Exact Solutions for Some Singular Linear Ordinary Differential Equations of High Orders via NB1 Polynomials

Abstract

In this study, we numerically solve the singular linear ordinary differential equations (SLODEs) of higher order using the collocation method based on the NB1 polynomial. An operational matrix form of the given ordinary differential equations (ODEs), the relations of various solutions and the derivatives are obtained from NB1 polynomials. The proposed method reduces the given problem to a linear algebraic equation system, which removes the singularity of ordinary differential equations. The inverse matrix method is used to solve the resulting system to obtain the coefficients of NB1 polynomials. The obtained exact solutions to different problems of high orders show the reliability and accuracy of the presented method.

Keywords: Collocation method, NB1 polynomial, operational matrix form, singular differential equations.

1. Introduction

The equations Emden-Fowler with appropriate initial and boundary value problems appear in various fields of engineering and science, for instance, fluid mechanics and quantum, chemical reactors, optimal design, geophysics and so on, for more details, see [6, 9, 19, 31, 39, 43].

In this study, we will consider the singular linear Emden-Fowler of order $(m+1)$, $m \geq 1$ in the following general form:

$$\omega^{(m+1)}(r) + \frac{\mu_1}{r}\omega^{(m)}(r) + \frac{\mu_2}{r^2}\omega^{(m-1)}(r) + \mathrm{g}(r)\omega(r) = h(r), \qquad (1)$$

with initial conditions (ICs.)

$$\omega(0) = \omega'(0) = \omega''(0) = \cdots = \omega^{(m)}(0) = 0, \qquad (2)$$

where μ_1 and μ_2 are appropriate constants, g and h are given real values functions of the variable r and $\omega(r)$ is an unknown function.

We can derive some various order kinds of Emden-Fowler Eq. (1) from the following equation

$$r^{-t}\,\frac{d}{dr}\left[r^{t-1}\,\frac{d^s}{dr^s}(r\,\omega)\right] + \mathrm{g}(r)\omega - h(r) = 0, \quad t, s \geq 1. \qquad (3)$$

For $s = 1, 2, 3, \cdots, m$, we get the first kind, the second kind, and the third kind up to $(m+1)th$ kind respectively of the Emden-Fowler equation, as follows

$$\omega^{(2)} + \frac{t+1}{r}\omega^{(1)} + \frac{t-1}{r^2}\omega + \mathrm{g}(r)\omega - h(r) = 0, \qquad (4)$$

$$\omega^{(3)} + \frac{t+2}{r}\omega^{(2)} + \frac{2(t-1)}{r^2}\omega^{(1)} + \mathrm{g}(r)\omega - h(r) = 0, \qquad (5)$$

$$\omega^{(4)} + \frac{t+3}{r}\omega^{(3)} + \frac{3(t-1)}{r^2}\omega^{(2)} + \mathrm{g}(r)\omega - h(r) = 0, \qquad (6)$$

$$\vdots$$

$$\omega^{(m+1)} + \frac{t+m}{r}\omega^{(m)} + \frac{m(t-1)}{r^2}\omega^{(m-1)} + \mathrm{g}(r)\omega - h(r) = 0, \qquad (7)$$

where $\mu_1 = t + s$ and $\mu_2 = s(t-1)$, $s = 1, 2, \cdots, m$.

If $m = 1$ and $\mu_2 = 0$, Eq. (1) becomes

$$\omega'' + \frac{\mu_1}{r}\omega' + \mathrm{g}(r)\omega - h(r) = 0, \qquad (8)$$

which called the Lane- Emden equation. The standard linear Lane- Emden equation is obtained from Eq. (8) When $\mu_1 = 2$, $\mathrm{g}(r) = 1$ and $h(r) = 0$. Many models of Eq. (8) with some appropriate fixed values of μ_1 as astrophysics phenomena and a mathematical physics you can find in [10, 13, 17, 20, 24, 25, 26, 28, 40, 41, 45]. The singularity at $r = 0$ makes a solution of these types of equations is difficult.

