X. A lot has been done on application of contraction mapping in Fixed Point Theorem on metric spaces such as cantor set with the contraction constant of $\frac{1}{3}$, the Sierpinski triangle also with contraction constant of $\frac{1}{2}$. On the other hand, a mapping $T: X \to X$ on (X, d) such that $\forall x, y \in X: d(Tx, Ty) \geq d(x, y)$ is called an expansive mapping. There are various types of expansive mappings such as; isometry expansive mapping, proper/strict expansive mapping and anti-contraction expansive mapping. From the available literature, Fixed Point Theorem has been derived using contraction mapping approach. In this paper, we establish that it is also possible to derive Fixed point theorem using expansive mapping approach.

Cantor, Singer, in capital letters.

Fixed-Point Theorem, in capital letters.

Keywords: *Fixed-point theorem; Expansive mapping; Contractive mapping*

1. INTRODUCTION

Definition 1.1 Expansive mapping (Mustafa and Sims [4])

Let (S, l) be a metric space. A mapping $R: S \to S$ on (S, l) such that $\forall\, x, y \in S: l(R_x, R_y) \geq l(x, y)$ is called an expansive mapping.

The following result is a consequence of the definition of expansive mapping.

Types of expansive mappings

- **I.** An expansion $R: S \to S$ such that $\forall\, x, y \in S: l(R_x, R_y) = l(x, y)$ is called an isometry, which is the weakest form of expansive mappings.
- **II.** An expansion $R: S \to S$ such that $\forall\, x, y \in S\; x \neq y: l(R_x, R_y) > l(x, y)$ is called a strict/proper expansion
- **III.** An expansion $R: S \to S$ such that $\exists\, E > 1\; \forall\, x, y \in S: l(R_x, R_y) \geq El(x, y)$ is called an anti-contraction expansion constant E.
  Strict expansion can be anti-contraction expansion but the converse is not true.

Yeilkaya and Aydin[2]gave a good account and in depth analysis of expansive mapping, efforts can be directed towards the existence of fixed point in expansive mappings.

The aim here is to review the expansive mappings and its properties. As this question on expansive mappings is too general since we have different spaces, the following two properties are necessary;

a) Does the fixed point exist under expansive mapping?
b) If the fixed point exists, is it unique?

The goal here is to present the recent results in trying to answer the questions above. The key idea is to combine the iteration property and expansive mapping properties with an aim of achieving the results.

The following definition is useful in achieving the main results.

Definition 1.2 Contraction (Ahmed [1]) Let $R: S \to S$ be a contraction mapping on a complete metric space (S, l), then there is exactly one solution $x \in S$.

Types of contractive mappings

Abbas, M., and Rhoades, B. E. [3] presented the following types of contractive mapping, which are useful in achieving main results.

I. A mapping $R: S \to S$ such that $\forall\, x, y \in S: l(x, y) = l(R_x, R_y)$ is an isometry mapping since they preserve the isometric distance. This case the object distance is equal to the image distance.

II. A mapping $R: S \to S$ such that $\exists\, x, y \in S\ x \neq y: l(R_x, R_y) < l(x, y)$ is called a proper/strict contraction.

Remark: The main aim is to show that it is possible to extend the results of contraction mapping and utilize it in expansive mapping so as to achieve the main results of this study

Fixed-Point Theorem under expansive mapping approach.

Recall the definition of fixed-point iteration and expansion mapping that is useful in achieving the results.

Definition 1.3 Expansion(Huang and Wen, [9])

Let (S, l) be a complete metric space. If $R: S \to S$ is an onto mapping and $\exists$ a constant $k > 1$ such that $l\ (R_x, R_y) \geq k\ l\ (x, y)$ for each $x, y \in S$. Then R has a unique fixed point.

Definition 1.4 (Fixed-point iteration (Shahi,*et al* [7]).

Let f be a function defined on $\mathbb{R}$ and a point x_0 in the domain of f, the fixed-point iteration is $x_{n+1} = f(x_n)$, $n = 0,1,2 \ldots$which yields the sequence $x_0, x_1, x_2, \ldots$which converges to x

Definition 1.5. Isometry(Shahi P. *et al*, [7]) An expansion $T: X \to X$ such that $\forall x, y \in X, x \neq y: d\ (Tx, Ty) = d\ (x, y)$ which is the weakest form of expansive mappings.

Definition 1.6 Proper/strict expansion(Park and Rhoades, [10]) An expansion $T: X \to X$ such that $\exists x, y \epsilon X, x \neq y: d\ (T_x, T_y) > d\ (x, y)$.

Definition 1.7 Anti-contraction mapping(Sessa,S.[6]) An expansion $T: X \to X$ such that $\exists\ E > 1\ \forall x, y \in X: d\ (Tx, Ty)\ \geq E\ d\ (x, y)$ with expansion constant E.

