

# Generalized Horadam-Leonardo Numbers and Polynomials

**Abstract.** In this study, we define and investigate some linear third order polynomials called the generalized Horadam-Leonardo polynomials (with its two special cases, namely),  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials. We give Binet's formulas, generating functions, Simson formulas, and the sum formulas for these polynomial sequences. Also, we present some identities and matrices related to these polynomials. Furthermore, we present some special cases of generalized Horadam-Leonardo polynomials, namely, generalized Leonardo, generalized John, generalized Ernst, generalized Pisano, generalized Edouard and generalized Bigollo numbers.

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**Keywords.** Horadam polynomials, Fibonacci polynomials, Tribonacci polynomials, Horadam-Leonardo polynomials, third order recurrence relations.

## 1. Introduction: Horadam (Generalized Fibonacci) Polynomials

Before defining and investigating the generalized Horadam-Leonardo polynomials we recall the definition and some properties of Horadam polynomials and its two special cases. The generalized Fibonacci polynomials (or Horadam polynomials)

$$\{V_n(V_0(x), V_1(x); r(x), s(x))\}_{n \geq 0}$$

(or  $\{V_n(x)\}_{n \geq 0}$  or shortly  $\{V_n\}_{n \geq 0}$ ) is defined by

$$\begin{aligned} V_n(x) &= r(x)V_{n-1}(x) + s(x)V_{n-2}(x), \\ V_0(x) &= a(x), V_1(x) = b(x), \quad n \geq 2 \end{aligned} \tag{1.1}$$

where  $V_0(x), V_1(x)$  are arbitrary real (or complex) polynomials with real coefficients and  $r(x), s(x)$  are polynomials with real coefficients with  $r(x) \neq 0, s(x) \neq 0$ .

The sequence  $\{V_n\}_{n \geq 0}$  can be extended to negative indexes by defining

$$V_{-n}(x) = -\frac{r(x)}{s(x)}V_{-n+1}(x) + \frac{1}{s(x)}V_{-n+2}(x)$$

for  $n = 1, 2, 3, \dots$  when  $s(x) \neq 0$ . Thus, recurrence (1.1) holds for all integers  $n$ . Notice that  $V_{-n}(x)$  need not to be a polynomial in the ordinary sense.

We can give some references on special cases of second-order linear recurrence sequences of polynomials and numbers. For example, see [15,16,20,50,52,69,70] for papers and [1,14,26,24,25,59,67] for books.

Binet's formula of generalized Fibonacci polynomials can be given using its characteristic equation which is given as

$$y^2 - r(x)y - s(x) = 0. \quad (1.2)$$

The roots of characteristic equation are given as

$$\alpha(x) := \alpha = \frac{1}{2}(r(x) + \sqrt{r^2(x) + 4s(x)}), \beta(x) := \beta = \frac{1}{2}(r(x) - \sqrt{r^2(x) + 4s(x)}), \quad (1.3)$$

and the followings hold

$$\alpha + \beta = r(x), \quad \alpha\beta = -s(x).$$

If  $\alpha$  and  $\beta$  of characteristic equation (1.2) are distinct, i.e.,  $\alpha \neq \beta$  then  $r^2(x) + 4s(x) \neq 0$  and if  $\alpha$  and  $\beta$  of characteristic equation (1.2) are equal, i.e.,  $\alpha = \beta$  then (1.2) can be written as

$$\begin{aligned} y^2 - r(x)y - s(x) &= y^2 - 2\alpha y + \alpha^2 \\ &= (y - \alpha)^2 \\ &= 0 \end{aligned}$$

and, in this case,

$$\alpha = \frac{r(x)}{2}, r(x) = 2\alpha, s(x) = -\alpha^2 = -\frac{r^2(x)}{4}, r^2(x) + 4s(x) = 0.$$

Next, we can define two specific cases of the polynomials  $V_n(x)$ .  $(r(x), s(x))$ -Fibonacci polynomials  $\{M_n(0, 1; r(x), s(x))\}_{n \geq 0}$  (or shortly,  $M_n(x)$ ) and  $(r(x), s(x))$ -Lucas polynomials  $\{N_n(2, r(x); r(x), s(x))\}_{n \geq 0}$  (or shortly,  $N_n(x)$ ) are defined by the second-order recurrence relations

$$M_{n+2}(x) = r(x)M_{n+1} + s(x)M_n(x), \quad M_0(x) = 0, M_1(x) = 1, \quad (1.4)$$

$$N_{n+2}(x) = r(x)N_{n+1} + s(x)N_n(x), \quad N_0(x) = 2, N_1(x) = r(x), \quad (1.5)$$

respectively

$\{M_n(x)\}_{n \geq 0}$  and  $\{N_n(x)\}_{n \geq 0}$  can be extended to negative indexes by defining

$$\begin{aligned} M_{-n}(x) &= -\frac{r(x)}{s(x)}M_{-(n-1)}(x) + \frac{1}{s(x)}M_{-(n-2)}(x), \\ N_{-n}(x) &= -\frac{r(x)}{s(x)}N_{-(n-1)}(x) + \frac{1}{s(x)}N_{-(n-2)}(x), \end{aligned}$$

for  $n = 1, 2, 3, \dots$  respectively. Thus, recurrences (1.4) and (1.5) hold for all integers  $n$ .

NOTE: Throughout the rest of the paper, we use, respectively,

$$V_n, r, s, V_0, V_1, \alpha, \beta, M_n, N_n, M_0, M_1, N_0, N_1$$

instead of

$$V_n(x), r(x), s(x), V_0(x), V_1(x), \alpha(x), \beta(x), M_n(x), N_n(x), M_0(x), M_1(x), N_0(x), N_1(x)).$$

For instance, we write

$$V_n = rV_{n-1} + sV_{n-2}, \quad V_0 = a, V_1 = b, \quad n \geq 2$$

for the equation (1.1).

Using  $\alpha, \beta$  and recurrence relation (1.1), Binet's formula of  $V_n$  can be given as follows:

THEOREM 1.

(a): ( $\alpha \neq \beta$  : Distinct Roots Case) Binet's formula of generalized Fibonacci polynomials is

$$V_n = \frac{r_1 \alpha^n}{\alpha - \beta} + \frac{r_2 \beta^n}{\beta - \alpha} = \frac{r_1 \alpha^n - r_2 \beta^n}{\alpha - \beta} \quad (1.6)$$

where

$$r_1 = V_1 - \beta V_0, \quad r_2 = V_1 - \alpha V_0.$$

(b): ( $\alpha = \beta$  : Single Root Case) Binet's formula of generalized Fibonacci polynomials is

$$V_n = (D_2 n + D_1) \alpha^n \quad (1.7)$$

where

$$\begin{aligned} D_1 &= V_0, \\ D_2 &= \frac{1}{\alpha} (V_1 - \alpha V_0). \end{aligned}$$

Note that Binet's formulas of  $M_n$  and  $N_n$  can be given, respectively, as follows:

$$\begin{aligned} M_n &= \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta} & , \quad \text{if (Distinct Roots Case): } \alpha \neq \beta \\ n\alpha^{n-1} & , \quad \text{if (Single Root Case): } \alpha = \beta \end{cases} , \\ N_n &= \begin{cases} \alpha^n + \beta^n & , \quad \text{if (Distinct Roots Case): } \alpha \neq \beta \\ 2\alpha^n & , \quad \text{if (Single Root Case): } \alpha = \beta \end{cases} . \end{aligned}$$

Next, we define two sequences related to  $(r, s)$ -Fibonacci polynomials and  $(r, s)$ -Fibonacci-Lucas polynomials. For  $r, s$  satisfying Eq. (1.4)-(1.5),  $(r, s)$ -Horadam-Leonardo polynomials and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials are defined as

$$G_n(x) = rG_{n-1}(x) + sG_{n-2}(x) + 1 \quad \text{with } G_0(x) = 0, G_1(x) = 1, \quad n \geq 2, \quad (1.8)$$

and

$$H_n(x) = rH_{n-1}(x) + sH_{n-2}(x) + (1 - s - r) \quad \text{with } H_0(x) = 3, H_1(x) = r + 1, \quad n \geq 2, \quad (1.9)$$

respectively.

Note that  $G_2(x) = r+1$  and  $H_2(x) = r^2+2s+1$ . We can present the first few values of Horadam-Leonardo polynomials and Horadam-Leonardo-Lucas polynomials as

$$0, 1, r+1, r^2+r+s+1, r^3+r^2+r+2rs+s+1, \dots$$

and

$$3, r+1, r^2+2s+1, r^3+3sr+1, r^4+4r^2s+2s^2+1, \dots$$

respectively. Note also that from the equations (1.8) and (1.9), we get

$$sG_{n-3}(x) = G_{n-1}(x) - rG_{n-2}(x) - 1,$$

$$sH_{n-3}(x) = H_{n-1}(x) - rH_{n-2}(x) - (1-s-r),$$

and so the sequences  $\{G_n(x)\}$  and  $\{H_n(x)\}$  satisfy third order linear recurrences and given as

$$G_n(x) = (r+1)G_{n-1}(x) + (s-r)G_{n-2}(x) - sG_{n-3}(x), \quad (1.10)$$

$$H_n(x) = (r+1)H_{n-1}(x) + (s-r)H_{n-2}(x) - sH_{n-3}(x). \quad (1.11)$$

REMARK 2. Note that if  $1-s-r=0$ , i.e.,  $s=1-r, r=1-s$ , then we see from (1.9) and (1.11) that the sequence  $\{H_n(x)\}$  both have second order and third order linear relations. In this case, we get

$$H_2(x) = r^2 + 2s + 1 = r^2 - 2r + 3 = s^2 + 2.$$

In this paper, we also consider and investigate the case  $1-s-r=0$  by considering it in the general case, i.e. in (1.9) and (1.11) that is (2.3).

Note that if we define a sequence of polynomials as

$$Y_n(x) = rY_{n-1}(x) + sY_{n-2}(x) + c(x), \quad \text{with } Y_0(x) = d_1(x), Y_1(x) = d_2(x), \quad n \geq 2$$

where  $r, s$  satisfying (1.1) and  $Y_0(x), Y_1(x)$  are arbitrary real (or complex) polynomials with real coefficients and  $c(x)$  is a polynomial with real coefficients, then since

$$sY_{n-3}(x) = Y_{n-1}(x) - rY_{n-2}(x) - c(x),$$

we get

$$Y_n(x) = (r+1)Y_{n-1}(x) + (s-r)Y_{n-2}(x) - sY_{n-3}(x).$$

## 2. Generalized Horadam-Leonardo Polynomials

In this section, for  $r, s$  satisfying (1.1), we present and investigate a new sequence and its two special cases, namely the generalized Horadam-Leonardo,  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials.

For  $r, s$  satisfying (1.1), generalized Horadam-Leonardo polynomials  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2; r + 1, s - r, -s)\}_{n \geq 0}$  (or, shortly,  $\{W_n(x)\}_{n \geq 0}$ ) is defined by (the third-order recurrence relation)

$$W_n(x) = (r + 1)W_{n-1}(x) + (s - r)W_{n-2}(x) - sW_{n-3}(x) \quad (2.1)$$

with initial values  $W_0(x) = c_0(x), W_1(x) = c_1(x), W_2(x) = c_2(x)$  not all being zero and  $W_0(x), W_1(x), W_2(x)$  are arbitrary real (or complex) polynomials with real coefficients.

$\{W_n(x)\}_{n \geq 0}$  can be extended to negative indexes by defining

$$W_{-n}(x) = \frac{s-r}{s}W_{-(n-1)}(x) + \frac{r+1}{s}W_{-(n-2)}(x) - \frac{1}{s}W_{-(n-3)}(x)$$

for  $n = 1, 2, 3, \dots$ . Thus, recurrence (2.1) holds for all integer  $n$ . Notice that  $W_{-n}(x)$  need not to be a polynomial in the ordinary sense for  $n \geq 1$ .

Generalized Horadam-Leonardo polynomial are special cases of generalized Tribonacci polynomials. See [8,28,30,32] for some references on generalized Tribonacci polynomials and its special cases,.

Note that the sequences  $\{G_n(x)\}$  and  $\{H_n(x)\}$ , defined in the section Introduction, are the special cases of the generalized Horadam-Leonardo polynomials  $\{W_n(x)\}$ . For the benefit, we give the definition of these two special cases of the sequence  $\{W_n(x)\}$  in this section too.  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials are defined, by (the third-order recurrence relations)

$$G_n(x) = (r + 1)G_{n-1}(x) + (s - r)G_{n-2}(x) - sG_{n-3}(x), \quad (2.2)$$

$$G_0(x) = 0, G_1(x) = 1, G_2(x) = r + 1,$$

and

$$H_n(x) = (r + 1)H_{n-1}(x) + (s - r)H_{n-2}(x) - sH_{n-3}(x), \quad (2.3)$$

$$H_0(x) = 3, H_1(x) = r + 1, H_2(x) = r^2 + 2s + 1,$$

respectively. The sequences  $\{G_n(0, 1, r + 1; r + 1, s - r, -s)\}_{n \geq 0}$  and  $\{H_n(3, r + 1, r^2 + 2s + 1; r + 1, s - r, -s)\}_{n \geq 0}$  can be extended to negative indexes by defining

$$G_{-n}(x) = \frac{s-r}{s}G_{-(n-1)}(x) + \frac{r+1}{s}G_{-(n-2)}(x) - \frac{1}{s}G_{-(n-3)}(x),$$

$$H_{-n}(x) = \frac{s-r}{s}H_{-(n-1)}(x) + \frac{r+1}{s}H_{-(n-2)}(x) - \frac{1}{s}H_{-(n-3)}(x),$$

for  $n = 1, 2, 3, \dots$  respectively.

REMARK 3. Throughout the rest of the paper, we use, respectively,

$$W_n, W_0, W_1, W_2, G_n, G_0, G_1, G_2, H_n, H_0, H_1, H_2,$$

instead of

$$W_n(x), W_0(x), W_1(x), W_2(x), G(x), G_0(x), G_1(x), G_2(x), H(x), H_0(x), H_1(x), H_2(x))$$

unless otherwise stated. For instance, we write

$$W_n = (r + 1)W_{n-1} + (s - r)W_{n-2} - sW_{n-3}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, \quad n \geq 3$$

for the equation (2.1). Also we write  $K_n, K_0, K_1, K_2$  instead of  $K_n(x)$  with initial conditions  $K_0(x), K_1(x), K_2(x)$  for any subsequence  $\{K_n(x)\}$  of  $\{W_n\}$ .

When  $r, s, W_0, W_1, W_2$  are real numbers we call generalized Horadam-Leonardo,  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials as generalized Horadam-Leonardo,  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas numbers (sequences).

Now, we give a few special cases of generalized Horadam-Leonardo sequence are given as follows (Table 1):

Table 1 Several cases of generalized Horadam-Leonardo sequence.

No	Numbers (Sequences)	$r, s$	Notation	References
1	Generalized Leonardo	$r = 1, s = 1$	$\{W_n(W_0, W_1, W_2; 2, 0, -1)\}$	[53]
2	Generalized John	$r = 2, s = 1$	$\{W_n(W_0, W_1, W_2; 3, -1, -1)\}$	[54]
3	Generalized Ernst	$r = 1, s = 2$	$\{W_n(W_0, W_1, W_2; 2, 1, -2)\}$	[55]
4	Generalized Pisano	$r = 1, s = -\frac{1}{4}$	$\{W_n(W_0, W_1, W_2; 2, -\frac{5}{4}, \frac{1}{4})\}$	[56]
5	Generalized Edouard	$r = 6, s = -1$	$\{W_n(W_0, W_1, W_2; 7, -7, 1)\}$	[57]
6	Generalized Bigollo	$r = 3, s = -2$	$\{W_n(W_0, W_1, W_2; 4, -5, 2)\}$	[58]

Next, we give several values of the  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials with positive and negative indexes (Table 2):

Table 2. Several special third-order numbers with positive and negative indexes.

$n$	0	1	2	3	4
$G_n$	0	1	$r + 1$	$r^2 + r + s + 1$	$r^3 + r^2 + r + 2rs + s + 1$
$G_{-n}$		0	$-\frac{1}{s}$	$\frac{1}{s^2}(r - s)$	$-\frac{1}{s^3}(s - rs + r^2 + s^2)$
$H_n$	3	$r + 1$	$r^2 + 2s + 1$	$r^3 + 3sr + 1$	$r^4 + 4r^2s + 2s^2 + 1$
$H_{-n}$		$-\frac{1}{s}(r - s)$	$\frac{1}{s^2}(r^2 + s^2 + 2s)$	$-\frac{1}{s^3}(r^3 + 3rs - s^3)$	$\frac{1}{s^4}(r^4 + 4r^2s + s^4 + 2s^2)$

Several special cases of  $(r, s)$ -Horadam-Leonardo sequence  $\{G_n(0, 1, r + 1; r + 1, s - r, -s)\}$  and  $(r, s)$ -Horadam-Leonardo-Lucas sequence  $\{H_n(3, r + 1, r^2 + 2s + 1; r + 1, s - r, -s)\}$  are given as follows:

- (1)  $G_n(0, 1, 2; 2, 0, -1) = G_n$ , modified Leonardo sequence, see [53].
- (2)  $H_n(3, 2, 4; 2, 0, -1) = H_n$ , Leonardo-Lucas sequence, see [53].
- (3)  $G_n(0, 1, 3; 3, -1, -1) = J_n$ , John sequence, see [54].

- (4)  $H_n(3, 3, 7; 3, -1, -1) = H_n$ , John-Lucas sequence, see [54].  
 (5)  $G_n(0, 1, 2; 2, 1, -2) = E_n$ , Ernst sequence, see [55].  
 (6)  $H_n(3, 2, 6; 2, 1, -2) = H_n$ , Ernst-Lucas sequence, see [55].  
 (7)  $G_n(0, 1, 2; 2, -\frac{5}{4}, \frac{1}{4}) = P_n$ , Pisano sequence, see [56].  
 (8)  $H_n(3, 2, \frac{3}{2}; 2, -\frac{5}{4}, \frac{1}{4}) = R_n$ , Pisano-Lucas sequence, see [56].  
 (9)  $G_n(0, 1, 7; 7, -7, 1) = E_n$ , Edouard sequence, see [57].  
 (10)  $H_n(3, 7, 35; 7; 7, -7, 1) = K_n$ , Edouard-Lucas sequence, see [57].  
 (11)  $G_n(0, 1, 4; 4, -5, 2) = B_n$ , Bigollo sequence, see [58].  
 (12)  $H_n(3, 4, 6; 1, 4, -5, 2) = C_n$ , Bigollo-Lucas sequence, see [58].

The characteristic equation of  $W_n$  is given by

$$y^3 - (r+1)y^2 - (s-r)y + s = (y^2 - ry - s)(y-1) = 0. \quad (2.4)$$

and the roots of characteristic equation are given as

$$\alpha = \frac{r + \sqrt{r^2 + 4s}}{2}, \quad \beta = \frac{r - \sqrt{r^2 + 4s}}{2}, \quad \gamma = 1,$$

where  $\alpha$  and  $\beta$  are as in (1.3).