The Emden-Fowler and Lane-Emden equations are solved analytically and numerically by various methods. [15, 16, 27, 39, 42] used the Adomian decomposition method, [3, 7, 32] employed the homotopy analysis method, [36, 44, 46] used the variational iteration method. You can find more different methods in [5, 11, 29, 30, 33, 35, 37, 47]. In the 20th century, the Taylor wavelet method solved the linear and nonlinear Lane-Emden Equations [14]. In [18], Kashem solved the nonlinear Lane-Emden problem by modifying Hermite operational matrix method. Haar wavelet quasilinearization method was used to solve the Emden–Fowler type equations [34]. In [22], Said-Ball Polynomials were employed to solve some linear delay differential equations. The DP-Ball and Wang-Ball polynomials [21, 23] were used to solve singular ordinary differential equations. In [12], Bessel matrix method was used to solve linear and nonlinear singular differential equations. In [38], Wang et al. used the Chebyshev collocation method to solve singular ordinary differential equations. The least square method depends on Wang Ball function to find an approximate solution of ODEs for higher order [4]. Lastly, Said-Ball Polynomials were used to solve singular ordinary differential equations in [2].

The main purpose of this research is to use the method of an operational matrix depending on NB1 Polynomials to solve high order of singular linear ordinary differential equations (SLODEs).

We organize this paper as follows: Some concepts of NB1 polynomials and NB1 monomial formulas are presented in section 2. Section 3 shows the fundamental matrix relations. Numerical examples are presented in section 4. Section 5 is a conclusion of this study.

2. The NB1curve and NB1 monomial formulas:

2.1 Definition: An NB1 polynomial $N_i^n(r)$ of degree n $(n > 3)$ is defined as [8]

$$N_i^n(r) = \begin{cases} \binom{\lfloor \frac{n}{2} \rfloor + i - 1}{i} r^i (1-r)^{\lceil \frac{n}{2} \rceil}, & 0 \le i \le \lfloor \frac{n}{2} \rfloor - 2, \\ \binom{2(\lfloor \frac{n}{2} \rfloor - 1)}{\lfloor \frac{n}{2} \rfloor - 1} r^{\lfloor \frac{n}{2} \rfloor - 1} (1-r)^{1+\lceil \frac{n}{2} \rceil}, & i = \lfloor \frac{n}{2} \rfloor - 1, \\ 2\binom{2(\lfloor \frac{n}{2} \rfloor - 1)}{\lfloor \frac{n}{2} \rfloor - 1} r^{\lfloor \frac{n}{2} \rfloor} (1-r)^{\lceil \frac{n}{2} \rceil}, & i = \lfloor \frac{n}{2} \rfloor, \\ N_{n-i}^n(1-r), & \lfloor \frac{n}{2} \rfloor + 1 \le i \le n, \end{cases} \tag{9}$$

where $\lfloor k \rfloor$ denotes the greatest integer less than or equal to k and $\lceil k \rceil$ denotes the least integer greater than or equal to k .

2.2 Definition: An NB1curve $N^n(r)$ of degree n with $n+1$ control points, denoted by $\{\xi_i\}_{i=0}^n$, can be expressed power basis as [1]

$$N^n(r) = \sum_{i=0}^{n} \sum_{j=0}^{n} \xi_i . \grave{u}_{\,i,j} . r^j , \tag{10}$$

where

$$\grave{u}_{i,j} = \begin{cases} (-1)^{-i+j}\binom{\left\lfloor\frac{n}{2}\right\rfloor + i - 1}{i}\binom{\left\lfloor\frac{n}{2}\right\rfloor}{j-i}, & 0 \le i \le -2 + \left\lfloor\frac{n}{2}\right\rfloor, \\ (-1)^{-i+j}\binom{2i}{i}\binom{i+2}{-i+j}, & i = -1 + \left\lfloor\frac{n}{2}\right\rfloor, \\ (-1)^{-i+j}\, 2\binom{2(i-1)}{i-1}\binom{-i+n}{-i+j}, & i = \left\lfloor\frac{n}{2}\right\rfloor, \\ (-1)^{-i+j}\, 2\binom{2(-i+n-1)}{-i+n-1}\binom{-i+n}{-i+j}, & i = \left\lceil\frac{n}{2}\right\rceil, \\ (-1)^{i+j-n}\binom{2(-i+n)}{-i+n}\binom{-i+n}{i+j-n-2}, & i = \left\lceil\frac{n}{2}\right\rceil + 1, \\ (-1)^{j-\left\lfloor\frac{n}{2}\right\rfloor}\binom{-i+n+\left\lfloor\frac{n}{2}\right\rfloor - 1}{-i+n}\binom{-i+n}{j-\left\lfloor\frac{n}{2}\right\rfloor}, & \left\lceil\frac{n}{2}\right\rceil + 2 \le i \le n. \end{cases} \qquad (11)$$