By utilizing the definitions above then the main aim is to determining the existence of unique fixed point under expansive mapping, the following are the main results;

2. MAIN RESULTS

Lemma 2.1(Sahin and Telci, [5])

Let H be a Hilbert space and let $T: H \to H$ be a continuous map, which is expansive.

i.e. $\parallel T_x - T_y \geq \parallel x - y \parallel$ for all $x, y \in H$ and $T\theta = \theta$ suppose T maps a neighborhood of the origin onto a neighbourhood of the origin. Does T Maps Honto H?

Let $T_x = p$ for every $p \in H$ thus if we consider hT instead of T withoutloss of generality, we may assume T is expansive mapping with constant $h > 1$; that is

$\parallel T_x - T_y \parallel \geq h \parallel x - y \parallel$for all $x, y \in H$.

The lemma above tryto apply the restricted expansive mapping on Hilbert spaces.

Theorem 2.2 (Suantai*et al,*[8])

Let $T: E \to E$ be either a contraction or an expansion with constant $h > 1$. If $T(E) = E$ then $(1 - T)(E) = E$ furthermore, if T is an expansion with constant $h > 2$, then $(1 - T)(E) = E$ implies $T(E) = E$. Assume T is an expansion for $y \in E$define $T_y: E \to E$ by T_yx $= T_x + y$ then T_y maps E onto itself for $h > 2$, let $S = 1 - T$ then $\parallel S_x - S_y \parallel T_x - T_y \parallel x$-$y \parallel \geq$(h-1)$\parallel x - y \parallel$ for all $x, y \in E$ thus S is expansive since $h - 1 > 1$.

Theorem 2.1

Let $R: S \to S$ be an anti-contraction expansive mapping on a complete metric space S and R is onto. Then R has a fixed point in S.

Proof

Let $x_0 \in S$, since R is bijective(inverse exist), then $\exists$ an element x_1 satisfying $x_1 \in R^{-1}(x_0)$. By the same way, construct a sequence $x_n \in R^{-1}(x_{n-1})$ where ($n = 1, 2, 3, \dots$) here the concept of iteration as $S_n = R(S_{n-1})$ is applied.

If $x_{m-1} = x_m$ for some m, then x_m is a fixed point of R,suppose $x_n \neq x_{n-1}$ for every n. So,

From iteration $S_n = R(S_{n-1})$ it implies;

$$l(x_{n+1}, x_{n+2}) = l\big(R(x_n), R(x_{n+1})\big)$$

Since this is an expansive mapping then

$l(R(x_n), R(x_{n+1})) \geq kl(x_n, x_{n+1})$ Where $k > 1$ is expansive constant

$$l(x_n, x_{n+1}) \leq \frac{1}{k} l(R_{n-1}, Rx_n) \leq \left(\frac{1}{k}\right)^n l(x_0, x_1)$$

According to triangular property that is $\{d(a,c) \leq d(a,b) + d(b,c)\}$then ;

$$l(x_n, x_{n+m}) \leq l(x_n, x_{n+1}) + \cdots + l(x_{n+m-1}, x_{n+m}) \leq$$

$$\left(\frac{1}{k}\right)^{n+1} l(x_0, x_1) + \cdots + \left(\frac{1}{k}\right)^{n+m} l(x_0, x_1)$$

Since this will form a geometric series with the common ratio $\frac{1}{k}$ and the first term as $\left(\frac{1}{k}\right)^2$ then;

$$\leq \frac{\left(\frac{1}{k}\right)^{n+1}}{1 - \frac{1}{k}} l_N(x_0, x_1)$$

Let $\varepsilon > 0$, be arbitrary since $k \in \mathbb{N}$, find a large $N \in \mathbb{N}$ so that

$$k^N > \frac{\varepsilon(1 - \frac{1}{k})}{l(x_1, x_{0)}}$$

Therefore, by choosing m and n greater than N write as :

$$l(x_m, x_n) \leq K^n l(x_1, x_0)(\frac{1}{1-\frac{1}{K}}) < \left\{\frac{\varepsilon\left(1-\frac{1}{K}\right)}{l(x_1, x_0)} l(x_1, x_0)(\frac{1}{1-\frac{1}{K}}) = \varepsilon\right.$$

Based on the above it shows that the sequence $\{S_n\}$ is Cauchy since every point on it converges to a limit on S .Since (S, l) is complete then the limit $x^* \in S$ exist, by utilizing the definition of completeness. Moreover, x^* must be a fixed point of R

$$x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} R(x_{n-1}) = R(\lim_{n\to\infty} x_{n-1}) = R(x^*)$$

Since R is bijective then continuity is assured since the inverse exist cannot have more than one fixed point in (S, l)since it is onto and one to one. Taking any pair of distinct points p_1 and p_2 as their fixed point will end up contradicting the expansion of R. Such that;

$$l\{R(p_1), R(p_2)\} = l(p_1, p_2) < kl(p_1, p_2)$$

It then follows that the points p_1 and p_2 must not be distinct, that is $p_1 = p_2$. Hence R has a unique fixed point.

3. CONCLUSION

Unique Fixed point exist in a metric space under restricted expansive mappings for it to converge.(Prospects for future research?)

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