Next, we give Binet's formula of generalized Horadam-Leonardo polynomials.

**COROLLARY 4.** *According to the roots of characteristic equation (2.4), Binet's formula of generalized Horadam-Leonardo polynomials is given by:*

**(a):** ( $\alpha \neq \beta \neq \gamma = 1$ , *Three Distinct Roots Case*)

$$\begin{aligned} W_n &= \frac{W_2 - (\beta + 1)W_1 + \beta W_0}{(\alpha - \beta)(\alpha - 1)} \alpha^n + \frac{W_2 - (\alpha + 1)W_1 + \alpha W_0}{(\beta - \alpha)(\beta - 1)} \beta^n + \frac{W_2 + (-(r+1) + 1)W_1 + (-s)W_0}{(-s) - (r+1) + 2} \\ &= \frac{(\alpha W_2 + \alpha(-(r+1) + \alpha)W_1 + (-s)W_0)}{(r+1)\alpha^2 + 2((s-r) - r)\alpha + 3(-s)} \alpha^n + \frac{(\beta W_2 + \beta(-(r+1) + \beta)W_1 + (-s)W_0)}{(r+1)\beta^2 + 2((s-r) - r)\beta + 3(-s)} \beta^n \\ &\quad + \frac{W_2 + (-(r+1) + 1)W_1 + (-s)W_0}{(r+1) + 2((s-r) - r) + 3(-s)}. \end{aligned}$$

**(b):** ( $\alpha \neq \beta = \gamma = 1$ , *Two Distinct Roots Case*)

$$W_n = \frac{1}{(1 - (-s))^2} ((W_2 - 2W_1 + W_0)\alpha^n + (-W_2 + 2W_1 + (-s)((-s) - 2)W_0) + (1 - (-s))(W_2 - (1 + (-s))W_1 + (-s)W_0)n).$$

**(c):** ( $\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$ , *Single Root Case*)

$$W_n = \frac{1}{2}(n(n-1)W_2 - 2n(n-2)W_1 + (n-1)(n-2)W_0).$$

**Proof.** In [51, Corollary 7.], replace  $r$ ,  $s$  and  $t$  with  $r+1$ ,  $s-r$ ,  $-s$ , respectively.  $\square$

Now, we present the ordinary generating function  $\sum_{n=0}^{\infty} W_n z^n$  of the sequence  $W_n$ .

LEMMA 5. Assume that  $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$  is the ordinary generating function of the generalized Horadam-Leonardo polynomials  $\{W_n\}_{n \geq 0}$ . Then,  $\sum_{n=0}^{\infty} W_n z^n$  is given as

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - (r+1)W_0)z + (W_2 - (r+1)W_1 - (s-r)W_0)z^2}{1 - (r+1)z - (s-r)z^2 - (-s)z^3}.$$

Proof. Replace  $r$ ,  $s$  and  $t$  with  $r+1$ ,  $s-r$ ,  $-s$ , respectively, in [51, Lemma 9].  $\square$

As particular examples (generating functions of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials), from the last Lemma, we get

COROLLARY 6. The generating functions of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials are

$$\begin{aligned} \sum_{n=0}^{\infty} G_n z^n &= \frac{z}{1 - (r+1)z - (s-r)z^2 - (-s)z^3}, \\ \sum_{n=0}^{\infty} H_n z^n &= \frac{3 - 2(r+1)z - (s-r)z^2}{1 - (r+1)z - (s-r)z^2 - (-s)z^3}, \end{aligned}$$

respectively.

Proof. In the last Lemma, put  $W_n = G_n$  with  $G_0 = 0, G_1 = 1, G_2 = (r+1)$  and take  $W_n = H_n$  with  $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$ , respectively.  $\square$

Next, as a special case of Corollary 4, we present Binet formulas of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials.

COROLLARY 7. Binet's formulas of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials, for all integers  $n$ , are given as follows

(a): ( $\alpha \neq \beta \neq \gamma = 1$ , Three Distinct Roots Case)

$$\begin{aligned} G_n &= \frac{\alpha^{n+1}}{(\alpha - \beta)(\alpha - 1)} + \frac{\beta^{n+1}}{(\beta - \alpha)(\beta - 1)} + \frac{1}{(1 - \alpha)(1 - \beta)} \\ &= \frac{\alpha^{n+2}}{(r+1)\alpha^2 + 2(s-r)\alpha + 3(-s)} + \frac{\beta^{n+2}}{(r+1)\beta^2 + 2(s-r)\beta + 3(-s)} + \frac{1}{(r+1) + 2(s-r) + 3(-s)}, \\ H_n &= \alpha^n + \beta^n + 1. \end{aligned}$$

(b): ( $\alpha \neq \beta = \gamma = 1$ , Two Distinct Roots Case)

$$G_n = \frac{\alpha^{n+1} + ((1 - \alpha)n - \alpha)}{(1 - \alpha)^2}, \quad H_n = \alpha^n + 2.$$

(c): ( $\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$ , Single Root Case)

$$G_n = \frac{n(n+1)}{2}, \quad H_n = 3.$$

Proof. In Corollary 4, put  $W_n = G_n$  with  $G_0 = 0, G_1 = 1, G_2 = (r+1)$  and take  $W_n = H_n$  with  $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$ , respectively.  $\square$

### 3. Some Specific Cases of Generalized Horadam-Leonardo Polynomials

In this section, we give literature review and present on some special cases of generalized Horadam-Leonardo polynomials.

**3.1. Generalized Leonardo Sequence.** In this subsection, it is considered that the special case  $r = 1, s = 1$ . In this case,  $r + 1 = 2, s - r = 0, -s = -1$ . A generalized Leonardo sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$  is defined by (the third-order recurrence relations)

$$W_n = 2W_{n-1} - W_{n-3} \quad (3.1)$$

with the initial values  $W_0 = c_0, W_1 = c_1, W_2 = c_2$ . Next, define three special cases of the sequence  $\{W_n\}$ . Modified Leonardo sequence  $\{G_n\}_{n \geq 0}$ , Leonardo-Lucas sequence  $\{H_n\}_{n \geq 0}$  and Leonardo sequence  $\{l_n\}_{n \geq 0}$  are defined, (by the third-order recurrence relations)

$$G_n = 2G_{n-1} - G_{n-3}, \quad (3.2)$$

$$G_0 = 0, G_1 = 1, G_2 = 2,$$

$$H_n = 2H_{n-1} - H_{n-3}, \quad (3.3)$$

$$H_0 = 3, H_1 = 2, H_2 = 4,$$

$$l_n = 2l_{n-1} - l_{n-3}, \quad (3.4)$$

$$l_0 = 1, l_1 = 1, l_2 = 3,$$

respectively. Modified Leonardo, Leonardo-Lucas, Leonardo numbers and Fibonacci, Lucas numbers satisfy the following interrelations:

$$G_n = F_{n+2} - 1, H_n = L_n + 1, l_n = 2F_{n+1} - 1,$$

and

$$5G_n = 3L_{n+1} + L_n - 5, H_n = 2F_{n+1} - F_n + 1, 5l_n = 2L_{n+1} + 4L_n - 5.$$

where the sequence of Fibonacci numbers  $\{F_n\}$  and the sequence of Lucas numbers  $\{L_n\}$  are defined, respectively, as

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2,$$

$$F_0 = 0, F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2,$$

$$L_0 = 2, L_1 = 1,$$

For more information on generalized Leonardo number, see [53].

**3.2. Generalized John Sequence.** In this subsection, it is considered that the special case  $r = 2, s = 1$ . In this case,  $r + 1 = 3, s - r = -1, -s = -1$ . A generalized John sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$  is defined by the third-order recurrence relations

$$W_n = 3W_{n-1} - W_{n-2} - W_{n-3} \quad (3.5)$$

with the initial values  $W_0 = c_0, W_1 = c_1, W_2 = c_2$ . Next, define two special cases of the sequence  $\{W_n\}$ . John sequence  $\{J_n\}_{n \geq 0}$  and John-Lucas sequence  $\{H_n\}_{n \geq 0}$  are defined by (the third-order recurrence relations)

$$J_n = 3J_{n-1} - J_{n-2} - J_{n-3}, \quad (3.6)$$

$$J_0 = 0, J_1 = 1, J_2 = 3,$$

$$H_n = 3H_{n-1} - H_{n-2} - H_{n-3}, \quad (3.7)$$

$$H_0 = 3, H_1 = 3, H_2 = 7,$$

respectively. John and John-Lucas and Pell, Pell-Lucas numbers satisfy the following interrelations:

$$J_n = \frac{1}{2}(P_{n+2} - P_{n+1} - 1), H_n = Q_n + 1,$$

and

$$J_n = \frac{1}{4}(Q_{n+1} - 2), H_n = 2P_{n+1} - 2P_n + 1,$$

where Pell sequence  $\{P_n\}_{n \geq 0}$  and Pell-Lucas sequence  $\{Q_n\}_{n \geq 0}$  are defined, respectively, by (the second-order recurrence relations)

$$P_n = 2P_{n-1} + P_{n-2}, \quad (3.8)$$

$$P_0 = 0, P_1 = 1$$

and

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad (3.9)$$

$$Q_0 = 2, Q_1 = 2.$$

See [54] for more information on generalized John numbers.

**3.3. Generalized Ernst Sequence.** In this subsection, it is considered that the special case  $r = 1, s = 2$ . In this case,  $r + 1 = 2, s - r = 1, -s = -2$ . A generalized Ernst sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$  is defined by the third-order recurrence relation

$$W_n = 2W_{n-1} + W_{n-2} - 2W_{n-3} \quad (3.10)$$

with the initial values  $W_0 = c_0, W_1 = c_1, W_2 = c_2$ . Next, define two special cases of the sequence  $\{W_n\}$ . Ernst sequence  $\{E_n\}_{n \geq 0}$  and Ernst-Lucas sequence  $\{H_n\}_{n \geq 0}$  are defined by (the third-order recurrence

relations)

$$E_n = 2E_{n-1} + E_{n-2} - 2E_{n-3}, \quad (3.11)$$

$$E_0 = 0, E_1 = 1, E_3 = 2,$$

$$H_n = 2H_{n-1} + H_{n-2} - 2H_{n-3}, \quad (3.12)$$

$$H_0 = 3, H_1 = 2, H_2 = 6,$$

respectively. Ernst and Ernst-Lucas and Jacobsthal, Jacobsthal-Lucas numbers satisfy the following inter-relations:

$$E_n = \frac{1}{2}(J_{n+2} - 1), H_n = j_n + 1,$$

and

$$18E_n = 5j_{n+1} + 2j_n - 9, 2H_n = 4J_{n+1} - 2J_n + 2.$$

where Jacobsthal sequence  $\{J_n\}_{n \geq 0}$  and Jacobsthal-Lucas sequence  $\{j_n\}_{n \geq 0}$  are defined by (the second-order recurrence relations)

$$J_n = J_{n-1} + 2J_{n-2}, \quad (3.13)$$

$$J_0 = 0, J_1 = 1$$

and

$$j_n = j_{n-1} + 2j_{n-2}, \quad (3.14)$$

$$j_0 = 2, j_1 = 1,$$

respectively. See [55] for more information on generalized Ernst numbers.

**3.4. Generalized Pisano Sequence.** In this subsection, it is considered that the special case  $r = 1, s = -\frac{1}{4}$ . In this case,  $r + 1 = 2, s - r = -\frac{5}{4}, -s = \frac{1}{4}$ . A generalized Pisano sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$  is defined by (the third-order recurrence relation)

$$W_n = 2W_{n-1} - \frac{5}{4}W_{n-2} + \frac{1}{4}W_{n-3} \quad (3.15)$$

with the initial values  $W_0 = c_0, W_1 = c_1, W_2 = c_2$ . Next, define two special cases of the sequence  $\{W_n\}$ . Pisano sequence  $\{P_n\}_{n \geq 0}$  and Pisano-Lucas sequence  $\{R_n\}_{n \geq 0}$  are defined by (the third-order recurrence relations)

$$P_n = 2P_{n-1} - \frac{5}{4}P_{n-2} + \frac{1}{4}P_{n-3}, \quad (3.16)$$

$$P_0 = 0, P_1 = 1, P_2 = 2,$$

$$R_n = 2R_{n-1} - \frac{5}{4}R_{n-2} + \frac{1}{4}R_{n-3}, \quad (3.17)$$

$$R_0 = 3, R_1 = 2, R_2 = \frac{3}{2},$$

respectively. Pisano and Pisano–Lucas and modified Oresme, Oresme–Lucas, Oresme numbers satisfy the following interrelations:

$$nP_n = -(n+2)G_n + 4n, P_n = -(n+2)H_n + 4, nP_n = -2(n+2)O_n + 4n,$$

and

$$nR_n = G_n + n, R_n = H_n + 1, nR_n = 2O_n + n,$$

where modified Oresme sequence  $\{G_n\}_{n \geq 0}$ , Oresme–Lucas sequence  $\{H_n\}_{n \geq 0}$  and Oresme sequence  $\{O_n\}_{n \geq 0}$  are defined, respectively, by (the second-order recurrence relations)

$$G_{n+2} = G_{n+1} - \frac{1}{4}G_n, \quad G_0 = 0, G_1 = 1, \quad (3.18)$$

$$H_{n+2} = H_{n+1} - \frac{1}{4}H_n, \quad H_0 = 2, H_1 = 1, \quad (3.19)$$

$$O_{n+2} = O_{n+1} - \frac{1}{4}O_n, \quad O_0 = 0, O_1 = \frac{1}{2}. \quad (3.20)$$

See [56] for more information on generalized Pisano numbers.

**3.5. Generalized Edouard Sequence.** In this subsection, it is considered that the special case  $r = 6, s = -1$ . In this case,  $r + 1 = 7, s - r = -7, -s = 1$ . A generalized Edouard sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$  is defined by (the third-order recurrence relations)

$$W_n = 7W_{n-1} - 7W_{n-2} + W_{n-3} \quad (3.21)$$

with the initial values  $W_0 = c_0, W_1 = c_1, W_2 = c_2$ . Next, define two special cases of the sequence  $\{W_n\}$ . Edouard sequence  $\{E_n\}_{n \geq 0}$  and Edouard–Lucas sequence  $\{K_n\}_{n \geq 0}$  are defined by (the third-order recurrence relations)

$$E_n = 7E_{n-1} - 7E_{n-2} + E_{n-3}, \quad (3.22)$$

$$E_0 = 0, E_1 = 1, E_2 = 7,$$

$$K_n = 7K_{n-1} - 7K_{n-2} + K_{n-3}, \quad (3.23)$$

$$K_0 = 3, K_1 = 7, K_2 = 35.$$

respectively. Edouard, Edouard–Lucas and balancing, modified Lucas–balancing, Lucas–balancing numbers satisfy the following interrelations:

$$E_n = \frac{1}{4}(B_{n+1} - B_n - 1), K_n = H_n + 1 = 2C_n + 1$$

and

$$32E_n = H_{n+1} + H_n - 8 = 2C_{n+1} + 2C_n - 8, K_n = 2B_{n+1} - 6B_n + 1,$$

where balancing sequence  $\{B_n\}_{n \geq 0}$ , modified Lucas-balancing sequence  $\{H_n\}_{n \geq 0}$  and Lucas-balancing sequence  $\{C_n\}_{n \geq 0}$  are defined, respectively, by (the second-order recurrence relations)

$$B_n = 6B_{n-1} - B_{n-2}, \quad (3.24)$$

$$B_0 = 0, B_1 = 1,$$

$$H_n = 6H_{n-1} - H_{n-2}, \quad (3.25)$$

$$H_0 = 2, H_1 = 6,$$

$$C_n = 6C_{n-1} - C_{n-2}, \quad (3.26)$$

$$C_0 = 1, C_1 = 3.$$

See [57] for more information on generalized Edouard numbers.

**3.6. Generalized Bigollo Sequence.** In this subsection, it is considered that the special case  $r = 3, s = -2$ . In this case,  $r + 1 = 4, s - r = -5, -s = 2$ . A generalized Bigollo sequence  $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$  is defined by (the third-order recurrence relations)

$$W_n = 4W_{n-1} - 5W_{n-2} + 2W_{n-3} \quad (3.27)$$

with the initial values  $W_0 = c_0, W_1 = c_1, W_2 = c_2$ . Next, define two special cases of the sequence  $\{W_n\}$ . Bigollo sequence  $\{B_n\}_{n \geq 0}$  and Bigollo-Lucas sequence  $\{C_n\}_{n \geq 0}$  are defined by (the third-order recurrence relations)

$$B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3},$$

$$B_0 = 0, B_1 = 1, B_2 = 4, \quad (3.28)$$

$$C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3},$$

$$C_0 = 3, C_1 = 4, C_2 = 6. \quad (3.29)$$

respectively. Bigollo and Bigollo-Lucas and Mersenne, Mersenne-Lucas numbers satisfy the following inter-relations:

$$B_n = 2M_n - n, C_n = H_n + 1,$$

and

$$B_n = 4H_{n+1} - 6H_n - n, 2C_n = 2M_{n+1} - 2M_n + 4$$

where Mersenne sequence  $\{M_n\}_{n \geq 0}$  and Mersenne-Lucas sequence  $\{H_n\}_{n \geq 0}$  are defined by (the second-order recurrence relations)

$$M_n = 3M_{n-1} - 2M_{n-2}, \quad (3.30)$$

$$M_0 = 0, M_1 = 1,$$

$$H_n = 3H_{n-1} - 2H_{n-2}, \quad (3.31)$$

$$H_0 = 2, H_1 = 3,$$

respectively. See [58] for more information on generalized Bigollo numbers.

#### 4. Simson Formulas of Horadam-Leonardo Polynomials

Next we present Simson's formula of the generalized Horadam-Leonardo polynomials  $\{W_n\}$ .

**THEOREM 8.** *For all integers  $n$ , we get*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = (-s)^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \quad (4.1)$$

*Proof.* Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Theorem 33.].  $\square$

The previous theorem gives the following results as particular examples.