The NB1 monomial matrix is given by

$$\Gamma_{(n+1)\times(n+1)} = \begin{bmatrix} \grave{u}_{0,0} & \grave{u}_{0,1} & \dots & \grave{u}_{0,n} \\ \grave{u}_{1,0} & \grave{u}_{1,1} & \dots & \grave{u}_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ \grave{u}_{n,0} & \grave{u}_{n,1} & \dots & \grave{u}_{n,n} \end{bmatrix}, \qquad (12)$$

where $\grave{u}_{i,j}$ is given in Eq. (11).

To obtain approximate solutions of Eq. (1) with some appropriate initial conditions, we will use NB1polynomials in the form:

$$\omega_n(r) = \sum_{i=0}^{n} b_i \; N_i^n(r), \qquad (13)$$

where $b_i, i = 0,1,\cdots,n$ are unknown NB1 coefficients to be determined and $N_i^n(r)$ are the NB1 polynomials.

3. Fundamental Matrix Relations

The approximate solution (13) will be written as:

$$\omega_n(r) = \boldsymbol{N}(r)\boldsymbol{B} \qquad (14)$$

Where $\boldsymbol{N}(r) = [N_0^n(r) \;\; N_1^n(r) \;\; \cdots \;\; N_n^n(r)]$ and $\boldsymbol{B} = [b_0 \;\; b_1 \;\; \cdots \;\; b_n]^T$.
Eq. **(**14**)** can be written as:

$$\omega_n(r) = \mathbf{X}_{\boldsymbol{n}}(r)\; \Gamma^T \boldsymbol{B} = \psi(r)\boldsymbol{B} \qquad (15)$$

Where $\psi(r) = \mathbf{X}_{\boldsymbol{n}}(r)\,\Gamma^T$, $\mathbf{X}_{\boldsymbol{n}}(r) = [1 \quad r \quad r^2 \quad \cdots \quad r^n]$ and Γ is the monomial matrix, that is defined in **(**12**)**.

3.1. The First Derivative of The Matrix Relation:

We will derive Eq. (14) as:

$$\begin{aligned} \omega_n'(r) &= \boldsymbol{N}'(r)\,\boldsymbol{B} \\ &= \psi'(r)\,\boldsymbol{B} \\ &= \mathbf{X'}_{\boldsymbol{n}}(r)\;\Gamma^T\,\boldsymbol{B} \\ &= \frac{d}{dt}\left(\mathbf{X}_{\boldsymbol{n}}(r)\right)\,\Gamma^T\,\boldsymbol{B} \end{aligned}$$

$$= [0 \quad 1 \quad 2r \quad \cdots \quad nr^{n-1}]\ \Gamma^T \boldsymbol{B}$$

$$= \begin{bmatrix} 1 & r & r^2 & \cdots & r^n \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \Gamma^T \boldsymbol{B}$$

$$= \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}\,\Gamma^T \boldsymbol{B},$$

where

$$\boldsymbol{V} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

Thus

$$\omega_n'(r) = \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}\ \Gamma^T \boldsymbol{B} \qquad (16)$$

Similarly,

$$\omega_n''(r) = \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{2}}\,\Gamma^T \boldsymbol{B} \qquad (17)$$

$$\vdots$$

$$\omega_n^{(\boldsymbol{m})}(r) = \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{m}}\,\Gamma^T \boldsymbol{B}, \qquad m = 0,1,2,\cdots \qquad (18)$$

Where

$$\boldsymbol{V}^0 = \boldsymbol{I}_{n+1} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{bmatrix}_{(n+1)\times(n+1)} \qquad when\ m = 0,$$

and $\omega_n^{(0)}(r) = \omega_n(r)$.