**COROLLARY 9.** *For all integers  $n$ , Simson's formula of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials are given by*

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -(-s)^{n-1}, \quad \begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = (r^2 + 4s)(r + s - 1)^2(-s)^{n-2}$$

*respectively.*

Notice that for all integers  $n, m$ , (4.1) can be written as

$$\begin{vmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{vmatrix} = (-s)^{n+m} \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}.$$

Next we define

$$\begin{aligned} \Lambda_W(n) &= W_{n+2}^3 - r(r-s+1)W_{n+1}^3 + (-s)^2W_n^3 - 2(r+1)W_{n+1}W_{n+2}^2 - (s-r)W_nW_{n+2}^2 \\ &\quad + (r^2 + 3r - s + 1)W_{n+2}W_{n+1}^2 + ((r+1)(-s) + (s-r)^2)W_nW_{n+1}^2 \\ &\quad + (r+1)(-s)W_n^2W_{n+2} + 2(s-r)(-s)W_n^2W_{n+1} - (r^2 + r - 4s - rs)W_{n+2}W_{n+1}W_n \end{aligned}$$

Then

$$\begin{aligned}\Lambda_W(0) &= W_2^3 - r(r-s+1)W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 \\ &\quad + (r^2+3r-s+1)W_2W_1^2 + ((r+1)(-s) + (s-r)^2)W_0W_1^2 \\ &\quad + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 - (r^2+r-4s-rs)W_2W_1W_0.\end{aligned}\tag{4.2}$$

Notice that Simson's formulas of  $W_n, G_n, H_n$  can be written in the following forms:

LEMMA 10. *For all integers  $n$ , we get*

(a):  $\Lambda_W(n) = (-s)^n \Lambda_W(0)$ , *that is*

$$\begin{aligned}&W_{n+2}^3 - r(r-s+1)W_{n+1}^3 + (-s)^2W_n^3 - 2(r+1)W_{n+1}W_{n+2}^2 - (s-r)W_nW_{n+2}^2 \\ &\quad + (r^2+3r-s+1)W_{n+2}W_{n+1}^2 + ((r+1)(-s) + (s-r)^2)W_nW_{n+1}^2 \\ &\quad + (r+1)(-s)W_n^2W_{n+2} + 2(s-r)(-s)W_n^2W_{n+1} - (r^2+r-4s-rs)W_{n+2}W_{n+1}W_n \\ &= (-s)^n(W_2^3 - r(r-s+1)W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 \\ &\quad + (r^2+3r-s+1)W_2W_1^2 + (r^2+s^2-s-3rs)W_0W_1^2 + (r+1)(-s)W_0^2W_2 \\ &\quad + 2(s-r)(-s)W_0^2W_1 - (r^2+r-4s-rs)W_2W_1W_0).\end{aligned}$$

(b):  $\Lambda_G(n) = (-s)^n \Lambda_G(0) = (-s)^{n+1}$ , *that is*

$$\begin{aligned}&G_{n+2}^3 - r(r-s+1)G_{n+1}^3 + (-s)^2G_n^3 - 2(r+1)G_{n+1}G_{n+2}^2 - (s-r)G_nG_{n+2}^2 \\ &\quad + (r^2+3r-s+1)G_{n+2}G_{n+1}^2 + (r^2+s^2-s-3rs)G_nG_{n+1}^2 \\ &\quad + (r+1)(-s)G_n^2G_{n+2} + 2(s-r)(-s)G_n^2G_{n+1} - (r^2+r-4s-rs)G_{n+2}G_{n+1}G_n \\ &= (-s)^{n+1}.\end{aligned}$$

(c):  $\Lambda_H(n) = (-s)^n \Lambda_H(0) = -(r^2+4s)(r+s-1)^2(-s)^n$ , *that is*

$$\begin{aligned}&H_{n+2}^3 - r(r-s+1)H_{n+1}^3 + (-s)^2H_n^3 - 2(r+1)H_{n+1}H_{n+2}^2 - (s-r)H_nH_{n+2}^2 \\ &\quad + ((r+1)^2 - (s-r))H_{n+2}H_{n+1}^2 + ((r+1)(-s) + (s-r)^2)H_nH_{n+1}^2 \\ &\quad + (r+1)(-s)H_n^2H_{n+2} + 2(s-r)(-s)H_n^2H_{n+1} - (r^2+r-4s-rs)H_{n+2}H_{n+1}H_n \\ &= -(r^2+4s)(r+s-1)^2(-s)^n.\end{aligned}$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Lemma 35].  $\square$

## 5. Identities of Generalized Horadam-Leonardo Polynomials

In this section, we get some identities of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials. First, we present a few basic relations between  $\{G_n\}$  and  $\{H_n\}$ .

LEMMA 11. *The following equalities are true:*

- (a):  $(-s)^3 H_n = (3rs + r^3 - s^3)G_{n+4} - (4r^2s - rs^3 + 3rs + r^3 + r^4 + 2s^2)G_{n+3} + (4r^2s + r^4 + 2s^2 + s^4)G_{n+2}$ .
- (b):  $(-s)^2 H_n = (2s + r^2 + s^2)G_{n+3} - (2s + rs^2 + 3rs + r^2 + r^3)G_{n+2} + (3rs + r^3 - s^3)G_{n+1}$ .
- (c):  $(-s)H_n = -(s - r)G_{n+2} + (rs - 2s - r - r^2)G_{n+1} + (r^2 + s^2 + 2s)G_n$ .
- (d):  $H_n = 3G_{n+1} - 2(r + 1)G_n - (s - r)G_{n-1}$ .
- (e):  $H_n = (r + 1)G_n + 2(s - r)G_{n-1} + 3(-s)G_{n-2}$ .
- (f):  $s(r^2 + 4s)(r + s - 1)^2 G_n = (-r^3 + r^2s + 2r^2 - 3rs - r + 4s^2 + 4s)H_{n+4} + (r^4 - r^3s - r^3 + 4r^2s - r^2 - 4rs^2 - 3rs + r + 2s^2 - 2s)H_{n+3} - (2s + 4r^2s + r^2s^2 - 6rs + r^2 - 2r^3 + r^4 + 2s^2 + 4s^3)H_{n+2}$ .
- (g):  $-(r^2 + 4s)(r + s - 1)^2 G_n = -2(r^2 - r + 3s + 1)H_{n+3} + (-r - 2s + 7rs - r^2 + 2r^3 + 2)H_{n+2} + (-r + 4s + r^2s - 3rs + 2r^2 - r^3 + 4s^2)H_{n+1}$ .
- (h):  $-(r^2 + 4s)(r + s - 1)^2 G_n = (rs - 8s - r - r^2)H_{n+2} + (r + 2s - r^2s + 5rs + r^3 - 2s^2)H_{n+1} + 2s(-r + 3s + r^2 + 1)H_n$ .
- (i):  $-(r^2 + 4s)(r + s - 1)^2 G_n = -2(3s + rs + r^2 + s^2)H_{n+1} + (2s + rs^2 + 5rs + r^2 + r^3 - 2s^2)H_n + s(r + 8s - rs + r^2)H_{n-1}$ .
- (j):  $-(r^2 + 4s)(r + s - 1)^2 G_n = -(4s + rs^2 + 2r^2s + 3rs + r^2 + r^3 + 4s^2)H_n + (-rs^2 + r^2s + 7rs + 2r^3 + 2s^2 - 2s^3)H_{n-1} + 2s(3s + rs + r^2 + s^2)H_{n-2}$ .

Proof. Replace  $r, s$  and  $t$  with  $r + 1, s - r, -s$ , respectively, in [51, Lemma 36].  $\square$

Now, we give some basic relations between  $\{G_n\}$  and  $\{W_n\}$ .

LEMMA 12. *The following equalities are true:*

- (a):  $(-W_2^2 - (r - s + 1)W_1^2 + sW_0^2 + (r + 2)W_1W_2 - rW_0W_2 + (r - 2s)W_0W_1)(-W_2 + rW_1 + sW_0)G_n = ((r + 1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s - r)W_0W_1)W_{n+2} + (W_2^2 - (r + 1)W_1W_2 - (s - r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n$ .
- (b):  $(-W_2^2 - (r - s + 1)W_1^2 + sW_0^2 + (r + 2)W_1W_2 - rW_0W_2 + (r - 2s)W_0W_1)(-W_2 + rW_1 + sW_0)G_n = (W_2^2 + (r + 1)^2W_1^2 + (r + 1)(-s)W_0^2 - 2(r + 1)W_1W_2 - (s - r)W_0W_2 + ((r + 1)(s - r) - (-s))W_0W_1)W_{n+1} + (((-s) + (r + 1)(s - r))W_1^2 + (s - r)(-s)W_0^2 - (s - r)W_1W_2 - (-s)W_0W_2 + (s - r)^2W_0W_1)W_n + ((r + 1)(-s)W_1^2 + (-s)^2W_0^2 - (-s)W_1W_2 + (s - r)(-s)W_0W_1)W_{n-1}$ .
- (c):  $(-W_2^2 - (r - s + 1)W_1^2 + sW_0^2 + (r + 2)W_1W_2 - rW_0W_2 + (r - 2s)W_0W_1)(-W_2 + rW_1 + sW_0)G_n = ((r + 1)W_2^2 + ((r + 1)^3 + (-s) + (r + 1)(s - r))W_1^2 + (-s)((s - r) + (r + 1)^2)W_0^2 - ((s - r) + 2(r + 1)^2)W_1W_2 - ((-s) + (r + 1)(s - r))W_0W_2 + ((r + 1)^2(s - r) + (s - r)^2 - (r + 1)(-s))W_0W_1)W_n + ((s - r)W_2^2 + (r + 1)((r + 1)(s - r) + (-s))W_1^2 + (-s)((-s) + (r + 1)(s - r))W_0^2 - ((-s) + 2(r + 1)(s - r))W_1W_2 - (s - r)^2W_0W_2 + (r + 1)(s - r)^2W_0W_1)W_{n-1} + ((-s)W_2^2 + (r + 1)^2(-s)W_1^2 + (r + 1)(-s)^2W_0^2 - 2(r + 1)(-s)W_1W_2 - (s - r)(-s)W_0W_2 + (-s)((r + 1)(s - r) - (-s))W_0W_1)W_{n-2}$ .
- (d):  $(-s)W_n = (W_2 - (r + 1)W_1 - (s - r)W_0)G_{n+2} + (-(r + 1)W_2 + (r + 1)^2W_1 + ((-s) + (r + 1)(s - r))W_0)G_{n+1} + (-(s - r)W_2 + ((-s) + (r + 1)(s - r))W_1 + ((s - r)^2 - (r + 1)(-s))W_0)G_n$ .
- (e):  $W_n = W_0G_{n+1} + (W_1 - (r + 1)W_0)G_n + (W_2 - (r + 1)W_1 - (s - r)W_0)G_{n-1}$ .
- (f):  $W_n = W_1G_n + (W_2 - (r + 1)W_1)G_{n-1} + (-s)W_0G_{n-2}$ .

Proof. Replace  $r, s$  and  $t$  with  $r + 1, s - r, -s$ , respectively, in [51, Lemma 37.].  $\square$

Next, we present some basic relations between  $\{H_n\}$  and  $\{W_n\}$ .

LEMMA 13. *The following equalities hold:*

- (a):  $(-W_2^2 - (r - s + 1)W_1^2 + sW_0^2 + (r + 2)W_1W_2 - rW_0W_2 + (r - 2s)W_0W_1)(-W_2 + rW_1 + sW_0)H_n =$   
 $(3W_2^2 + ((r + 1)^2 - (s - r))W_1^2 + (r + 1)(-s)W_0^2 - 4(r + 1)W_1W_2 - 2(s - r)W_0W_2 + ((r + 1)(s - r) -$   
 $3(-s))W_0W_1)W_{n+2} + (-2(r + 1)W_2^2 + 3(-s)W_1^2 - 2(s - r)W_1W_2 - 3(-s)W_0W_2 + 3(r + 1)(s - r)W_1^2 +$   
 $2(s - r)(-s)W_0^2 + 2(r + 1)^2W_1W_2 + 2(s - r)^2W_0W_1 + (r + 1)(s - r)W_0W_2 + 2(r + 1)(-s)W_0W_1)W_{n+1} +$   
 $(-s - r)W_2^2 + ((s - r)^2 + (r + 1)(-s))W_1^2 + 3(-s)^2W_0^2 + ((r + 1)(s - r) - 3(-s))W_1W_2 + 2(r +$   
 $1)(-s)W_0W_2 + 4(s - r)(-s)W_0W_1)W_n.$
- (b):  $(-W_2^2 - (r - s + 1)W_1^2 + sW_0^2 + (r + 2)W_1W_2 - rW_0W_2 + (r - 2s)W_0W_1)(-W_2 + rW_1 + sW_0)H_n =$   
 $((r + 1)^3W_1^2 + (r + 1)W_2^2 + 3(-s)W_1^2 + (r + 1)^2(-s)W_0^2 - 2(s - r)W_1W_2 - 3(-s)W_0W_2 + 2(r + 1)(s -$   
 $r)W_1^2 + 2(s - r)(-s)W_0^2 - 2(r + 1)^2W_1W_2 + 2(s - r)^2W_0W_1 - (r + 1)(s - r)W_0W_2 - (r + 1)(-s)W_0W_1 +$   
 $(r + 1)^2(s - r)W_0W_1)W_{n+1} + (3(-s)^2W_0^2 + 2(s - r)W_2^2 + (r + 1)^2(s - r)W_1^2 - 3(-s)W_1W_2 + (r +$   
 $1)(-s)W_1^2 - 2(s - r)^2W_0W_2 - 3(r + 1)(s - r)W_1W_2 + 2(r + 1)(-s)W_0W_2 + (s - r)(-s)W_0W_1 +$   
 $(r + 1)(s - r)(-s)W_0^2 + (r + 1)(s - r)^2W_0W_1)W_n + (3(-s)W_2^2 + (-s)((r + 1)^2 - (s - r))W_1^2 + (r +$   
 $1)(-s)^2W_0^2 - 4(r + 1)(-s)W_1W_2 - 2(s - r)(-s)W_0W_2 + (-s)((r + 1)(s - r) - 3(-s))W_0W_1)W_{n-1}.$
- (c):  $(-W_2^2 - (r - s + 1)W_1^2 + sW_0^2 + (r + 2)W_1W_2 - rW_0W_2 + (r - 2s)W_0W_1)(-W_2 + rW_1 + sW_0)H_n =$   
 $((r + 1)^2 + 2(s - r))W_2^2 + (r + 1)((r + 1)^3 + 3(r + 1)(s - r) + 4(-s))W_1^2 + (-s)((r + 1)^3 + 3(r +$   
 $1)(s - r) + 3(-s))W_0^2 - (3(-s) + 2(r + 1)^3 + 5(r + 1)(s - r))W_1W_2 - ((r + 1)^2(s - r) + (r + 1)(-s) +$   
 $2(s - r)^2)W_0W_2 + ((r + 1)^3(s - r) - (r + 1)^2(-s) + 3(r + 1)(s - r)^2 + (s - r)(-s))W_0W_1)W_n + (($   
 $3(-s) + (r + 1)(s - r))W_2^2 + ((r + 1)^3(s - r) + 2(r + 1)(s - r)^2 + 2(s - r)(-s) + (r + 1)^2(-s))W_1^2 + (-s)$   
 $((r + 1)(-s) + (r + 1)^2(s - r) + 2(s - r)^2)W_0^2 - 2(2(r + 1)(-s) + (s - r)^2 + (r + 1)^2(s - r))W_1$   
 $W_2 - (s - r)((r + 1)(s - r) + 5(-s))W_0W_2 + (2(s - r)^3 + (r + 1)^2(s - r)^2 - 3(-s)^2)W_0W_1)W_{n-1} +$   
 $((r + 1)(-s)W_2^2 + (-s)((r + 1)^3 + 3(-s) + 2(r + 1)(s - r))W_1^2 + (-s)^2(r^2 + 2s + 1)W_0^2 - 2(-s)((s -$   
 $r) + (r + 1)^2)W_1W_2 - (-s)(3(-s) + (r + 1)(s - r))W_0W_2 + (-s)(2(s - r)^2 + (r + 1)^2(s - r) - (r +$   
 $1)(-s))W_0W_1)W_{n-2}.$
- (d):  $-(r^2 + 4s)(r + s - 1)^2W_n = (-2((r + 1)^2 + 3(s - r))W_2 + (2(r + 1)^3 + 9(-s) + 7(r + 1)(s -$   
 $r))W_1 + (4(s - r)^2 - 3(r + 1)(-s) + (r + 1)^2(s - r))W_0)H_{n+2} + ((2(r + 1)^3 + 9(-s) + 7(r + 1)(s -$   
 $r))W_2 - 2((r + 1)^4 + 4(r + 1)^2(s - r) + 6(-s)(r + 1) + (s - r)^2)W_1 - (4(r + 1)(s - r)^2 + 6(-s)(s -$   
 $r) - (-s)(r + 1)^2 + (r + 1)^3(s - r))W_0)H_{n+1} + ((-3(r + 1)(-s) + 4(s - r)^2 + (r + 1)^2(s - r))W_2 -$   
 $(4(r + 1)(s - r)^2 + (r + 1)^3(s - r) - (r + 1)^2(-s) + 6(s - r)(-s))W_1 + (-(r + 1)^2(s - r)^2 + 2(r +$   
 $1)^3(-s) + 9(-s)^2 - 4(s - r)^3 + 10(r + 1)(s - r)(-s))W_0)H_n.$
- (e):  $-(r^2 + 4s)(r + s - 1)^2W_n = ((9(-s) + (r + 1)(s - r))W_2 - ((r + 1)^2(s - r) + 3(-s)(r + 1) + 2(s -$   
 $r)^2)W_1 - 2(-s)(3(s - r) + (r + 1)^2)W_0)H_{n+1} + (-(r + 1)^2(s - r) + 2(s - r)^2 + 3(r + 1)(-s))W_2 +$   
 $((r + 1)^3(s - r) + 3(r + 1)(s - r)^2 + (r + 1)^2(-s) + 3(s - r)(-s))W_1 + (-s)(9(-s) + 2(r + 1)^3 +$

$$7(r+1)(s-r)W_0H_n + (-2(-s)((r+1)^2 + 3(s-r))W_2 + (-s)(2(r+1)^3 + 9(-s) + 7(r+1)(s-r)))W_1 + (-s)(4(s-r)^2 + (r+1)^2(s-r) - 3(r+1)(-s))W_0H_{n-1}.$$

$$\begin{aligned} \mathbf{(f):} \quad & -(r^2 + 4s)(r + s - 1)^2W_n = (2(3(r+1)(-s) - (s-r)^2)W_2 + (3(s-r)(-s) + (r+1)(s-r)^2 - \\ & 2(r+1)^2(-s))W_1 + (-s)(9(-s) + (r+1)(s-r))W_0)H_n + (((r+1)(s-r)^2 - 2(r+1)^2(-s) + \\ & 3(s-r)(-s))W_2 + (2(r+1)^3(-s) - (r+1)^2(s-r)^2 + 4(r+1)(s-r)(-s) - 2(s-r)^3 + 9(-s)^2)W_1 - \\ & (-s)(2(s-r)^2 + 3(r+1)(-s) + (r+1)^2(s-r))W_0)H_{n-1} + ((-s)(9(-s) + (r+1)(s-r))W_2 - \\ & (-s)((r+1)^2(s-r) + 3(r+1)(-s) + 2(s-r)^2)W_1 - 2(-s)^2(3(s-r) + (r+1)^2)W_0)H_{n-2}. \end{aligned}$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Lemma 38].  $\square$

## 6. Recurrence Properties of Generalized Horadam-Leonardo Polynomials

Now, we give a formula for  $W_{-n}$ .

THEOREM 14. For  $n \in \mathbb{Z}$ , we have

$$W_{-n} = (-s)^{-n}(W_{2n} - H_nW_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Theorem 39].  $\square$

Next, we present the following corollary.

COROLLARY 15. For  $n \in \mathbb{Z}$ , we get

$$\mathbf{(a):} \quad G_{-n} = \frac{1}{(-s)^{n+1}}((2(r+1)(-s) - (s-r)^2)G_n^2 + (-s)G_{2n} + (s-r)G_{n+2}G_n - (3(-s) + (r+1)(s-r))G_{n+1}G_n).$$

$$\mathbf{(b):} \quad H_{-n} = \frac{1}{2(-s)^n}(H_n^2 - H_{2n}).$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Corollary 42] or put  $W_n = G_n$  with  $G_0 = 0, G_1 = 1, G_2 = (r+1)$  and take  $W_n = H_n$  with  $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$ , respectively, in the last Theorem.  $\square$

## 7. Generalized Horadam-Leonardo Polynomials by Matrix Methods

In this section, we give matrix representations of the sequences  $W_n, G_n, H_n$  and present Simson matrix and investigate its properties.

**7.1. Matrix related to the Sequences  $W_n, G_n$  and  $H_n$ .** We define the square matrix  $A$  of order 3 as:

$$A = \begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that  $\det A = (-s)$ . We also define

$$B_n = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & (s-r)W_n + (-s)W_{n-1} & (-s)W_n \\ W_n & (s-r)W_{n-1} + (-s)W_{n-2} & (-s)W_{n-1} \\ W_{n-1} & (s-r)W_{n-2} + (-s)W_{n-3} & (-s)W_{n-2} \end{pmatrix}.$$

**THEOREM 16.** *The following properties hold: for all integers  $m, n$ ,*

**(a):**  $B_n = A^n$ , *i.e.*,

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix}.$$

**(b):**  $D_1 A^n = A^n D_1$ .

**(c):**  $D_{n+m} = D_n B_m = B_m D_n$ , *i.e.*,

**(d):**

$$A^n = G_{n-1} A^2 + ((s-r)G_{n-2} + (-s)G_{n-3})A + (-s)G_{n-2}I,$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

*Proof.* Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Theorem 51].  $\square$

Now, we give matrix formulas for the generalized Horadam-Leonardo polynomials and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials.

**COROLLARY 17.** *For all integers  $n$ , the following formulas for generalized Horadam-Leonardo polynomials and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials hold.*

**(a):** *(Generalized Horadam-Leonardo polynomials).*

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$a_{11} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+3} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+2} + ((-s)W_1^2 - (-s)W_0W_2)W_{n+1},$$

$$a_{21} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+2} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n,$$

$$a_{31} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1},$$

$$a_{12} = (s-r)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+2} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n + (-s)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1},$$

$$a_{22} = (s-r)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1} + (-s)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_n + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-1} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-2},$$

$$a_{32} = (s-r)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_n + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-1} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-2} + (-s)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n-1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-2} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-3},$$

$$a_{13} = (-s)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+2} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n,$$

$$a_{23} = (-s)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1},$$

$$a_{33} = (-s)((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_n + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n-1} + ((-s)W_1^2 - (-s)W_0W_2)W_{n-2},$$

$\Lambda_W(0)$  given as in (4.2), i.e.,

$$\begin{aligned} \Lambda_W(0) &= W_2^3 - r(r-s+1)W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 \\ &\quad + (r^2 + 3r - s + 1)W_2W_1^2 + ((r+1)(-s) + (s-r)^2)W_0W_1^2 \\ &\quad + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 - (r^2 + r - 4s - rs)W_2W_1W_0. \end{aligned}$$

(b):  $((r, s)$ -Horadam-Leonardo-Lucas polynomials).

$$\begin{aligned} &\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \\ &= \frac{1}{4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s)} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \end{aligned}$$

where

$$b_{11} = (9(-s) + (r+1)(s-r))H_{n+3} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+2} - 2(-s)((r+1)^2 + 3(s-r))H_{n+1},$$

$$b_{21} = (9(-s) + (r+1)(s-r))H_{n+2} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+1} - 2(-s)((r+1)^2 + 3(s-r))H_n,$$

$$b_{31} = (9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1},$$

$$b_{12} = (s-r)((9(-s) + (r+1)(s-r))H_{n+2} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+1} - 2(-s)((r+1)^2 + 3(s-r))H_n) + (-s)((9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1}),$$

$$b_{22} = (s-r)((9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1}) + (-s)((9(-s) + (r+1)(s-r))H_n - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-1} - 2(-s)((r+1)^2 + 3(s-r))H_{n-2}),$$

$$b_{32} = (s-r)((9(-s) + (r+1)(s-r))H_n - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-1} - 2(-s)((r+1)^2 + 3(s-r))H_{n-2}) + (-s)((9(-s) + (r+1)(s-r))H_{n-1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-2} - 2(-s)((r+1)^2 + 3(s-r))H_{n-3}),$$

$$b_{13} = (-s)((9(-s) + (r+1)(s-r))H_{n+2} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n+1} - 2(-s)((r+1)^2 + 3(s-r))H_n),$$

$$b_{23} = (-s)((9(-s) + (r+1)(s-r))H_{n+1} - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_n - 2(-s)((r+1)^2 + 3(s-r))H_{n-1}),$$

$$b_{33} = (-s)((9(-s) + (r+1)(s-r))H_n - ((r+1)^2(s-r) + 2(s-r)^2 + 3(r+1)(-s))H_{n-1} - 2(-s)((r+1)^2 + 3(s-r))H_{n-2}).$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Corollary 52].  $\square$

Next, we give an identity for  $W_{n+m}$ .

**THEOREM 18.** (*Honsberger Formula*) *We have*

$$\begin{aligned} W_{n+m} &= W_n G_{m+1} + W_{n-1}((s-r)G_m + (-s)G_{m-1}) + (-s)W_{n-2}G_m \\ &= W_n G_{m+1} + ((s-r)W_{n-1} + (-s)W_{n-2})G_m + (-s)W_{n-1}G_{m-1}, \end{aligned}$$

for all integers  $m$  and  $n$ .

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Theorem 53].  $\square$

**COROLLARY 19.** *We have the following properties:*

$$\begin{aligned} G_{n+m} &= G_n G_{m+1} + G_{n-1}((s-r)G_m + (-s)G_{m-1}) + (-s)G_{n-2}G_m, \\ H_{n+m} &= H_n G_{m+1} + H_{n-1}((s-r)G_m + (-s)G_{m-1}) + (-s)H_{n-2}G_m, \end{aligned}$$

for all integers  $m$  and  $n$ .

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Corollary 54].  $\square$

COROLLARY 20. *We have the following properties:*

$$\begin{aligned} W_{mn+j} &= G_{mn-1}W_{j+2} + ((s-r)G_{mn-2} + (-s)G_{mn-3})W_{j+1} + (-s)G_{mn-2}W_j, \\ G_{mn+j} &= G_{mn-1}G_{j+2} + ((s-r)G_{mn-2} + (-s)G_{mn-3})G_{j+1} + (-s)G_{mn-2}G_j, \\ H_{mn+j} &= G_{mn-1}H_{j+2} + ((s-r)G_{mn-2} + (-s)G_{mn-3})H_{j+1} + (-s)G_{mn-2}H_j, \end{aligned}$$

for all integers  $m$  and  $n$ .

Proof. Replace  $r$ ,  $s$  and  $t$  with  $r+1$ ,  $s-r$ ,  $-s$ , respectively, in [51, Corollary 55].  $\square$

**7.2. The Simson Matrix.** For  $n \in \mathbb{Z}$ , we define

$$f_W(n) = \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix}.$$

Let us call this matrix as Simson matrix of the sequence  $W_n$ . Then, as special cases of  $W_n$ ,

$$f_G(n) = \begin{pmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{pmatrix}, \quad f_H(n) = \begin{pmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{pmatrix},$$

are Simson matrices of the sequences  $G_n$  and  $H_n$ , respectively.

LEMMA 21. *The followings hold: For all integers  $n, m$  and  $j$ ,*

(a):  $f_W(n) = (r+1)f_W(n-1) + (s-r)f_W(n-2) + (-s)f_W(n-3).$

(b):

$$f_W(n) = Af_W(n-1),$$

$$f_W(n) = A^n f_W(0).$$

(c):

$$f_W(n+m) = A^n f_W(m),$$

$$f_W(n+m) = A^m f_W(n),$$

$$f_W(n) = A^m f_W(n-m).$$

(d):

$$f_W(mn+j) = A^{mn} f_W(j)$$

and

$$f_W(mn+j) = (G_{n-1}A^2 + ((s-r)G_{n-2} + (-s)G_{n-3})A + (-s)G_{n-2}I)^m f_W(j).$$

(e):

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}.$$

(f):

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}$$

where  $a_{33}, a_{23}, a_{13}, a_{32}, a_{22}, a_{12}, a_{31}, a_{21}, a_{11}$ , and  $\Lambda_W(0)$  are as in Corollary 17 (a) (in the last identity above, we replace  $n$  with  $m$  in  $a_{33}, a_{23}, a_{13}, a_{32}, a_{22}, a_{12}, a_{31}, a_{21}, a_{11}$ ).

(g):

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}}{-(r^2 + 4s)(r + s - 1)^2}$$

where  $b_{33}, b_{23}, b_{13}, b_{32}, b_{22}, b_{12}, b_{31}, b_{21}, b_{11}$  are as in Corollary 17 (b) (in the last identity above, we replace  $n$  with  $m$  in  $b_{33}, b_{23}, b_{13}, b_{32}, b_{22}, b_{12}, b_{31}, b_{21}, b_{11}$ ).

Proof. Replace  $r, s$  and  $t$  with  $r + 1, s - r, -s$ , respectively, in [51, Lemma 56].  $\square$

By taking the determinant of both sides of the identities given in Lemma 21, one get the following Theorem.

**THEOREM 22.** For all  $n, m \in \mathbb{Z}$ , the following identities are true.

(a): Catalan's Formula:

$$\det(f_W(n + m)) = (-s)^n \det(f_W(m)), \quad \det(f_W(n)) = (-s)^m \det(f_W(n - m)).$$

(b): Cassini's(or Simson's) Formula (see Theorem 8):

$$\det(f_W(n)) = (-s)^n \det(f_W(0)).$$

Proof. Replace  $r, s$  and  $t$  with  $r + 1, s - r, -s$ , respectively, in [51, Theorem 57].  $\square$

By using the last Theorem, one get the following Corollary which produces determinantal formulas of  $(r, s)$ -Horadam-Leonardo polynomials (put  $W_n = G_n$  with  $G_0 = 0, G_1 = 1, G_2 = (r + 1)$ ).

**COROLLARY 23.** For all  $n, m \in \mathbb{Z}$ , the following identities are true.

(a): Catalan's Formula:

$$\det(f_G(n + m)) = (-s)^n \det(f_G(m)), \quad \det(f_G(n)) = (-s)^m \det(f_G(n - m)).$$

(b): *Cassini's(or Simson's) Formula:*

$$\det(f_G(n)) = (-s)^n \det(f_G(0)), \text{ i.e., } \begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -(-s)^{n-1}.$$

By putting  $W_n = H_n$  with  $H_0 = 3, H_1 = (r + 1), H_2 = r^2 + 2s + 1$  in the last Theorem, one get the following Corollary which produces determinantal formulas of  $(r, s)$ -Horadam-Leonardo-Lucas polynomials.

COROLLARY 24. *For all  $n, m \in \mathbb{Z}$ , the following identities are true.*

(a): *Catalan's Formula:*

$$\det(f_H(n + m)) = (-s)^n \det(f_H(m)), \det(f_H(n)) = (-s)^m \det(f_H(n - m)).$$

(b): *Cassini's(or Simson's) Formula:*

$$\det(f_H(n)) = (-s)^n \det(f_H(0)).$$

## 8. The Sum Formula $\sum_{k=0}^n z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials

Now, we present the sum formula  $\sum_{k=0}^n z^k W_{mk+j}$  of generalized Horadam-Leonardo polynomials.

**8.1. The Sum Formula  $\sum_{k=0}^n z^k W_{mk+j}$  of Generalized Horadam-Leonardo Polynomials in Terms of Generalized Horadam-Leonardo Polynomials.** We can give the sum formula  $\sum_{k=0}^n z^k W_{mk+j}$  of generalized Horadam-Leonardo polynomials (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials).

THEOREM 25. *For all  $m, j \in \mathbb{Z}$ , one get the following sum formulas.*

(a): *If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$  then*

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_W(z)}{\Gamma_W(z)}$$

where

$$\begin{aligned} \Theta_W(z) &= z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6, \\ z^{n+3}\Theta_1 &= z^{n+3}(-W_j W_{m+2}^2 W_{m+mn+2} + (-W_{j+1} + (r+1)W_j)W_{m+2}^2 W_{m+mn+1} + (-W_{j+2} + (r+1)W_{j+1} + (s-r)W_j)W_{m+2}^2 W_{m+mn} + (-W_{j+2} + (s-r)W_j)W_{m+1}^2 W_{m+mn+2} - ((-s) + (r+1)(s-r))W_j W_{m+1}^2 W_{m+mn+1} + ((s-r)W_{j+2} - ((-s) + (r+1)(s-r))W_{j+1} - (s-r)^2 W_j)W_{m+1}^2 W_{m+mn} - (-s)W_m^2 W_{j+1} W_{m+mn+2} + (-s)(-W_{j+2} + (r+1)W_{j+1})W_m^2 W_{m+mn+1} - (-s)^2 W_j W_m^2 W_{m+mn} + (W_{j+1} + (r+1)W_j)W_{m+2} W_{m+1} W_{m+mn+2} + (W_{j+2} - (r+1)W_{j+1})W_{m+2} W_m W_{m+mn+2} + (-s-r)W_{j+1} + (-s)W_j)W_{m+1} W_m W_{m+mn+2} + (W_{j+2} - (r+1)^2 W_j)W_{m+2} W_{m+1} W_{m+mn+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} + (-s)W_j)W_{m+2} W_m W_{m+mn+1} + (-(s-r)W_{j+2} + ((-s) + (r+1)(s-r))W_{j+1} - (r+1)(-s)W_j)W_{m+1} W_m W_{m+mn+1} + (r+1)W_{j+2} W_{m+1} W_{m+2} W_{m+mn} + (-s)W_j W_{m+1} W_{m+2} W_{m+mn} - (r+1)(s-r)W_j W_{m+1} W_{m+2} W_{m+mn} - (r+1)^2 W_{j+1} W_{m+1} W_{m+2} \end{aligned}$$

$$W_{m+mn} + (-s)(W_{j+1} - (r+1)W_j)W_{m+2}W_mW_{m+mn} + (-s)(W_{j+2} - (r+1)W_{j+1} - 2(s-r)W_j)W_{m+1}W_mW_{m+mn},$$

$$\begin{aligned} z^{n+2}\Theta_2 = & z^{n+2}((-W_0W_{j+2} + ((r+1)W_0 - W_1)W_{j+1} + (2W_2 - (r+1)W_1)W_j)W_{m+2}W_{m+mn+2} + \\ & (2W_1W_{j+2} + ((s-r)W_0 - W_2)W_{j+1} - ((r+1)W_2 + 2(s-r)W_1 + (-s)W_0)W_j)W_{m+1}W_{m+mn+2} + \\ & (-W_2W_{j+2} + ((r+1)W_2 + (s-r)W_1 + 2(-s)W_0)W_{j+1} - (-s)W_1W_j)W_mW_{m+mn+2} + (((r+1)W_0 - \\ & W_1)W_{j+2} + (2W_2 - (r+1)^2W_0 - (s-r)W_0)W_{j+1} + (-2(r+1)W_2 + (r+1)^2W_1 - (-s)W_0)W_j)W_{m+2} \\ & W_{m+mn+1} + ((-W_2 + (s-r)W_0)W_{j+2} - ((r+1)(s-r)W_0 + (-s)W_0)W_{j+1} + ((r+1)^2W_2 + 2((-s) + (r+1) \\ & (s-r))W_1 + (r+1)(-s)W_0)W_j)W_{m+1}W_{m+mn+1} + (((r+1)W_2 + (s-r)W_1 + 2(-s)W_0)W_{j+2} - ((r+1) \\ & ^2W_2 + (r+1)(s-r)W_1 + 2(r+1)(-s)W_0 + (s-r)W_2 + (-s)W_1)W_{j+1} + (-s)((r+1)W_1 - W_2)W_j) \\ & W_mW_{m+mn+1} + ((2W_2 - (r+1)W_1)W_{j+2} + ((r+1)^2W_1 - 2(r+1)W_2 - (-s)W_0)W_{j+1} + ((r+1)(s-r) \\ & W_1 - 2(s-r)W_2 - (-s)W_1 + (r+1)(-s)W_0)W_j)W_{m+2}W_{m+mn} + (-((r+1)W_2 + 2(s-r)W_1 + \\ & (-s)W_0)W_{j+2} + ((r+1)^2W_2 + 2((-s) + (r+1)(s-r))W_1 + (r+1)(-s)W_0)W_{j+1} + (((r+1)(s-r) \\ & - (-s))W_2 + 2(s-r)^2W_1 + 2(s-r)(-s)W_0)W_j)W_{m+1}W_{m+mn} + (-s)(-W_1W_{j+2} + ((r+1)W_1 - \\ & W_2)W_{j+1} + ((r+1)W_2 + 2(s-r)W_1 + 2(-s)W_0)W_j)W_mW_{m+mn}), \end{aligned}$$

$$\begin{aligned} z^{n+1}\Theta_3 = & z^{n+1}(((W_0W_2 - W_1^2)W_{j+2} + (-(-s)W_0^2 + W_1W_2 - (r+1)W_0W_2 - (s-r)W_0W_1) \\ & )W_{j+1} + (-W_2^2 + (s-r)W_1^2 + (r+1)W_1W_2 + (-s)W_0W_1)W_j)W_{m+mn+2} + ((-(-s)W_0^2 + W_1W_2 - \\ & (r+1)W_0W_2 - (s-r)W_0W_1)W_{j+2} + (-W_2^2 + (r+1)(-s)W_0^2 + ((r+1)^2 + (s-r))W_0W_2 + ((-s) + \\ & (r+1)(s-r))W_0W_1)W_{j+1} + ((r+1)W_2^2 - ((r+1)(s-r) + (-s))W_1^2 - (r+1)^2W_1W_2 + (-s)W_0W_2 - \\ & (r+1)(-s)W_0W_1)W_j)W_{m+mn+1} + ((-W_2^2 + (s-r)W_1^2 + (r+1)W_1W_2 + (-s)W_0W_1)W_{j+2} + ( \\ & (r+1)W_2^2 - ((r+1)(s-r) + (-s))W_1^2 - (r+1)^2W_1W_2 + (-s)W_0W_2 - (r+1)(-s)W_0W_1)W_{j+1} + \\ & ((s-r)W_2^2 - (s-r)^2W_1^2 - (-s)^2W_0^2 + ((-s) - (r+1)(s-r))W_1W_2 - (r+1)(-s)W_0W_2 - 2(s-r) \\ & (-s)W_0W_1)W_j)W_{m+mn}), \end{aligned}$$