By using Eqs. (15) and (18) in Eq. (1), we get

$$\mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{m+1}}\,\Gamma^T \boldsymbol{B} + \frac{\mu_1}{r}\ \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{m}}\,\Gamma^T \boldsymbol{B} + \frac{\mu_2}{r^2}\ \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{m-1}}\,\Gamma^T \boldsymbol{B} + \mathrm{g}(r)\,\mathbf{X}_{\boldsymbol{n}}(r)\,\Gamma^T \boldsymbol{B} = h(r) \qquad (19)$$

Eq. (19) can be written as the following:

$$\left(\mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{m+1}} + \frac{\mu_1}{r}\ \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{m}} + \frac{\mu_2}{r^2}\ \mathbf{X}_{\boldsymbol{n}}(r)\boldsymbol{V}^{\boldsymbol{m-1}} + \mathrm{g}(r)\,\mathbf{X}_{\boldsymbol{n}}(r)\right)\Gamma^T \boldsymbol{B} = h(r) \qquad (20)$$

By using the following collocation points:

$$r_i = \frac{i+1}{n}\ , \qquad i = 1,2,\cdots,n, \qquad (21)$$

we obtain

$$\left(\boldsymbol{X}\,\boldsymbol{V}^{\boldsymbol{m+1}} + \boldsymbol{P}_0\,\boldsymbol{X}\,\boldsymbol{V}^{\boldsymbol{m}} + \boldsymbol{P}_1\,\boldsymbol{X}\,\boldsymbol{V}^{\boldsymbol{m-1}} + \boldsymbol{G}\,\boldsymbol{X}\right)\Gamma^T \boldsymbol{B} = \boldsymbol{H}, \qquad (22)$$

where

$$\boldsymbol{X} = \begin{bmatrix} 1 & r_0 & r_0^{\ 2} & \dots & r_0^{\ n} \\ 1 & r_1 & r_1^{\ 2} & \dots & r_1^{\ n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & r_n & r_n^{\ 2} & \dots & r_n^{\ n} \end{bmatrix}_{(n+1)\times(n+1)} ,\boldsymbol{G} = \begin{bmatrix} g(r_0) & 0 & \dots & 0 \\ 0 & g(r_1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & g(r_n) \end{bmatrix}_{(n+1)\times(n+1)}$$

$$\boldsymbol{P}_0=\begin{bmatrix}\frac{\mu_1}{r_0} & 0 & \dots & 0\\ 0 & \frac{\mu_1}{r_1} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & \frac{\mu_1}{r_n}\end{bmatrix}_{(n+1)\times(n+1)} ,\boldsymbol{P}_1=\begin{bmatrix}\frac{\mu_2}{r_0} & 0 & \dots & 0\\ 0 & \frac{\mu_2}{r_1} & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 0 & \frac{\mu_2}{r_n}\end{bmatrix}_{(n+1)\times(n+1)},$$

and $\boldsymbol{H}=[h(r_0)\quad h(r_1)\quad \dots\quad h(r_n)]^T$.

We can write Eq. (22) as the short form:

$$\boldsymbol{W}\boldsymbol{B}=\boldsymbol{H}\quad \text{or}\quad [\boldsymbol{W};\boldsymbol{H}], \tag{23}$$

where

$$\boldsymbol{W}=\boldsymbol{X}\,\boldsymbol{V}^{\boldsymbol{m+1}}\,\Gamma^T+\boldsymbol{P}_0\,\boldsymbol{X}\,\boldsymbol{V}^{\boldsymbol{m}}\,\Gamma^T+\boldsymbol{P}_1\,\boldsymbol{X}\,\boldsymbol{V}^{\boldsymbol{m-1}}\,\Gamma^T+\boldsymbol{G}\,\boldsymbol{X}\,\Gamma^T$$

Now, we apply the proposed method to initial conditions, we have

$$\omega^{(k)}(0)=\boldsymbol{X}(0)\boldsymbol{V}^{(k)}\,\Gamma^T\,\boldsymbol{B},\quad k=0,1,\cdots,m\,. \tag{24}$$

We can write Eq. (24) in the following matrix form:

$$\boldsymbol{U}\boldsymbol{B}=\boldsymbol{O}\quad \text{or}\quad [\boldsymbol{U};\boldsymbol{O}] \tag{25}$$