$$\begin{aligned} z^2\Theta_4 = & z^2((W_0W_{j+2} + (W_1 - (r+1)W_0)W_{j+1} + (W_2 - (r+1)W_1 - (s-r)W_0)W_j)W_{m+2}^2 + ((W_2 - \\ & (s-r)W_0)W_{j+2} + ((-s)W_0 + (r+1)(s-r)W_0)W_{j+1} + ((s-r)^2W_0 + ((r+1)(s-r) + (-s))W_1 - \\ & (s-r)W_2)W_j)W_{m+1}^2 + (-s)(W_1W_{j+2} + (W_2 - (r+1)W_1)W_{j+1} + (-s)W_0W_j)W_m^2 + (-W_1 + (r+1) \\ & W_0)W_{j+2} + ((r+1)^2W_0 - W_2)W_{j+1} + (-(r+1)W_2 + (r+1)^2W_1 + ((r+1)(s-r) - (-s))W_0)W_j) \\ & W_{m+1}W_{m+2} + (((r+1)W_1 - W_2)W_{j+2} + ((r+1)W_2 - ((s-r) + (r+1)^2)W_1 - (-s)W_0)W_{j+1} + \\ & (-s)((r+1)W_0 - W_1)W_j)W_{m+2}W_m + (((s-r)W_1 - (-s)W_0)W_{j+2} + ((s-r)W_2 - ((r+1)(s-r) + \\ & (-s))W_1 + (r+1)(-s)W_0)W_{j+1} + (-s)(-W_2 + (r+1)W_1 + 2(s-r)W_0)W_j)W_{m+1}W_m), \end{aligned}$$

$$\begin{aligned} z\Theta_5 = & z(((W_1^2 - W_0W_2)W_{j+2} + ((-s)W_0^2 - W_1W_2 + (r+1)W_0W_2 + (s-r)W_0W_1)W_{j+1} + \\ & (-2W_2^2 - (r+1)^2W_1^2 - (-s)(r+1)W_0^2 + 3(r+1)W_1W_2 + 2(s-r)W_0W_2 + (2(-s) - (s-r)(r+1) \\ & )W_0W_1)W_j)W_{m+2} + (((-s)W_0^2 - W_1W_2 + (r+1)W_0W_2 + (s-r)W_0W_1)W_{j+2} + (W_2^2 - ((r+1)^2 + \\ & (s-r))W_0W_2 - ((r+1)(s-r) + (-s))W_0W_1)W_{j+1} + ((r+1)W_2^2 - 2((-s) + (r+1)(s-r))W_1^2 - 2(s-r) \\ & (-s)W_0^2 + (2(s-r) - (r+1)^2)W_1W_2 + (2(-s) - (r+1)(s-r))W_0W_2 - (2(s-r)^2 + (r+1)(-s))W_0W_1) \\ & W_j)W_{m+1} + ((W_2^2 - (s-r)W_1^2 - (r+1)W_1W_2 - (-s)W_0W_1)W_{j+2} + (-(r+1)W_2^2 + ((r+1)(s-r) + \end{aligned}$$

$$(-s)W_1^2 + (r+1)^2W_1W_2 - (-s)W_0W_2 + (r+1)(-s)W_0W_1W_{j+1} + (-s)(-(r+1)W_1^2 - 2(-s)W_0^2 + 2W_1W_2 - (r+1)W_0W_2 - 2(s-r)W_0W_1)W_jW_m - (r+1)(-s)W_0^2W_{j+1}W_{m+1}),$$

$$\Theta_6 = (W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 + ((r+1)^2 - (s-r))W_1^2W_2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2)W_j,$$

and

$$\Gamma_W(z) = z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4,$$

$$z^3\Gamma_1 = z^3(-(-s)^m(W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 + ((r+1)^2 - (s-r))W_1^2W_2 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 + (r+1)(-s)W_0^2W_2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2)),$$

$$z^2\Gamma_2 = z^2((3W_2 - 2(r+1)W_1 - (s-r)W_0)W_{m+2}^2 + (((r+1)^2 - (s-r))W_2 + (3(r+1)(s-r) + 3(-s))W_1 + ((s-r)^2 + (r+1)(-s))W_0)W_{m+1}^2 + (-s)((r+1)W_2 + 2(s-r)W_1 + 3(-s)W_0)W_m^2 + (-4(r+1)W_2 + 2((r+1)^2 - (s-r))W_1 + ((r+1)(s-r) - 3(-s))W_0)W_{m+2}W_{m+1} + (-2(s-r)W_2 + ((r+1)(s-r) - 3(-s))W_1 + 2(r+1)(-s)W_0)W_{m+2}W_m + (((r+1)(s-r) - 3(-s))W_2 + 2((s-r)^2 + (r+1)(-s))W_1 + 4(s-r)(-s)W_0)W_{m+1}W_m),$$

$$z\Gamma_3 = z((-3W_2^2 + ((s-r) - (r+1)^2)W_1^2 - (-s)(r+1)W_0^2 + 4(r+1)W_1W_2 + 2(s-r)W_0W_2 + (3(-s) - (s-r)(r+1))W_0W_1)W_{m+2} + (2(r+1)W_2^2 - (3(r+1)(s-r) + 3(-s))W_1^2 - 2(s-r)(-s)W_0^2 + (2(s-r) - 2(r+1)^2)W_1W_2 + (3(-s) - (r+1)(s-r))W_0W_2 - 2((s-r)^2 + (r+1)(-s))W_0W_1)W_{m+1} + ((s-r)W_2^2 - ((s-r)^2 + (r+1)(-s))W_1^2 - 3(-s)^2W_0^2 + (3(-s) - (r+1)(s-r))W_1W_2 - 2(r+1)(-s)W_0W_2 - 4(s-r)(-s)W_0W_1)W_m),$$

$$\Gamma_4 = W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 + ((r+1)^2 - (s-r))W_1^2W_2 - (s-r)W_0W_2^2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2.$$

(b): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Theorem 61].  $\square$

Next, we present some special cases of the last Theorem.

**THEOREM 26.** *One get the following summing formulas.*

(a): ( $m = 1, j = 0$ ).

(i): If  $z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1 \neq 0$  then

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_1}{z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1}$$

where

$$\Omega_1 = z^{n+3}(-1)(-s)W_n + z^{n+2}((r+1)W_{n+1} - W_{n+2}) + z^{n+1}(-1)W_{n+1} + z^2(W_2 - (r+1)W_1 - (s-r)W_0) + z(W_1 - (r+1)W_0) + W_0.$$

(ii): If  $z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_2}{3z^2(-(-s)) + 2z(-1)(s-r) + (-1)(r+1)}$$

where

$$\Omega_2 = (n+3)z^{n+2}(-1)(-s)W_n + (n+2)z^{n+1}((r+1)W_{n+1} - W_{n+2}) + (n+1)z^n(-1)W_{n+1} + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0).$$

(iii): If  $z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_3}{6z(-(-s)) + 2(-1)(s-r)}$$

where

$$\Omega_3 = (n+3)(n+2)z^{n+1}(-1)(-s)W_n + (n+2)(n+1)z^n((r+1)W_{n+1} - W_{n+2}) + (n+1)nz^{n-1}(-1)W_{n+1} + 2(W_2 - (r+1)W_1 - (s-r)W_0)$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_4}{3z^2(-(-s)) + 2z(-1)(s-r) + (-1)(r+1)}$$

where

$$\Omega_4 = (n+3)z^{n+2}(-1)(-s)W_n + (n+2)z^{n+1}((r+1)W_{n+1} - W_{n+2}) + (n+1)z^n(-1)W_{n+1} + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0).$$

(iv): If  $z^3(-(-s)) + z^2(-1)(s-r) + z(-1)(r+1) + 1 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_5}{6(-(-s))}$$

where

$$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)(-s)W_n + (n+2)(n+1)nz^{n-1}((r+1)W_{n+1} - W_{n+2}) + (n+1)n(n-1)z^{n-2}(-1)W_{n+1}.$$

(b): ( $m = 2, j = 0$ ).

(i): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 \neq 0$  then

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_1}{z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1}$$

where

$$\begin{aligned} \Omega_1 = & z^{n+3}(-1)(-s)^2 W_{2n} + z^{n+2}((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + \\ & z^{n+1}(-1)W_{2n+2} + z^2(-s-r)W_2 + ((-s) + (r+1)(s-r))W_1 + ((s-r)^2 - (r+1)(-s))W_0 + \\ & z(W_2 - ((r+1)^2 + 2(s-r))W_0) + W_0. \end{aligned}$$

(ii): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_2}{3z^2(-(-s)^2) + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)(r^2 + 2s + 1)}$$

where

$$\begin{aligned} \Omega_2 = & (n+3)z^{n+2}(-1)(-s)^2 W_{2n} + (n+2)z^{n+1}((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - \\ & (r+1)(-s)W_{2n}) + (n+1)z^n(-1)W_{2n+2} + 2z(-s-r)W_2 + ((-s) + (r+1)(s-r))W_1 + ((s-r)^2 - \\ & (r+1)(-s))W_0 + (W_2 - ((r+1)^2 + 2(s-r))W_0). \end{aligned}$$

(iii): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_3}{6z(-(-s)^2) + 2(-2(r+1)(-s) + (s-r)^2)}$$

where

$$\begin{aligned} \Omega_3 = & (n+3)(n+2)z^{n+1}(-1)(-s)^2 W_{2n} + (n+2)(n+1)z^n((s-r)W_{2n+2} - ((r+1)(s-r) + \\ & (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + (n+1)nz^{n-1}(-1)W_{2n+2} + 2(-s-r)W_2 + ((-s) + (r+1)(s-r)) \\ & W_1 + ((s-r)^2 - (r+1)(-s))W_0 \end{aligned}$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_4}{3z^2(-(-s)^2) + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)(r^2 + 2s + 1)}$$

where

$$\Omega_4 = (n+3)z^{n+2}(-1)(-s)^2W_{2n} + (n+2)z^{n+1}((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + (n+1)z^n(-1)W_{2n+2} + 2z(-s-r)W_2 + ((-s) + (r+1)(s-r))W_1 + ((s-r)^2 - (r+1)(-s))W_0 + (W_2 - ((r+1)^2 + 2(s-r))W_0).$$

(iv): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_5}{6(-(-s)^2)}$$

where

$$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)(-s)^2W_{2n} + (n+2)(n+1)nz^{n-1}((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + (n+1)n(n-1)z^{n-2}(-1)W_{2n+2}.$$

(c): ( $m = 2, j = 1$ ).

(i): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 \neq 0$  then

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_1}{z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1}$$

where

$$\Omega_1 = z^{n+3}(-(-s)^2W_{2n+1}) + z^{n+2}(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + z^{n+1}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + z^2(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + z((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0) + W_1.$$

(ii): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-(-s)^2) + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)(r^2 + 2s + 1)}$$

where

$$\Omega_2 = (n+3)z^{n+2}(-(-s)^2W_{2n+1}) + (n+2)z^{n+1}(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)z^n(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2z(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0).$$

(iii): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_3}{6z(-(-s)^2) + 2(-2(r+1)(-s) + (s-r)^2)}$$

where

$$\Omega_3 = (n+3)(n+2)z^{n+1}(-(-s)^2W_{2n+1}) + (n+2)(n+1)z^n(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)nz^{n-1}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2(-s)(W_2 - (r+1)W_1 - (s-r)W_0)$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-(-s)^2) + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)(r^2 + 2s + 1)}$$

where

$$\begin{aligned} \Omega_4 = & (n+3)z^{n+2}(-(-s)^2 W_{2n+1}) + (n+2)z^{n+1}(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + \\ & (s-r)(-s)W_{2n}) + (n+1)z^n(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2z(-s)(W_2 - \\ & (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0). \end{aligned}$$

(iv): If  $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2 + 2s + 1) + 1 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_5}{6(-(-s)^2)}$$

where

$$\begin{aligned} \Omega_5 = & (n+3)(n+2)(n+1)z^n(-(-s)^2 W_{2n+1}) + (n+2)(n+1)nz^{n-1}(-(-s)W_{2n+2} + ((s-r)^2 - \\ & (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)n(n-1)z^{n-2}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + \\ & (-s)W_{2n}). \end{aligned}$$

**8.2. The Sum Formula  $\sum_{k=0}^n z^k W_{mk+j}$  of Generalized Horadam-Leonardo Polynomials (in Terms of Generalized Horadam-Leonardo Polynomials and  $(r, s)$ -Horadam-Leonardo Polynomials).** The sum formula  $\sum_{k=0}^n z^k W_{mk+j}$  of generalized Horadam-Leonardo polynomials (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and  $(r, s)$ -Horadam-Leonardo polynomials) can be given as follows.

**THEOREM 27.** For all  $m, j \in \mathbb{Z}$ , one get the following summing formulas.

(a): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$  then

$$\begin{aligned} \sum_{k=0}^n z^k W_{mk+j} &= \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} \\ &= \frac{\Theta_G(z)}{\Gamma_G(z)} \end{aligned}$$

where

$$\begin{aligned} \Theta_G(z) &= z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6, \\ z^{n+3}\Theta_1 &= z^{n+3}((-W_j G_{m+2}^2 + (-W_{j+2} + (s-r)W_j)G_{m+1}^2 - (-s)W_{j+1}G_m^2 + (W_{j+1} + (r+1)W_j) \\ & G_{m+2}G_{m+1} + (W_{j+2} - (r+1)W_{j+1})G_{m+2}G_m + (-s-r)W_{j+1} + (-s)W_j)G_{m+1}G_m)G_{m+mn+2} \\ & + ((-W_{j+1} + (r+1)W_j)G_{m+2}^2 - ((-s) + (r+1)(s-r))W_j G_{m+1}^2 + (-s)(-W_{j+2} + (r+1)W_{j+1})G_m^2 + \\ & (W_{j+2} - (r+1)^2 W_j)G_{m+2}G_{m+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} + (-s)W_j)G_{m+2}G_m + \\ & (-s-r)W_{j+2} + ((-s) + (r+1)(s-r))W_{j+1} - (r+1)(-s)W_j)G_{m+1}G_m)G_{m+mn+1} + ((-W_{j+2} + \\ & (r+1)W_{j+1} + (s-r)W_j)G_{m+2}^2 + ((s-r)W_{j+2} - ((-s) + (r+1)(s-r))W_{j+1} - (s-r)^2 W_j)G_{m+1}^2 - \end{aligned}$$

$$W_j(-s)^2G_m^2 + (W_{j+2}(r+1) - (r+1)^2W_{j+1} + ((-s) - (r+1)(s-r))W_j)G_{m+2}G_{m+1} + (-s)(W_{j+1} - (r+1)W_j)G_{m+2}G_m + (-s)(W_{j+2} - (r+1)W_{j+1} - 2(s-r)W_j)G_{m+1}G_m)G_{m+mn},$$

$$\begin{aligned} z^{n+2}\Theta_2 = & z^{n+2}(((r+1)W_j - W_{j+1})G_{m+2} + (2W_{j+2} - (r+1)W_{j+1} - ((r+1)^2 + 2(s-r))W_j)G_{m+1} + \\ & (-r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} - (-s)W_j)G_m)G_{m+mn+2} + ((2(r+1)W_{j+1} - W_{j+2} - (r+1)^2W_j)G_{m+2} + \\ & (-r+1)W_{j+2} + ((r+1)^3 + 2(-s) + 2(r+1)(s-r))W_j)G_{m+1} + (((r+1)^2 + (s-r))W_{j+2} - \\ & ((r+1)^3 + 2(r+1)(s-r) + (-s))W_{j+1})G_m)G_{m+mn+1} + (((r+1)W_{j+2} - (r+1)^2W_{j+1} - ((r+1)(s-r) + \\ & (-s))W_j)G_{m+2} + (-((r+1)^2 + 2(s-r))W_{j+2} + ((r+1)^3 + 2(r+1)(s-r) + 2(-s))W_{j+1} + ((r+1)^2(s-r) + \\ & 2(s-r)^2 - (r+1)(-s))W_j)G_{m+1} + 2(s-r)(-s)W_jG_m - (-s)G_mW_{j+2} + (r+1)^2(-s)W_jG_m) \\ & G_{m+mn}), \end{aligned}$$

$$\begin{aligned} z^{n+1}\Theta_3 = & z^{n+1}((-W_{j+2} + (r+1)W_{j+1} + (s-r)W_j)G_{m+mn+2} + ((r+1)W_{j+2} - (r+1)^2W_{j+1} - \\ & ((r+1)(s-r) + (-s))W_j)G_{m+mn+1} + ((s-r)W_{j+2} - ((r+1)(s-r) + (-s))W_{j+1} - ((s-r)^2 - \\ & (r+1)(-s))W_j)G_{m+mn}), \end{aligned}$$

$$\begin{aligned} z^2\Theta_4 = & z^2(G_{m+2}^2W_{j+1} + ((r+1)W_{j+2} + (-s)W_j)G_{m+1}^2 + (-s)G_m^2W_{j+2} - ((r+1)W_{j+1} + \\ & W_{j+2})G_{m+1}G_{m+2} - ((s-r)W_{j+1} + (-s)W_j)G_mG_{m+2} + ((s-r)W_{j+2} - (-s)W_{j+1})G_mG_{m+1}), \end{aligned}$$

$$\begin{aligned} z\Theta_5 = & z((W_{j+2} - (r+1)W_{j+1})G_{m+2} + (-r+1)W_{j+2} + (r+1)^2W_{j+1} - 2(-s)W_j)G_{m+1} + \\ & (-s-r)W_{j+2} + ((-s)W_{j+1} + (r+1)(s-r)W_{j+1}) + (r+1)(-s)W_j)G_m), \end{aligned}$$

$$\Theta_6 = (-s)W_j,$$

and

$$\Gamma_G(z) = \Gamma_1(z) + \Gamma_2(z) + \Gamma_3(z) + \Gamma_4(z),$$

$$z^3\Gamma_1 = z^3(-s)^{m+1},$$

$$\begin{aligned} z^2\Gamma_2 = & z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + ((r+1)^2(-s) + 2(s-r) \\ & (-s))G_m^2 - 2((r+1)^2 + (s-r))G_{m+1}G_{m+2} - ((r+1)(s-r) + 3(-s))G_mG_{m+2} + ((r+1)^2(s-r) \\ & + 2(s-r)^2 - (r+1)(-s))G_mG_{m+1}), \end{aligned}$$

$$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} + (2(r+1)(-s) - (s-r)^2)G_m),$$

$$\Gamma_4 = (-s).$$

(b): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Theorem 63].  $\square$

Some special cases of the last Theorem can be given as follows.

**THEOREM 28.** *One get the following sum formulas.*

(a): ( $m = 1, j = 0$ ).