Next, we replace the last rows of the augmented matrix (23) with the rows of the initial conditions matrix (25), then the system (23) becomes

$$\widetilde{\boldsymbol{W}}\boldsymbol{B}=\widetilde{\boldsymbol{H}}\quad \text{or}\quad [\widetilde{\boldsymbol{W}};\widetilde{\boldsymbol{H}}], \tag{26}$$

where

$$\left[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{H}}\right]=\begin{bmatrix} w_{0,1} & w_{0,2} & \cdots & w_{0,n} & ; & h(r_0)\\ w_{1,1} & w_{1,2} & \cdots & w_{1,n} & ; & h(r_1)\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ w_{n-(m+1),1} & w_{n-(m+1),2} & \cdots & w_{n-(m+1),n} & ; & h(r_n)\\ u_{0,1} & u_{0,2} & \cdots & u_{0,n} & ; & 0\\ u_{1,1} & u_{1,2} & \cdots & u_{1,n} & ; & 0\\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots\\ u_{m,1} & u_{m,2} & \cdots & u_{m,n} & ; & 0\end{bmatrix}$$

If $\det(\widetilde{\boldsymbol{W}})\neq 0$, we get

$$\boldsymbol{B}=(\widetilde{\boldsymbol{W}})^{-1}\widetilde{\boldsymbol{H}}, \tag{27}$$

which is determined uniquely.

By using Eq. (27), we can solve the linear system (26) of $(n+1)$ unknown coefficients $b_i, i=0,1,\cdots,n$ of NB1 polynomials in $(n+1)$ equations to get the unknowns $b_i, i=0,1,\cdots,n$. Finally, we obtained the solution of the studied problems.

4. Numerical Examples

In this section, we present four examples of the singular linear ordinary differential equations (SLODEs) of various high orders.

Matlab program has carried out the computations of the examples.

Example 1: Consider the Lane-Emden equation **[23]**

$$\omega''+\frac{8}{r}\omega'+r\,\omega=r^5-r^4+44\,r^2-30r, \tag{28}$$

with ICs.

$$\omega(0) = 0, \qquad \omega'(0) = 0. \tag{29}$$

The exact solution to this problem is $\omega(r) = r^4 - r^3$.
Here, $g(r) = r$ and $h(r) = r^5 - r^4 + 44\,r^2 - 30r$.
We applied the mentioned method for $n = 4$, the collocation points are $r_0 = \frac{1}{4}, r_1 = \frac{1}{2}$, $r_2 = \frac{3}{4}$, $r_3 = 1$, $r_4 = \frac{5}{4}$, and the matrix equation (22) becomes

$$\left[\boldsymbol{X}_4(r)\boldsymbol{V}^2\,\Gamma^T + \boldsymbol{P}_0\,\boldsymbol{X}_4(r)\boldsymbol{V}\,\Gamma^T + \boldsymbol{G}\boldsymbol{X}_4(r)\,\Gamma^T\right]\boldsymbol{B} = \boldsymbol{H}, \tag{30}$$

where

$$\Gamma^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 \\ 1 & -6 & 4 & 0 & 0 \\ 0 & 6 & -8 & 2 & 0 \\ 0 & -2 & 4 & -2 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{V}^2 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{P}_0 = \begin{bmatrix} 32 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 0 & 0 \\ 0 & 0 & 32/3 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 32/5 \end{bmatrix}, \boldsymbol{G} = \begin{bmatrix} 1/4 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 3/4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5/4 \end{bmatrix}, \boldsymbol{H} = \begin{bmatrix} -4867/1024 \\ -129/32 \\ 1283/591 \\ 14 \\ 7073/222 \end{bmatrix},$$

$$\boldsymbol{X}_4 = \begin{bmatrix} 1 & 1/4 & 1/16 & 1/64 & 1/256 \\ 1 & 1/2 & 1/4 & 1/8 & 1/16 \\ 1 & 3/4 & 9/16 & 27/64 & 81/256 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 5/4 & 25/16 & 125/64 & 625/256 \end{bmatrix}.$$

Therefore, the augmented matrix (23) becomes

$$[\boldsymbol{W};\boldsymbol{H}] = \begin{bmatrix} -2935/64 & -2277/512 & 5897/256 & 4867/512 & 513/16 & ; & -4867/1024 \\ -111/8 & -127/16 & -31/8 & 129/16 & 65/4 & ; & -129/32 \\ -631/192 & -1765/1536 & -2277/256 & -2223/512 & 539/48 & ; & 1283/591 \\ 2 & 0 & 8 & -28 & 9 & ; & 14 \\ 1689/320 & -19837/2560 & 12157/256 & -32625/512 & 637/80 & ; & 7073/222 \end{bmatrix}. \tag{31}$$

By using initial conditions (29), we have from Eq. (25), the matrix form

$$[\,\boldsymbol{U};\boldsymbol{O}\,] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0;\ 0 \\ -2 & 2 & 0 & 0 & 1;\ 0 \end{bmatrix} \tag{32}$$

Now, replace the last two rows of the augmented matrix (31) by matrix (32), and we get

$$[\tilde{\boldsymbol{W}};\tilde{\boldsymbol{H}}] = \begin{bmatrix} -2935/64 & -2277/512 & 5897/256 & 4867/512 & 513/16 & ; & -4867/1024 \\ -111/8 & -127/16 & -31/8 & 129/16 & 65/4 & ; & -129/32 \\ -631/192 & -1765/1536 & -2277/256 & -2223/512 & 539/48 & ; & 1283/591 \\ 1 & 0 & 0 & 0 & 0 & ; & 0 \\ -2 & 2 & 0 & 0 & 1 & ; & 0 \end{bmatrix}. \tag{33}$$

By using Eq. (27), the solution of system (33) gives the coefficients of NB1 polynomials

$$b_0 = 0.0,\ b_1 = 0.0,\ b_2 = 0.0,\ b_3 = -0.5 \ and\ b_4 = 0.0\ .$$

That is,

$$\boldsymbol{B} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ -0.5 \\ 0.0 \end{bmatrix}$$

Substituting these values into Eq. (14), then the approximate solution is

$$\begin{aligned} \omega_4(r) &\cong \boldsymbol{N}(r)\boldsymbol{B} \\ &= \mathbf{X}_4(r)\ \Gamma^T\ \boldsymbol{B} \\ &= \begin{bmatrix} 1 & r & r^2 & r^3 & r^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 \\ 1 & -6 & 4 & 0 & 0 \\ 0 & 6 & -8 & 2 & 0 \\ 0 & -2 & 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} \\ &= -r^3 + r^4 = \omega(r) \quad \text{(exact solution)} \end{aligned}$$

Example 2: Consider the Emden-Fowler equation **[27]**

$$\omega^{(4)} + \frac{7}{r}\omega^{(3)} + \frac{9}{r^2}\omega^{(2)} + \omega = 300 + r^4, \tag{34}$$

with ICs.

$$\omega(0) = \omega'(0) = \omega''(0) = \omega'''(0) = 0, \tag{35}$$

$\omega(r) = r^4$ is the exact solution.

As above example, we apply the proposed method for $n = 4$. Then Eq. (34) takes the following form as Eq. (20):

$$\left[\boldsymbol{X}_4(r)\boldsymbol{V}^4\,\Gamma^T + \frac{7}{r}\,\boldsymbol{X}_4(r)\boldsymbol{V}^3\,\Gamma^T + \frac{9}{r^2}\,\boldsymbol{X}_4(r)\boldsymbol{V}^2\,\Gamma^T + \boldsymbol{X}_4(r)\,\Gamma^T\right]\boldsymbol{B} = 300 + r^4. \tag{36}$$