(i): If  $z^3(-1)(-s) + z^2(-(s-r)) + z(-(r+1)) + 1 \neq 0$  then

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_1}{z^3(-1)(-s) + z^2(-(s-r)) + z(-(r+1)) + 1}$$

where

$$\begin{aligned} \Omega_1 = & z^{n+3}((-W_2 + (r+1)W_1 + (s-r)W_0)G_{n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((r+1)(s-r) \\ & + (-s))W_0)G_{n+1} + ((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 + ((r+1)(-s) - (s-r)^2)W_0)G_n \\ & + z^{n+2}((-W_1 + (r+1)W_0)G_{n+2} - (W_2 - 2(r+1)W_1 + (r+1)^2W_0)G_{n+1} + ((r+1)W_2 - (r+1)^2W_1 \\ & - ((r+1)(s-r) + (-s))W_0)G_n) + z^{n+1}(-W_0G_{n+2} + (-W_1 + (r+1)W_0)G_{n+1} + (-W_2 + \\ & (r+1)W_1 + (s-r)W_0)G_n) + z^2(W_2 - (r+1)W_1 - (s-r)W_0) + z(W_1 - (r+1)W_0) + W_0. \end{aligned}$$

(ii): If  $z^3(-1)(-s) + z^2(-(s-r)) + z(-(r+1)) + 1 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_2}{3z^2(-1)(-s) + 2z(-(s-r)) + (-(r+1))}$$

where

$$\begin{aligned} \Omega_2 = & (n+3)z^{n+2}((-W_2 + (r+1)W_1 + (s-r)W_0)G_{n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((r+1)(s-r) \\ & + (-s))W_0)G_{n+1} + ((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 + ((r+1)(-s) - (s-r)^2)W_0)G_n \\ & + (n+2)z^{n+1}((-W_1 + (r+1)W_0)G_{n+2} - (W_2 - 2(r+1)W_1 + (r+1)^2W_0)G_{n+1} + ((r+1)W_2 - \\ & (r+1)^2W_1 - ((r+1)(s-r) + (-s))W_0)G_n) + (n+1)z^n(-W_0G_{n+2} + (-W_1 + (r+1)W_0)G_{n+1} + \\ & (-W_2 + (r+1)W_1 + (s-r)W_0)G_n) + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0). \end{aligned}$$

(iii): If  $z^3(-1)(-s) + z^2(-(s-r)) + z(-(r+1)) + 1 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_3}{6z(-1)(-s) + 2(-(s-r))}$$

where

$$\begin{aligned} \Omega_3 = & (n+3)(n+2)z^{n+1}((-W_2 + (r+1)W_1 + (s-r)W_0)G_{n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((r+1)(s-r) \\ & + (-s))W_0)G_{n+1} + ((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 + ((r+1)(-s) - (s-r)^2)W_0)G_n \\ & + (n+2)(n+1)z^n((-W_1 + (r+1)W_0)G_{n+2} - (W_2 - 2(r+1)W_1 + (r+1)^2W_0)G_{n+1} + ((r+1)W_2 - \\ & (r+1)^2W_1 - ((r+1)(s-r) + (-s))W_0)G_n) + (n+1)z^{n-1}(-W_0G_{n+2} + (-W_1 + (r+1)W_0)G_{n+1} + \\ & (-W_2 + (r+1)W_1 + (s-r)W_0)G_n) + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0). \end{aligned}$$

$$1)W_2 - (r+1)^2W_1 - ((r+1)(s-r) + (-s))W_0)G_n) + (n+1)nz^{n-1}(-W_0G_{n+2} + (-W_1 + (r+1)W_0)G_{n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_n) + 2(W_2 - (r+1)W_1 - (s-r)W_0)$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_4}{3z^2(-1)(-s) + 2z(-(s-r)) + (-(r+1))}$$

where

$$\begin{aligned} \Omega_4 = & (n+3)z^{n+2}((-W_2 + (r+1)W_1 + (s-r)W_0)G_{n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((r+1)(s-r) \\ & + (-s))W_0)G_{n+1} + ((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 + ((r+1)(-s) - (s-r)^2)W_0)G_n \\ & ) + (n+2)z^{n+1}((-W_1 + (r+1)W_0)G_{n+2} - (W_2 - 2(r+1)W_1 + (r+1)^2W_0)G_{n+1} + ((r+1)W_2 - \\ & (r+1)^2W_1 - ((r+1)(s-r) + (-s))W_0)G_n) + (n+1)z^n(-W_0G_{n+2} + (-W_1 + (r+1)W_0)G_{n+1} + \\ & (-W_2 + (r+1)W_1 + (s-r)W_0)G_n) + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0) \end{aligned}$$

(iv): If  $z^3(-1)(-s) + z^2(-(s-r)) + z(-(r+1)) + 1 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_k = \frac{\Omega_5}{6(-1)(-s)}$$

where

$$\begin{aligned} \Omega_5 = & (n+3)(n+2)(n+1)z^n((-W_2 + (r+1)W_1 + (s-r)W_0)G_{n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((r+1)(s-r) \\ & + (-s))W_0)G_{n+1} + ((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 + ((r+1)(-s) - (s-r)^2)W_0)G_n \\ & ) + (n+2)(n+1)nz^{n-1}((-W_1 + (r+1)W_0)G_{n+2} - (W_2 - 2(r+1)W_1 + (r+1)^2W_0)G_{n+1} + \\ & ((r+1)W_2 - (r+1)^2W_1 - ((r+1)(s-r) + (-s))W_0)G_n) + (n+1)n(n-1)z^{n-2}(-W_0G_{n+2} + \\ & (-W_1 + (r+1)W_0)G_{n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_n) \end{aligned}$$

(b): ( $m = 2, j = 0$ ).

(i): If  $z^3(-1)(-s)^2 + z^2((s-r)^2 - 2(r+1)(-s)) + z(-((r+1)^2 + 2(s-r))) + 1 \neq 0$  then

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_1}{z^3(-1)(-s)^2 + z^2((s-r)^2 - 2(r+1)(-s)) + z(-((r+1)^2 + 2(s-r))) + 1}$$

where

$$\begin{aligned} \Omega_1 = & z^{n+3}(-s)((-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((-s) + \\ & (r+1)(s-r))W_0)G_{2n+1} + ((s-r)W_2 - ((-s) + (r+1)(s-r))W_1 + ((r+1)(-s) - (s-r)^2)W_0) \\ & G_{2n}) + z^{n+2}((-r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n+2} + (((r+1)^2 + (s-r))W_2 - \\ & ((r+1)^3 + 2(r+1)(s-r) + (-s))W_1)G_{2n+1} + (-s)(-W_2 + 2(s-r)W_0 + (r+1)^2W_0)G_{2n}) + \\ & z^{n+1}(-W_1G_{2n+2} + ((r+1)W_1 - W_2)G_{2n+1} - (-s)W_0G_{2n}) + z^2(-s-r)W_2 + ((r+1)(s-r) + \\ & (-s))W_1 + ((s-r)^2 - (r+1)(-s))W_0 + z(W_2 - ((r+1)^2 + 2(s-r))W_0) + W_0. \end{aligned}$$

(ii): If  $z^3(-1)(-s)^2 + z^2((s-r)^2 - 2(r+1)(-s)) + z(-((r+1)^2 + 2(s-r))) + 1 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$

then

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_2}{3z^2(-1)(-s)^2 + 2z((s-r)^2 - 2(r+1)(-s)) + (-((r+1)^2 + 2(s-r)))}$$

where

$$\begin{aligned} \Omega_2 = & (n+3)z^{n+2}(-s)((-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((-s) + \\ & (r+1)(s-r))W_0)G_{2n+1} + ((s-r)W_2 - ((-s) + (r+1)(s-r))W_1 + ((r+1)(-s) - (s-r)^2)W_0) \\ & G_{2n}) + (n+2)z^{n+1}((-r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n+2} + (((r+1)^2 + \\ & (s-r))W_2 - ((r+1)^3 + 2(r+1)(s-r) + (-s))W_1)G_{2n+1} + (-s)(-W_2 + 2(s-r)W_0 + (r+ \\ & 1)^2W_0)G_{2n}) + (n+1)z^n(-W_1G_{2n+2} + ((r+1)W_1 - W_2)G_{2n+1} - (-s)W_0G_{2n}) + 2z(-s-r)W_2 + \\ & ((r+1)(s-r) + (-s))W_1 + ((s-r)^2 - (r+1)(-s))W_0 + (W_2 - ((r+1)^2 + 2(s-r))W_0). \end{aligned}$$

(iii): If  $z^3(-1)(-s)^2 + z^2((s-r)^2 - 2(r+1)(-s)) + z(-((r+1)^2 + 2(s-r))) + 1 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_3}{6z(-1)(-s)^2 + 2((s-r)^2 - 2(r+1)(-s))}$$

where

$$\begin{aligned} \Omega_3 = & (n+3)(n+2)z^{n+1}(-s)((-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n+2} + ((r+1)W_2 - (r+1)^2W_1 - \\ & ((-s) + (r+1)(s-r))W_0)G_{2n+1} + ((s-r)W_2 - ((-s) + (r+1)(s-r))W_1 + ((r+1)(-s) - (s-r)^2)W_0) \\ & G_{2n}) + (n+2)(n+1)z^n((-r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n+2} + (((r+1)^2 + \\ & (s-r))W_2 - ((r+1)^3 + 2(r+1)(s-r) + (-s))W_1)G_{2n+1} + (-s)(-W_2 + 2(s-r)W_0 + (r+ \\ & 1)^2W_0)G_{2n}) + (n+1)nz^{n-1}(-W_1G_{2n+2} + ((r+1)W_1 - W_2)G_{2n+1} - (-s)W_0G_{2n}) + 2(-s-r) \\ & W_2 + ((r+1)(s-r) + (-s))W_1 + ((s-r)^2 - (r+1)(-s))W_0 \end{aligned}$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_4}{3z^2(-1)(-s)^2 + 2z((s-r)^2 - 2(r+1)(-s)) + (-((r+1)^2 + 2(s-r)))}$$

where

$$\begin{aligned} \Omega_4 = & (n+3)z^{n+2}(-s)((-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n+2} + ((r+1)W_2 - (r+1)^2W_1 - ((-s) + \\ & (r+1)(s-r))W_0)G_{2n+1} + ((s-r)W_2 - ((-s) + (r+1)(s-r))W_1 + ((r+1)(-s) - (s-r)^2)W_0) \\ & G_{2n}) + (n+2)z^{n+1}((-r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n+2} + (((r+1)^2 + \\ & (s-r))W_2 - ((r+1)^3 + 2(r+1)(s-r) + (-s))W_1)G_{2n+1} + (-s)(-W_2 + 2(s-r)W_0 + (r+ \\ & 1)^2W_0)G_{2n}) + (n+1)z^n(-W_1G_{2n+2} + ((r+1)W_1 - W_2)G_{2n+1} - (-s)W_0G_{2n}) + 2z(-s-r)W_2 + \\ & ((r+1)(s-r) + (-s))W_1 + ((s-r)^2 - (r+1)(-s))W_0 + (W_2 - ((r+1)^2 + 2(s-r))W_0). \end{aligned}$$

(iv): If  $z^3(-1)(-s)^2 + z^2((s-r)^2 - 2(r+1)(-s)) + z(-((r+1)^2 + 2(s-r))) + 1 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_{2k} = \frac{\Omega_5}{6(-1)(-s)^2}$$

where

$$\begin{aligned} \Omega_5 = & (n+3)(n+2)(n+1)z^n(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1- \\ & ((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0) \\ & G_{2n}) + (n+2)(n+1)nz^{n-1}((-r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2- \\ & ((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n}) \\ & + (n+1)n(n-1)z^{n-2}(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n}). \end{aligned}$$

(c): ( $m = 2, j = 1$ ).

(i): If  $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 \neq 0$  then

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_1}{z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1}$$

where

$$\begin{aligned} \Omega_1 = & z^{n+3}(-s)^2(-W_0G_{2n+2}+((r+1)W_0-W_1)G_{2n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_{2n}) \\ & + z^{n+2}(((s-r)W_2-((r+1)(s-r)+(-s))W_1-(r+1)(-s)W_0)G_{2n+2}+(-((r+1)(s-r)+ \\ & (-s))W_2+(s-r)((r+1)^2+(s-r))W_1+(-s)((s-r)+(r+1)^2)W_0)G_{2n+1}+(-s)(-(r+1)W_2+ \\ & ((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n})+z^{n+1}(-W_2G_{2n+2}-((s-r)W_1+(-s)W_0)G_{2n+1}-(-s)W_1 \\ & G_{2n})+z^2(-s)(W_2-(r+1)W_1-(s-r)W_0)+z((r+1)W_2-((r+1)^2+(s-r))W_1+(-s)W_0)+W_1. \end{aligned}$$

(ii): If  $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-1)(-s)^2 + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)((r+1)^2 + 2(s-r))}$$

where

$$\begin{aligned} \Omega_2 = & (n+3)z^{n+2}(-s)^2(-W_0G_{2n+2}+((r+1)W_0-W_1)G_{2n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_{2n}) \\ & + (n+2)z^{n+1}(((s-r)W_2-((r+1)(s-r)+(-s))W_1-(r+1)(-s)W_0)G_{2n+2}+(-((r+1)(s-r)+ \\ & (-s))W_2+(s-r)((r+1)^2+(s-r))W_1+(-s)((s-r)+(r+1)^2)W_0)G_{2n+1}+(-s)(-(r+1)W_2+((r+1)^2+(s-r))W_1- \\ & (-s)W_0)G_{2n})+(n+1)z^n(-W_2G_{2n+2}-((s-r)W_1+(-s)W_0)G_{2n+1}-(-s)W_1 \\ & G_{2n})+2z(-s)(W_2-(r+1)W_1-(s-r)W_0)+((r+1)W_2-((r+1)^2+(s-r))W_1+(-s)W_0). \end{aligned}$$

(iii): If  $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_3}{6z(-1)(-s)^2 + 2(-2(r+1)(-s) + (s-r)^2)}$$

where

$$\begin{aligned} \Omega_3 = & (n+3)(n+2)z^{n+1}(-s)^2(-W_0G_{2n+2}+((r+1)W_0-W_1)G_{2n+1}+(-W_2+(r+1)W_1+ \\ & (s-r)W_0)G_{2n}) + (n+2)(n+1)z^n(((s-r)W_2-((r+1)(s-r)+(-s))W_1-(r+1)(-s)W_0) \\ & G_{2n+2}+(-((r+1)(s-r)+(-s))W_2+(s-r)((r+1)^2+(s-r))W_1+(-s)((s-r)+(r+1)^2)W_0) \\ & G_{2n+1}+(-s)(-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n})+(n+1)nz^{n-1}(-W_2G_{2n+2}- \\ & ((s-r)W_1+(-s)W_0)G_{2n+1}-(-s)W_1G_{2n})+2(-s)(W_2-(r+1)W_1-(s-r)W_0) \end{aligned}$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-1)(-s)^2 + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)((r+1)^2 + 2(s-r))}$$

where

$$\begin{aligned} \Omega_4 = & (n+3)z^{n+2}(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) \\ & + (n+2)z^{n+1}(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0)G_{2n+2} + (-((r+1)(s-r) + \\ & (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0)G_{2n+1} + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) \\ & + (n+1)z^n(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1G_{2n}) + 2z(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0). \end{aligned}$$

(iv): If  $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_{2k+1} = \frac{\Omega_5}{6(-1)(-s)^2}$$

where

$$\begin{aligned} \Omega_5 = & (n+3)(n+2)(n+1)z^n(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) \\ & + (n+2)(n+1)nz^{n-1}(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0)G_{2n+2} + (-((r+1)(s-r) + (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0)G_{2n+1} \\ & + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) + (n+1)n(n-1)z^{n-2}(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1G_{2n}). \end{aligned}$$

**8.3. The Summing Formula  $\sum_{k=0}^n z^k W_{mk+j}$  of Generalized Horadam-Leonardo Polynomials (in Terms of Generalized Horadam-Leonardo Polynomials and  $(r, s)$ -Horadam-Leonardo-Lucas Polynomials).** The sum formula  $\sum_{k=0}^n z^k W_{mk+j}$  of generalized Horadam-Leonardo polynomials (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials) can be given as follows.

**THEOREM 29.** For all  $m, j \in \mathbb{Z}$ , one get the following summing formulas.

(a): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$  then

$$\begin{aligned} \sum_{k=0}^n z^k W_{mk+j} &= \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} \\ &= \frac{\Theta_W(z)}{\Gamma_W(z)} \end{aligned}$$

where

$$\begin{aligned} \Theta_W(z) &= z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6, \\ z^{n+3}\Theta_1 &= z^{n+3}((-W_j H_{m+2}^2 + (-W_{j+2} + (s-r)W_j)H_{m+1}^2 - (-s)W_{j+1}H_m^2 + (W_{j+1} + (r+1)W_j)H_{m+2}H_{m+1} + (W_{j+2} - (r+1)W_{j+1})H_{m+2}H_m + (-s-r)W_{j+1} + (-s)W_j)H_{m+1}H_m)H_{m+mn+2} + \\ &((-W_{j+1} + (r+1)W_j)H_{m+2}^2 - ((-s) + (r+1)(s-r))W_j H_{m+1}^2 + (-s)(-W_{j+2} + (r+1)W_{j+1})H_m^2 + \end{aligned}$$

$$(W_{j+2} - (r+1)^2 W_j) H_{m+2} H_{m+1} + (-(r+1) W_{j+2} + ((r+1)^2 + (s-r)) W_{j+1} + (-s) W_j) H_{m+2} H_m + (-s) W_j H_{m+2} H_m + (-(s-r) W_{j+2} + ((r+1)(s-r) + (-s)) W_{j+1} - (r+1)(-s) W_j) H_{m+1} H_m H_{m+mn+1} + ((-W_{j+2} + (r+1) W_{j+1} + (s-r) W_j) H_{m+2}^2 + ((s-r) W_{j+2} - ((r+1)(s-r) + (-s)) W_{j+1} - (s-r)^2 W_j) H_{m+1}^2 - W_j (-s)^2 H_m^2 + ((r+1) W_{j+2} - (r+1)^2 W_{j+1} + ((-s) - (r+1)(s-r)) W_j) H_{m+2} H_{m+1} + (-s)(W_{j+1} - (r+1) W_j) H_{m+2} H_m + (-s)(W_{j+2} - (r+1) W_{j+1} - 2(s-r) W_j) H_{m+1} H_m) H_{m+mn}),$$

$$z^{n+2} \Theta_2 = z^{n+2} (((-3W_{j+2} + 2(r+1)W_{j+1} + ((r+1)^2 + 4(s-r))W_j)H_{m+2} + (2(r+1)W_{j+2} + ((s-r) - (r+1)^2)W_{j+1} - ((r+1)^3 + 4(r+1)(s-r) + 3(-s))W_j)H_{m+1} + (-(r+1)^2 + 2(s-r))W_{j+2} + ((r+1)^3 + 3(r+1)(s-r) + 6(-s))W_{j+1} - (r+1)(-s)W_j)H_m)H_{m+mn+2} + ((2(r+1)W_{j+2} - ((r+1)^2 - (s-r))W_{j+1} - ((r+1)^3 + 3(-s) + 4(r+1)(s-r))W_j)H_{m+2} + (-(r+1)^2 - (s-r))W_{j+2} - 3((-s) + (r+1)(s-r))W_{j+1} + (r+1)((r+1)^3 + 4(r+1)(s-r) + 5(-s))W_j)H_{m+1} + (((r+1)^3 + 3(r+1)(s-r) + 6(-s))W_{j+2} - ((r+1)^4 + 4(r+1)^2(s-r) + 2(s-r)^2 + 7(r+1)(-s))W_{j+1} - 2(s-r)(-s)W_j)H_m)H_{m+mn+1} + (((r+1)^2 + 4(s-r))W_{j+2} - ((r+1)^3 + 4(r+1)(s-r) + 3(-s))W_{j+1} - ((r+1)^2(s-r) - 2(r+1)(-s) + 4(s-r)^2)W_j)H_{m+2} + (-(r+1)^3 + 4(r+1)(s-r) + 3(-s))W_{j+2} + ((r+1)^4 + 4(r+1)^2(s-r) + 5(r+1)(-s))W_{j+1} + ((r+1)^3(s-r) - (r+1)^2(-s) + 4(r+1)(s-r)^2 + 4(s-r)(-s))W_j)H_{m+1} + (-s)((-r+1)W_{j+2} - 2(s-r)W_{j+1} + ((r+1)^3 + 4(r+1)(s-r) + 6(-s))W_j)H_m)H_{m+mn}),$$

$$z^{n+1} \Theta_3 = z^{n+1} ((2(3(s-r) + (r+1)^2)W_{j+2} - (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))W_{j+1} - ((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)W_j)H_{m+mn+2} + (-(2(r+1)^3 + 9(-s) + 7(r+1)(s-r))W_{j+2} + 2((r+1)^4 + 4(r+1)^2(s-r) + 6(r+1)(-s) + (s-r)^2)W_{j+1} + ((r+1)^3(s-r) - (r+1)^2(-s) + 4(r+1)(s-r)^2 + 6(s-r)(-s))W_j)H_{m+mn+1} + (-(r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)W_{j+2} + ((r+1)^3(s-r) - (r+1)^2(-s) + 4(r+1)(s-r)^2 + 6(s-r)(-s))W_{j+1} + (-2(r+1)^3(-s) + (r+1)^2(s-r)^2 + 4(s-r)^3 - 9(-s)^2 - 10(r+1)(s-r)(-s))W_j)H_{m+mn}),$$

$$z^2 \Theta_4 = z^2 ((3W_{j+2} - 2(r+1)W_{j+1} - (s-r)W_j)H_{m+2}^2 + (((r+1)^2 - (s-r))W_{j+2} + 3((r+1)(s-r) + (-s))W_{j+1} + ((s-r)^2 + (r+1)(-s))W_j)H_{m+1}^2 + (-s)((r+1)W_{j+2} + 2(s-r)W_{j+1} + 3(-s)W_j)H_m^2 + (-4(r+1)W_{j+2} + 2((r+1)^2 - (s-r))W_{j+1} + ((r+1)(s-r) - 3(-s))W_j)H_{m+2}H_{m+1} + (-2(s-r)W_{j+2} + ((r+1)(s-r) - 3(-s))W_{j+1} + 2(r+1)(-s)W_j)H_{m+2}H_m + (((r+1)(s-r) - 3(-s))W_{j+2} + 2((s-r)^2 + (r+1)(-s))W_{j+1} + 4(s-r)(-s)W_j)H_{m+1}H_m),$$

$$z \Theta_5 = z((-2((r+1)^2 + 3(s-r))W_{j+2} + (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))W_{j+1} + ((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)W_j)H_{m+2} + ((2(r+1)^3 + 7(r+1)(s-r) + 9(-s))W_{j+2} - 2((r+1)^4 + 4(r+1)^2(s-r) + 6(r+1)(-s) + (s-r)^2)W_{j+1} - ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))W_j)H_{m+1} + (((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))W_{j+2} - (4(r+1)(s-r)^2 + (r+1)^3(s-r) - (r+1)^2(-s) + 6(s-r)(-s))W_{j+1} - 2(-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))W_j)H_m),$$

$$\Theta_6 = (4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s))W_j,$$

and

$$\Gamma_W(z) = \Gamma_1(z) + \Gamma_2(z) + \Gamma_3(z) + \Gamma_4(z),$$

$$z^3 \Gamma_1 = z^3 (-(s)^m (4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2)),$$

$$z^2\Gamma_2 = z^2(((r+1)^2+3(s-r))H_{m+2}^2+((r+1)^4+4(r+1)^2(s-r)+(s-r)^2+6(r+1)(-s))H_{m+1}^2+(-s)((r+1)^3+4(r+1)(s-r)+9(-s))H_m^2-(2(r+1)^3+7(r+1)(s-r)+9(-s))H_{m+2}H_{m+1}-((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))H_{m+2}H_m+(r+1)(s-r)((r+1)^2+4(s-r))H_{m+1}H_m-(-s)((r+1)^2-6(s-r))H_{m+1}H_m),$$

$$z\Gamma_3 = z(-4(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-27(-s)^2-18(r+1)(s-r)(-s))H_m,$$

$$\Gamma_4 = 4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2.$$

(b): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$  for some  $u, a, b, c \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b \neq c$ , i.e.,  $z = a$  or  $z = b$  or  $z = c$  then

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$  for some  $u, a, b \in \mathbb{C}$  with  $u \neq 0$  and  $a \neq b$ , i.e.,  $z = a$  or  $z = b$  then for  $z = a$  we get

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for  $z = b$  we get

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d): If  $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$  for some  $u, a \in \mathbb{C}$  with  $u \neq 0$ , i.e.,  $z = a$ , then

$$\sum_{k=0}^n z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Theorem 65].  $\square$

## 9. Generalized Horadam-Leonardo Polynomials: Generating Function

Now, we give generating function of the sequence  $W_{mn+j}$  and some special cases of its.

**9.1. Generating Function of Generalized Horadam-Leonardo Polynomials via Generalized Horadam-Leonardo Polynomials.** Next, we give the ordinary generating function  $\sum_{n=0}^{\infty} W_{mn+j}z^n$  of the sequence  $W_{mn+j}$  (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials).

LEMMA 30. Suppose that  $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, 1\}$ . Suppose that  $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$  is the generating function of the generalized Horadam-Leonardo polynomials  $\{W_{mn+j}\}$ . Then,  $\sum_{n=0}^{\infty} W_{mn+j}z^n$  is given by

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 25 (a))

$$z^2\Theta_4 = z^2((W_0W_{j+2} + (W_1 - (r+1)W_0)W_{j+1} + (W_2 - (r+1)W_1 - (s-r)W_0)W_j)W_{m+2}^2 + ((W_2 - (s-r)W_0)W_{j+2} + ((-s)W_0 + (r+1)(s-r)W_0)W_{j+1} + ((s-r)^2W_0 + ((r+1)(s-r) + (-s))W_1 - (s-r)W_2)W_j)W_{m+1}^2 +$$

$$(-s)(W_1W_{j+2}+(W_2-(r+1)W_1)W_{j+1}+(-s)W_0W_j)W_m^2+(-(W_1+(r+1)W_0)W_{j+2}+((r+1)^2W_0-W_2)W_{j+1}+(-r+1)W_2+(r+1)^2W_1+((r+1)(s-r)-(-s))W_0)W_j)W_{m+1}W_{m+2}+(((r+1)W_1-W_2)W_{j+2}+((r+1)W_2-((s-r)+(r+1)^2)W_1-(-s)W_0)W_{j+1}+(-s)((r+1)W_0-W_1)W_j)W_{m+2}W_m+(((s-r)W_1-(-s)W_0)W_{j+2}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+(r+1)(-s)W_0)W_{j+1}+(-s)(-W_2+(r+1)W_1+2(s-r)W_0)W_j)W_{m+1}W_m),$$

$$z\Theta_5 = z(((W_1^2-W_0W_2)W_{j+2}+((-s)W_0^2-W_1W_2+(r+1)W_0W_2+(s-r)W_0W_1)W_{j+1}+(-2W_2^2-(r+1)^2W_1^2-(-s)(r+1)W_0^2+3(r+1)W_1W_2+2(s-r)W_0W_2+(2(-s)-(-s-r)(r+1))W_0W_1)W_j)W_{m+2}+(((s-r)W_0^2-W_1W_2+(r+1)W_0W_2+(s-r)W_0W_1)W_{j+2}+(W_2^2-((r+1)^2+(s-r))W_0W_2-((r+1)(s-r)+(-s))W_0W_1)W_{j+1}+((r+1)W_2^2-2((-s)+(r+1)(s-r))W_1^2-2(s-r)(-s)W_0^2+(2(s-r)-(r+1)^2)W_1W_2+(2(-s)-(r+1)(s-r))W_0W_2-2(s-r)^2+(r+1)(-s))W_0W_1)W_j)W_{m+1}+((W_2^2-(s-r)W_1^2-(r+1)W_1W_2-(-s)W_0W_1)W_{j+2}+(-(r+1)W_2^2+((r+1)(s-r)+(-s))W_1^2+(r+1)^2W_1W_2-(-s)W_0W_2+(r+1)(-s)W_0W_1)W_{j+1}+(-s)(-(r+1)W_1^2-2(-s)W_0^2+2W_1W_2-(r+1)W_0W_2-2(s-r)W_0W_1)W_j)W_m-(r+1)(-s)W_0^2W_{j+1}W_{m+1}),$$

$$\Theta_6 = (W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3-2(r+1)W_1W_2^2-(s-r)W_0W_2^2+((r+1)^2-(s-r))W_1^2W_2+((s-r)^2+(r+1)(-s))W_0W_1^2+2(s-r)(-s)W_0^2W_1+(r+1)(-s)W_0^2W_2+((r+1)(s-r)-3(-s))W_0W_1W_2)W_j,$$

and

$$z^3\Gamma_1 = z^3(-(-s)^m(W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3+((r+1)^2-(s-r))W_1^2W_2-2(r+1)W_1W_2^2-(s-r)W_0W_2^2+(r+1)(-s)W_0^2W_2+((s-r)^2+(r+1)(-s))W_0W_1^2+2(s-r)(-s)W_0^2W_1+((r+1)(s-r)-3(-s))W_0W_1W_2)),$$

$$z^2\Gamma_2 = z^2((3W_2-2(r+1)W_1-(s-r)W_0)W_{m+2}^2+(((r+1)^2-(s-r))W_2+(3(r+1)(s-r)+3(-s))W_1+((s-r)^2+(r+1)(-s))W_0)W_{m+1}^2+(-s)((r+1)W_2+2(s-r)W_1+3(-s)W_0)W_m^2+(-4(r+1)W_2+2((r+1)^2-(s-r))W_1+((r+1)(s-r)-3(-s))W_0)W_{m+2}W_{m+1}+(-2(s-r)W_2+((r+1)(s-r)-3(-s))W_1+2(r+1)(-s)W_0)W_{m+2}W_m+(((r+1)(s-r)-3(-s))W_2+2((s-r)^2+(r+1)(-s))W_1+4(s-r)(-s)W_0)W_{m+1}W_m),$$

$$z\Gamma_3 = z((-3W_2^2+((s-r)-(r+1)^2)W_1^2-(-s)(r+1)W_0^2+4(r+1)W_1W_2+2(s-r)W_0W_2+(3(-s)-(s-r)(r+1))W_0W_1)W_{m+2}+(2(r+1)W_2^2-(3(r+1)(s-r)+3(-s))W_1^2-2(s-r)(-s)W_0^2+(2(s-r)-2(r+1)^2)W_1W_2+(3(-s)-(r+1)(s-r))W_0W_2-2((s-r)^2+(r+1)(-s))W_0W_1)W_{m+1}+(((s-r)W_2^2-((s-r)^2+(r+1)(-s))W_1^2-3(-s)^2W_0^2+(3(-s)-(r+1)(s-r))W_1W_2-2(r+1)(-s)W_0W_2-4(s-r)(-s)W_0W_1)W_m),$$

$$\Gamma_4 = W_2^3+((-s)+(r+1)(s-r))W_1^3+(-s)^2W_0^3-2(r+1)W_1W_2^2+((r+1)^2-(s-r))W_1^2W_2-(s-r)W_0W_2^2+((s-r)^2+(r+1)(-s))W_0W_1^2+(r+1)(-s)W_0^2W_2+2(s-r)(-s)W_0^2W_1+((r+1)(s-r)-3(-s))W_0W_1W_2.$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Lemma 66].  $\square$

Now, we consider special cases of the last Lemma.

**COROLLARY 31.** *The ordinary generating functions of the sequences  $W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1}$  are given as follows:*

(a): ( $m = 1, j = 0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, 1\}$ ).

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^2(W_2 - (r+1)W_1 - (s-r)W_0) + z(W_1 - (r+1)W_0) + W_0}{z^3 s + z^2(-1)(s-r) + z(-1)(r+1) + 1}.$$

(b): ( $m = 2, j = 0, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, 1\}$ ).

$$\sum_{n=0}^{\infty} W_{2n} z^n = \frac{z^2(-(s-r)W_2 - r(r-s+1)W_1 + (r^2 - rs + s^2 + s)W_0) + z(W_2 - (r^2 + 2s + 1)W_0) + W_0}{z^3(-s^2) + z^2(r^2 + s^2 + 2s) + z(-1)(r^2 + 2s + 1) + 1}.$$

(c): ( $m = 2, j = 1, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, 1\}$ ).

$$\sum_{n=0}^{\infty} W_{2n+1} z^n = \frac{z^2(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + z((r+1)W_2 - (r^2 + r + s + 1)W_1 + (-s)W_0) + W_1}{z^3(-s^2) + z^2(r^2 + s^2 + 2s) + z(-1)(r^2 + 2s + 1) + 1}.$$

(d): ( $m = -1, j = 0, |z| < \min\{|\alpha|, |\beta|, 1\}$ ).

$$\sum_{n=0}^{\infty} W_{-n} z^n = \frac{z^2 W_1 + z(W_2 - (r+1)W_1) + (-s)W_0}{z^3(-1) + z^2(r+1) + z(s-r) + (-s)}.$$

(e): ( $m = -2, j = 0, |z| < \min\{|\alpha|^2, |\beta|^2, 1\}$ ).

$$\sum_{n=0}^{\infty} W_{-2n} z^n = \frac{z^2 W_2 + z(-(s-r)W_2 - r(r-s+1)W_1 + (r+1)(-s)W_0) + s^2 W_0}{z^3(-1) + z^2(r^2 + 2s + 1) + z(-1)(r^2 + s^2 + 2s) + s^2}.$$

(f): ( $m = -2, j = 1, |z| < \min\{|\alpha|^2, |\beta|^2, 1\}$ ).

$$\sum_{n=0}^{\infty} W_{-2n+1} z^n = \frac{z^2((r+1)W_2 + (s-r)W_1 - sW_0) + z(-sW_2 - (r^2 + s^2 + s - rs)W_1 - s(r-s)W_0) + s^2 W_1}{z^3(-1) + z^2(r^2 + 2s + 1) + z(-1)(r^2 + s^2 + 2s) + s^2}.$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Corollary 67].  $\square$

As particular examples (generating functions of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials), from the last Lemma, we obtain the following results

COROLLARY 32. *Suppose that  $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, 1\}$ . One give the generating functions of  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials, respectively, as follows:*

(a):

$$\sum_{n=0}^{\infty} G_{mn+j} z^n = \frac{z^2 \Theta_4 + z \Theta_5 + \Theta_6}{z^3 \Gamma_1 + z^2 \Gamma_2 + z \Gamma_3 + \Gamma_4}$$

where (as Theorem 25 (a))

$$z^2 \Theta_4 = z^2(G_{m+2}^2 G_{j+1} + (r+1)G_{m+1}^2 G_{j+2} + (-s)G_{m+1}^2 G_j + (-s)G_m^2 G_{j+2} - G_{m+1} G_{m+2} G_{j+2} + (s-r)G_m G_{m+1} G_{j+2} - (r+1)G_{m+1} G_{m+2} G_{j+1} - (s-r)G_m G_{m+2} G_{j+1} - (-s)G_m G_{m+1} G_{j+1} - (-s)G_m G_{m+2} G_j),$$

$$z \Theta_5 = z(G_{m+2} G_{j+2} - (r+1)G_{m+1} G_{j+2} - (r+1)G_{m+2} G_{j+1} - (s-r)G_m G_{j+2} + (r+1)^2 G_{m+1} G_{j+1} - 2(-s)G_j G_{m+1} + ((-s) + (r+1)(s-r))G_m G_{j+1} + (r+1)(-s)G_m G_j),$$

$$\Theta_6 = (-s)G_j,$$

and

$$z^3 \Gamma_1 = z^3(-(-s)^{m+1}),$$

$$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + (-s)((r+1)^2 + 2(s-r))G_m^2 - 2((r+1)^2 + (s-r))G_{m+2}G_{m+1} - (3(-s) + (r+1)(s-r))G_{m+2}G_m + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_{m+1}G_m),$$

$$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} - ((s-r)^2 - 2(r+1)(-s))G_m),$$

$$\Gamma_4 = (-s).$$

(b):

$$\sum_{n=0}^{\infty} H_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as Theorem 25 (a))

$$z^2\Theta_4 = z^2((3H_{j+2} - 2(r+1)H_{j+1} - (s-r)H_j)H_{m+2}^2 + (((r+1)^2 - (s-r))H_{j+2} + 3((r+1)(s-r) + (-s))H_{j+1} + ((s-r)^2 + (r+1)(-s))H_j)H_{m+1}^2 + (-s)((r+1)H_{j+2} + 2(s-r)H_{j+1} + 3(-s)H_j)H_m^2 + (-4(r+1)H_{j+2} + 2((r+1)^2 - (s-r))H_{j+1} + ((r+1)(s-r) - 3(-s))H_j)H_{m+2}H_{m+1} + (-2(s-r)H_{j+2} + ((r+1)(s-r) - 3(-s))H_{j+1} + 2(r+1)(-s)H_j)H_{m+2}H_m + (((r+1)(s-r) - 3(-s))H_{j+2} + 2((s-r)^2 + (r+1)(-s))H_{j+1} + 4(s-r)(-s)H_j)H_{m+1}H_m),$$

$$z\Theta_5 = z((-2((r+1)^2 + 3(s-r))H_{j+2} + (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{j+1} + ((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))H_j)H_{m+2} + ((2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{j+2} - (2(r+1)^4 + 8(r+1)^2(s-r) + 3(r+1)(-s) + 2(s-r)^2)H_{j+1} - ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))H_j)H_{m+1} + (((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)H_{j+2} - ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))H_{j+1} - 2(-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))H_j)H_m - 9(r+1)(-s)H_{j+1}H_{m+1}),$$

$$\Theta_6 = (4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s))H_j,$$

and

$$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2)),$$

$$z^2\Gamma_2 = z^2(((r+1)^2 + 3(s-r))H_{m+2}^2 + ((r+1)^4 + 4(r+1)^2(s-r) + (s-r)^2 + 6(r+1)(-s))H_{m+1}^2 + (-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))H_m^2 - (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{m+2}H_{m+1} - ((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)H_{m+2}H_m + ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))H_{m+1}H_m),$$

$$z\Gamma_3 = z(-4(r+1)^3(-s) + (r+1)^2(s-r)^2 + 4(s-r)^3 - 27(-s)^2 - 18(r+1)(s-r)(-s))H_m,$$

$$\Gamma_4 = 4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s).$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Corollary 68.].  $\square$

Some special cases of the last two Corollaries can be given as follows.