By using the collocation points, Eq. (36) becomes

$$\boldsymbol{W}\boldsymbol{B} = \boldsymbol{H}, \tag{37}$$

where $\boldsymbol{W} = \boldsymbol{X}\,\boldsymbol{V}^4\,\Gamma^T + \boldsymbol{P}_0\,\boldsymbol{X}\,\boldsymbol{V}^3\,\Gamma^T + \boldsymbol{P}_1\,\boldsymbol{X}\,\boldsymbol{V}^2\,\Gamma^T + \boldsymbol{G}\,\boldsymbol{X}\,\Gamma^T$ such that

$$\boldsymbol{P}_0 = diag\left(28,\ 14,\ 28/3,\ 7,\ 28/5\right),$$

$$\boldsymbol{P}_1 = diag\left(144,\ 36,\ 16,\ 9,\ 144/25\right),$$

$$\boldsymbol{G} = diag\left(1,\ 1,\ 1,\ 1,\ 1\right),$$

$$\boldsymbol{H} = \begin{bmatrix} 76801/256 & 4801/16 & 23725/79 & 301 & 10283/34 \end{bmatrix}^T,$$

$\boldsymbol{X}$ and Γ^T are the same as example 1.

Replacing the last four rows of matrix (37) by the rows of initial conditions matrix (35), and we use Eq. (27) to get the vector matrix $\boldsymbol{B}$ as the following

$$\boldsymbol{B} = [0 \quad -1/2 \quad -3/4 \quad -3/2 \quad 1]^T$$

Thus

$$\begin{aligned} \omega_4(r) &\cong \boldsymbol{N}(r)\boldsymbol{B} \\ &= \mathbf{X}_4(r)\ \Gamma^T\ \boldsymbol{B} \\ &= \begin{bmatrix} 1 & r & r^2 & r^3 & r^4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 1 \\ 1 & -6 & 4 & 0 & 0 \\ 0 & 6 & -8 & 2 & 0 \\ 0 & -2 & 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1/2 \\ -3/4 \\ -3/2 \\ 1 \end{bmatrix} \\ &= r^4 = \omega(r). \end{aligned}$$

Example 3: Consider (SLODE) **[15]**

$$\omega^{(5)} + \frac{3}{r}\omega^{(4)} + 5!\,\omega = 4 + r^5, \tag{38}$$

with ICs.

$$\omega(0) = \omega'(0) = \omega''(0) = \omega'''(0) = \omega^{(4)}(0) = 0. \tag{39}$$

The exact solution is $\omega(r) = \frac{1}{5!}r^5$.

Applying the proposed method for $n = 5$ yields that

$$\boldsymbol{B} = [0 \quad 0 \quad -1/480 \quad 0 \quad -1/80 \quad 1/120]^T,$$

so we obtain

$$\begin{aligned}\omega_5(r) &\cong \boldsymbol{N}(r)\boldsymbol{B} \\ &= \mathbf{X}_5(r)\ \Gamma^T\ \boldsymbol{B} \\ &= \begin{bmatrix} 1 & r & r^2 & r^3 & r^4 & r^5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 & 0 \\ 1 & -6 & 4 & 0 & 0 & 1 \\ 0 & 6 & -12 & 4 & 2 & 0 \\ 0 & -2 & 12 & -8 & -2 & 0 \\ 0 & 0 & -4 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1/480 \\ 0 \\ -1/80 \\ 1/120 \end{bmatrix} \\ &= \frac{1}{5!}r^5 = \omega(r).\end{aligned}$$

Example 4: Consider (SLODE) **[15]**

$$\omega^{(6)} + \frac{1}{r}\omega^{(5)} + 6!\,\omega = -r^6 + 2, \tag{40}$$

with ICs.

$$\omega(0) = \omega'(0) = \omega''(0) = \omega'''(0) = \omega^{(4)}(0) = \omega^{(5)}(0) = 0. \tag{41}$$

The exact solution is $\omega(r) = \frac{1}{6!}r^6$.

In the same manner, we applied the proposed method when $n = 6$ and obtained

$$\boldsymbol{B} = [0 \quad 0 \quad 1/690 \quad 1/1080 \quad 1/720 \quad -1/360 \quad 1/720]^T,$$

after that we substituted these values of $\boldsymbol{B}$ in Eq. (15) thus getting the exact solution

$$\omega(r) = \omega_6(r) = \frac{1}{6!}r^6.$$

5. Conclusion

NB1 Polynomials with collocation method have been used to solve the singular linear ordinary differential equations (SLODEs) of various high orders. This method reduces the problem with suitable initial conditions into a system of linear algebraic equations, which we solved by using the inverse matrix method. The proposed method gave the exact solutions for all studied problems, that showing the potency, reliability and accuracy of the presented method.

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