**COROLLARY 33.** *One gives the ordinary generating functions of the sequences  $G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1}$  and  $H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1}$  as follows:*

(a): ( $m = 1, j = 0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, 1\}$ ).

$$\begin{aligned}\sum_{n=0}^{\infty} G_n z^n &= \frac{z}{1 - (r+1)z - (s-r)z^2 + sz^3}, \\ \sum_{n=0}^{\infty} H_n z^n &= \frac{3 - 2(r+1)z - (s-r)z^2}{1 - (r+1)z - (s-r)z^2 + sz^3}.\end{aligned}$$

(b): ( $m = 2, j = 0, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, 1\}$ ).

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n} z^n &= \frac{(r+1)z + (-s)z^2}{1 - (r^2 + 2s + 1)z + (r^2 + s^2 + 2s)z^2 - s^2 z^3}, \\ \sum_{n=0}^{\infty} H_{2n} z^n &= \frac{3 - 2(r^2 + 2s + 1)z + (r^2 + s^2 + 2s)z^2}{1 - (r^2 + 2s + 1)z + (r^2 + s^2 + 2s)z^2 - s^2 z^3}.\end{aligned}$$

(c): ( $m = 2, j = 1, |z| < \min\{|\alpha|^{-2}, |\beta|^{-2}, 1\}$ ).

$$\begin{aligned}\sum_{n=0}^{\infty} G_{2n+1} z^n &= \frac{1 - (s-r)z}{1 - (r^2 + 2s + 1)z + (r^2 + s^2 + 2s)z^2 - s^2 z^3}, \\ \sum_{n=0}^{\infty} H_{2n+1} z^n &= \frac{(r+1) + (rs - 2s - r - r^2)z - s(r-s)z^2}{1 - (r^2 + 2s + 1)z + (r^2 + s^2 + 2s)z^2 - s^2 z^3}.\end{aligned}$$

(d): ( $m = -1, j = 0, |z| < \min\{|\alpha|, |\beta|, 1\}$ ).

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-n} z^n &= \frac{z^2}{(-s) + (s-r)z + (r+1)z^2 - z^3}, \\ \sum_{n=0}^{\infty} H_{-n} z^n &= \frac{3(-s) + 2(s-r)z + (r+1)z^2}{(-s) + (s-r)z + (r+1)z^2 - z^3}.\end{aligned}$$

(e): ( $m = -2, j = 0, |z| < \min\{|\alpha|^2, |\beta|^2, 1\}$ ).

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n} z^n &= \frac{(-s)z + (r+1)z^2}{s^2 - (r^2 + s^2 + 2s)z + (r^2 + 2s + 1)z^2 - z^3}, \\ \sum_{n=0}^{\infty} H_{-2n} z^n &= \frac{3s^2 - 2(r^2 + s^2 + 2s)z + (r^2 + 2s + 1)z^2}{s^2 - (r^2 + s^2 + 2s)z + (r^2 + 2s + 1)z^2 - z^3}.\end{aligned}$$

(f): ( $m = -2, j = 1, |z| < \min\{|\alpha|^2, |\beta|^2, |\gamma|^2\}$ ).

$$\begin{aligned}\sum_{n=0}^{\infty} G_{-2n+1} z^n &= \frac{s^2 - (r^2 + s^2 + 2s)z + (r^2 + r + s + 1)z^2}{s^2 - (r^2 + s^2 + 2s)z + (r^2 + 2s + 1)z^2 - z^3}, \\ \sum_{n=0}^{\infty} H_{-2n+1} z^n &= \frac{(r+1)(-s)^2 - (r^3 + r^2 + 3rs + rs^2 + 2s)z + (r^3 + 3rs + 1)z^2}{s^2 - (r^2 + s^2 + 2s)z + (r^2 + 2s + 1)z^2 - z^3}.\end{aligned}$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Corollary 69].  $\square$

**9.2. Generating Function of Generalized Horadam-Leonardo Polynomials(via Generalized Horadam-Leonardo Polynomials and  $(r, s)$ -Horadam-Leonardo Polynomials).** (In terms of elements of the sequence of generalized Horadam-Leonardo polynomials and  $(r, s)$ -Horadam-Leonardo polynomials) The ordinary generating function  $\sum_{n=0}^{\infty} W_{mn+j}z^n$  of the sequence  $W_{mn+j}$  can be given as follows.

LEMMA 34. *Suppose that  $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$ . Assume that  $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$  is the generating function of the generalized Horadam-Leonardo polynomials  $\{W_{mn+j}\}$ . Then,  $\sum_{n=0}^{\infty} W_{mn+j}z^n$  is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 27 (a))

$$z^2\Theta_4 = z^2(G_{m+2}^2W_{j+1} + ((r+1)W_{j+2} + (-s)W_j)G_{m+1}^2 + (-s)G_m^2W_{j+2} - ((r+1)W_{j+1} + W_{j+2})G_{m+1}G_{m+2} - ((s-r)W_{j+1} + (-s)W_j)G_mG_{m+2} + ((s-r)W_{j+2} - (-s)W_{j+1})G_mG_{m+1}),$$

$$z\Theta_5 = z((W_{j+2} - (r+1)W_{j+1})G_{m+2} + (-(r+1)W_{j+2} + (r+1)^2W_{j+1} - 2(-s)W_j)G_{m+1} + (-(s-r)W_{j+2} + ((-s)W_{j+1} + (r+1)(s-r)W_{j+1}) + (r+1)(-s)W_j)G_m),$$

$$\Theta_6 = (-s)W_j,$$

and

$$z^3\Gamma_1 = z^3(-(-s)^{m+1}),$$

$$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + ((r+1)^2(-s) + 2(s-r)(-s))G_m^2 - 2((r+1)^2 + (s-r))G_{m+1}G_{m+2} - ((r+1)(s-r) + 3(-s))G_mG_{m+2} + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_mG_{m+1}),$$

$$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} + (2(r+1)(-s) - (s-r)^2)G_m),$$

$$\Gamma_4 = (-s).$$

Proof. Replace  $r, s$  and  $t$  with  $r+1, s-r, -s$ , respectively, in [51, Lemma 70].  $\square$

As particular example (generating functions of  $(r, s)$ -Horadam-Leonardo-Lucas polynomials), Lemma 34 gives the following result

COROLLARY 35. *Suppose that  $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$ . one can gives the generating function of  $(r, s)$ -Horadam-Leonardo-Lucas polynomials as follows:*

$$\sum_{n=0}^{\infty} H_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 27 (a))

$$z^2\Theta_4 = z^2(G_{m+2}^2H_{j+1} + ((r+1)H_{j+2} + (-s)H_j)G_{m+1}^2 + (-s)G_m^2H_{j+2} - ((r+1)H_{j+1} + H_{j+2})G_{m+1}G_{m+2} - ((s-r)H_{j+1} + (-s)H_j)G_mG_{m+2} + ((s-r)H_{j+2} - (-s)H_{j+1})G_mG_{m+1}),$$

$$z\Theta_5 = z((H_{j+2} - (r+1)H_{j+1})G_{m+2} + (-(r+1)H_{j+2} + (r+1)^2H_{j+1} - 2(-s)H_j)G_{m+1} + (-(s-r)H_{j+2} + ((-s)H_{j+1} + (r+1)(s-r)H_{j+1}) + (r+1)(-s)H_j)G_m),$$

$$\Theta_6 = (-s)H_j,$$

and

$$\begin{aligned} z^3\Gamma_1 &= z^3(-(-s)^{m+1}), \\ z^2\Gamma_2 &= z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + ((r+1)^2(-s) + 2(s-r)(-s))G_m^2 - 2((r+1)^2 + (s-r))G_{m+1}G_{m+2} - ((r+1)(s-r) + 3(-s))G_mG_{m+2} + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_mG_{m+1}), \\ z\Gamma_3 &= z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} + (2(r+1)(-s) - (s-r)^2)G_m), \\ \Gamma_4 &= (-s). \end{aligned}$$

Proof. Replace  $r$ ,  $s$  and  $t$  with  $r+1$ ,  $s-r$ ,  $-s$ , respectively, in [51, Corollary 71].  $\square$

**9.3. Generating Function of Generalized Horadam-Leonardo Polynomials(via Generalized Horadam-Leonardo Polynomials and  $(r, s)$ -Horadam-Leonardo-Lucas Polynomials).** (In terms of elements of the sequence of generalized Horadam-Leonardo polynomials and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials) The ordinary generating function  $\sum_{n=0}^{\infty} W_{mn+j}z^n$  of the sequence  $W_{mn+j}$  can be given as follows.

LEMMA 36. Let  $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$ . Assume that  $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$  is the generating function of the generalized Horadam-Leonardo polynomials  $\{W_{mn+j}\}$ . Then,  $\sum_{n=0}^{\infty} W_{mn+j}z^n$  is given by

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 29 (a))

$$\begin{aligned} z^2\Theta_4 &= z^2((3W_{j+2} - 2(r+1)W_{j+1} - (s-r)W_j)H_{m+2}^2 + (((r+1)^2 - (s-r))W_{j+2} + 3((r+1)(s-r) + (-s))W_{j+1} + ((s-r)^2 + (r+1)(-s))W_j)H_{m+1}^2 + (-s)((r+1)W_{j+2} + 2(s-r)W_{j+1} + 3(-s)W_j)H_m^2 + (-4(r+1)W_{j+2} + 2((r+1)^2 - (s-r))W_{j+1} + ((r+1)(s-r) - 3(-s))W_j)H_{m+2}H_{m+1} + (-2(s-r)W_{j+2} + ((r+1)(s-r) - 3(-s))W_{j+1} + 2(r+1)(-s)W_j)H_{m+2}H_m + (((r+1)(s-r) - 3(-s))W_{j+2} + 2((s-r)^2 + (r+1)(-s))W_{j+1} + 4(s-r)(-s)W_j)H_{m+1}H_m), \end{aligned}$$

$$\begin{aligned} z\Theta_5 &= z((-2((r+1)^2 + 3(s-r))W_{j+2} + (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))W_{j+1} + ((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)W_j)H_{m+2} + ((2(r+1)^3 + 7(r+1)(s-r) + 9(-s))W_{j+2} - 2((r+1)^4 + 4(r+1)^2(s-r) + 6(r+1)(-s) + (s-r)^2)W_{j+1} - ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))W_j)H_{m+1} + (((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))W_{j+2} - (4(r+1)(s-r)^2 + (r+1)^3(s-r) - (r+1)^2(-s) + 6(s-r)(-s))W_{j+1} - 2(-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))W_j)H_m), \end{aligned}$$

$$\Theta_6 = (4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s))W_j,$$

and

$$\begin{aligned} z^3\Gamma_1 &= z^3(-(-s)^m(4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2)), \\ z^2\Gamma_2 &= z^2(((r+1)^2 + 3(s-r))H_{m+2}^2 + ((r+1)^4 + 4(r+1)^2(s-r) + (s-r)^2 + 6(r+1)(-s))H_{m+1}^2 + (-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))H_m^2 - (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{m+2}H_{m+1} - ((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))H_{m+2}H_m + (r+1)(s-r)((r+1)^2 + 4(s-r))H_{m+1}H_m - (-s)((r+1)^2 - 6(s-r))H_{m+1}H_m), \\ z\Gamma_3 &= z(-4(r+1)^3(-s) + (r+1)^2(s-r)^2 + 4(s-r)^3 - 27(-s)^2 - 18(r+1)(s-r)(-s))H_m, \\ \Gamma_4 &= 4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2. \end{aligned}$$

Proof. Replace  $r, s$  and  $t$  with  $r + 1, s - r, -s$ , respectively, in [51, Lemma 72.].  $\square$

As particular example (generating function of  $(r, s)$ -Horadam-Leonardo polynomials), from Lemma 36, one gets the following result.

**COROLLARY 37.** *Suppose that  $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$ . One gives generating function of  $(r, s)$ -Horadam-Leonardo polynomials as follows:*

$$\sum_{n=0}^{\infty} G_{mn+j} z^n = \frac{z^2 \Theta_4 + z \Theta_5 + \Theta_6}{z^3 \Gamma_1 + z^2 \Gamma_2 + z \Gamma_3 + \Gamma_4}$$

where (as in Theorem 29 (a))

$$\begin{aligned} z^2 \Theta_4 = & z^2((3G_{j+2} - 2(r+1)G_{j+1} - (s-r)G_j)H_{m+2}^2 + (((r+1)^2 - (s-r))G_{j+2} + 3((r+1)(s-r) + (-s))G_{j+1} + \\ & ((s-r)^2 + (r+1)(-s))G_j)H_{m+1}^2 + (-s)((r+1)G_{j+2} + 2(s-r)G_{j+1} + 3(-s)G_j)H_m^2 + (-4(r+1)G_{j+2} + 2((r+1)^2 - (s-r))G_{j+1} + \\ & ((r+1)(s-r) - 3(-s))G_j)H_{m+2}H_{m+1} + (-2(s-r)G_{j+2} + ((r+1)(s-r) - 3(-s))G_{j+1} + 2(r+1)(-s)G_j)H_{m+2}H_m + \\ & (((r+1)(s-r) - 3(-s))G_{j+2} + 2((s-r)^2 + (r+1)(-s))G_{j+1} + 4(s-r)(-s)G_j)H_{m+1}H_m), \end{aligned}$$

$$\begin{aligned} z \Theta_5 = & z((-2((r+1)^2 + 3(s-r))G_{j+2} + (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))G_{j+1} + ((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)G_j)H_{m+2} + \\ & ((2(r+1)^3 + 7(r+1)(s-r) + 9(-s))G_{j+2} - 2((r+1)^4 + 4(r+1)^2(s-r) + 6(r+1)(-s) + (s-r)^2)G_{j+1} - \\ & ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))G_j)H_{m+1} + \\ & (((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))G_{j+2} - (4(r+1)(s-r)^2 + (r+1)^3(s-r) - (r+1)^2(-s) + 6(s-r)(-s))G_{j+1} - \\ & 2(-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))G_j)H_m), \end{aligned}$$

$$\Theta_6 = (4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s))G_j,$$

and

$$z^3 \Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2)),$$

$$\begin{aligned} z^2 \Gamma_2 = & z^2(((r+1)^2 + 3(s-r))H_{m+2}^2 + ((r+1)^4 + 4(r+1)^2(s-r) + (s-r)^2 + 6(r+1)(-s))H_{m+1}^2 + \\ & (-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))H_m^2 - (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{m+2}H_{m+1} - \\ & ((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))H_{m+2}H_m + (r+1)(s-r)((r+1)^2 + 4(s-r))H_{m+1}H_m - \\ & (-s)((r+1)^2 - 6(s-r))H_{m+1}H_m), \end{aligned}$$

$$z \Gamma_3 = z(-4(r+1)^3(-s) + (r+1)^2(s-r)^2 + 4(s-r)^3 - 27(-s)^2 - 18(r+1)(s-r)(-s))H_m,$$

$$\Gamma_4 = 4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2.$$

Proof. Replace  $r, s$  and  $t$  with  $r + 1, s - r, -s$ , respectively, in [51, Corollary 73.].  $\square$

## 10. Conclusions

The Fibonacci and Lucas sequences are sources of many nice and interesting identities. For the applications of these second order sequences in science and nature, one refer the citations in [24,25,26].

In this study, we define and investigate a linear third order polynomial (and two special cases of its). We called them the generalized Horadam-Leonardo polynomials (and  $(r, s)$ -Horadam-Leonardo and  $(r, s)$ -Horadam-Leonardo-Lucas polynomials). Binet's formulas, generating functions, Simson formulas, and the

summation formulas for these polynomial sequences are presented. Then, some identities and matrices related to these polynomials are given.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, some applications of second order sequences are given. For the applications of

- Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [2].
- Pell Numbers to the solutions of three-dimensional difference equation systems, see [5].
- Jacobsthal numbers to special matrices, see [68].
- generalized  $k$ -order Fibonacci numbers to hybrid quaternions, see [19].
- Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [64].
- generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [65].
- generalized Fibonacci numbers to binomial sums, see [63].
- generalized Jacobsthal numbers to hyperbolic numbers, see [35].
- generalized Fibonacci numbers to dual hyperbolic numbers, see [36].
- Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [11].
- generalized Fibonacci Matrices to Cryptography, see [31].
- higher order Jacobsthal numbers to quaternions, see [29].
- Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [17].
- Fibonacci numbers to lacunary statistical convergence, see [4].
- Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [21].
- Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [22].
- $k$ -Fibonacci and  $k$ -Lucas numbers to spinors, see [27].
- dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [61].
- Hyperbolic Fibonacci numbers to Quaternions, see [9].

Now, some applications of third order sequences are given. For the applications of

- third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [7] and [6], respectively.
- Tribonacci numbers to special matrices, see [66].
- Padovan numbers and Tribonacci numbers to coding theory, see [33] and [3], respectively.
- Pell-Padovan numbers to groups, see [10].
- adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [18].
- Gaussian Tribonacci numbers to various graphs, see [60].
- third-order Jacobsthal numbers to hyperbolic numbers, see [12].

- Narayan numbers to finite groups see [23].
- generalized third-order Jacobsthal sequence to binomial transform, see [37].
- generalized Generalized Padovan numbers to Binomial Transform, see [38].
- generalized Tribonacci numbers to Gaussian numbers, see [39].
- generalized Tribonacci numbers to Sedenions, see [40].
- Tribonacci and Tribonacci-Lucas numbers to matrices, see [41].
- generalized Tribonacci numbers to circulant matrix, see [42].
- Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [62].
- hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [13].

Next, list of some applications of fourth order sequences are given. For the applications of

- Tetranacci and Tetranacci-Lucas numbers to quaternions, see [43].
- generalized Tetranacci numbers to Gaussian numbers, see [44].
- Tetranacci and Tetranacci-Lucas numbers to matrices, see [45].
- generalized Tetranacci numbers to binomial transform, see [46].

Now, some applications of fifth order sequences are given. For the applications of

- Pentanacci numbers to matrices, see [34].
- generalized Pentanacci numbers to quaternions, see [48].
- generalized Pentanacci numbers to binomial transform, see [47].

Next, some applications of second order sequences of polynomials are given. For the applications of

- generalized Fibonacci Polynomials to the summation formulas, see [49].
- generalized Fibonacci Polynomials, see [50].

Now, some applications of third order sequences of polynomials are given. For the applications of

- generalized Tribonacci Polynomials, see [51].

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