Original Research Article

Generalized Horadam-Leonardo Numbers and Polynomials

Abstract. In this paper, we introduce and investigate some linear third order polynomials called the generalized Horadam-Leonardo polynomials (and its two special cases, namely), (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these polynomial sequences. Moreover, we give some identities and matrices related to these polynomials.

2020 Mathematics Subject Classification. 11B37, 11B39, 11B83.

Keywords. Horadam polynomials, Fibonacci polynomials, Tribonacci polynomials, Horadam-Leonardo polynomials, third order recurrence relations.

1. Introduction: Generalized Fibonacci (Horadam) Polynomials

Before defining and investigating the generalized Horadam-Leonardo polynomials we recall the definition and some properties of Horadam polynomials and its two special cases. The generalized Fibonacci polynomials (or Horadam polynomials or x-Horadam numbers or generalized $(r(x), s(x))$-polynomials or $(r(x), s(x))$ Horadam polynomials or 2-step Fibonacci polynomials)

$$\{V_n(V_0(x), V_1(x); r(x), s(x))\}_{n\geq 0}$$

(or $\{V_n(x)\}_{n\geq 0}$ or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n(x) = r(x)V_{n-1}(x) + s(x)V_{n-2}(x), \qquad V_0(x) = a(x), V_1(x) = b(x), \quad n \geq 2 \tag{1.1}$$

where $V_0(x), V_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $r(x)$ and $s(x)$ are polynomials with real coefficients with $r(x) \neq 0$, $s(x) \neq 0$.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n}(x) = -\frac{r(x)}{s(x)}V_{-(n-1)}(x) + \frac{1}{s(x)}V_{-(n-2)}(x)$$

for $n = 1, 2, 3, ...$ when $s(x) \neq 0$. Therefore, recurrence (1.1) holds for all integers n. Note that $V_{-n}(x)$ need not to be a polynomial in the ordinary sense.

For some references on special cases of second-order linear recurrence sequences of polynomials and numbers, see for instance [4,5,6,14,15,24,25] for papers and [1,3,7,8,9,13,23] for books.

Binet's formula of generalized Fibonacci (Horadam) polynomials can be calculated using its characteristic equation (the quadratic equation, polynomial) which is given as

$$y^2 - r(x)y - s(x) = 0. \tag{1.2}$$

The roots of characteristic equation are

$$\alpha(x) := \alpha = \frac{r(x) + \sqrt{r^2(x) + 4s(x)}}{2}, \quad \beta(x) := \beta = \frac{r(x) - \sqrt{r^2(x) + 4s(x)}}{2}, \tag{1.3}$$

and the followings hold

$$\begin{aligned}
\alpha + \beta &= r(x), \\
\alpha\beta &= -s(x), \\
(\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta = r^2(x) + 4s(x), \\
r(x) - 2\alpha &= -(\alpha - \beta), \\
r(x) - 2\beta &= \alpha - \beta, \\
(r(x) - 2\alpha)(r(x) - 2\beta) &= -(\alpha - \beta)^2 = -(r^2(x) + 4s(x)).
\end{aligned}$$

If the roots α and β of characteristic equation (1.2) are distinct, i.e., $\alpha \neq \beta$ then $r^2(x) + 4s(x) \neq 0$ and if the roots α and β of characteristic equation (1.2) are equal, i.e., $\alpha = \beta$ then (1.2) can be written as

$$y^2 - r(x)y - s(x) = (y - \alpha)^2 = y^2 - 2\alpha y + \alpha^2 = 0$$

and, in this case,

$$\alpha = \frac{r(x)}{2}, \ r(x) = 2\alpha, \ s(x) = -\alpha^2 = -\frac{r^2(x)}{4}, \ r^2(x) + 4s(x) = 0.$$

Now, we define two special cases of the polynomials $V_n(x)$. $(r(x), s(x))$-Fibonacci polynomials $\{M_n(0, 1; r(x), s(x))\}_{n\geq 0}$ (or shortly $M_n(x)$) and $(r(x), s(x))$-Lucas polynomials $\{N_n(2, r(x); r(x), s(x))\}_{n\geq 0}$ (or shortly $N_n(x)$) are defined, respectively, by the second-order recurrence relations

$$M_{n+2}(x) = r(x)M_{n+1} + s(x)M_n(x), \qquad M_0(x) = 0, M_1(x) = 1, \tag{1.4}$$

$$N_{n+2}(x) = r(x)N_{n+1} + s(x)N_n(x), \qquad N_0(x) = 2, N_1(x) = r(x). \tag{1.5}$$

The (sequences of polinomials) $\{M_n(x)\}_{n\geq 0}$ and $\{N_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} M_{-n}(x) &= -\frac{r(x)}{s(x)}M_{-(n-1)}(x) + \frac{1}{s(x)}M_{-(n-2)}(x), \\ N_{-n}(x) &= -\frac{r(x)}{s(x)}N_{-(n-1)}(x) + \frac{1}{s(x)}N_{-(n-2)}(x), \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.4) and (1.5) hold for all integers n.

NOTE: For the sake of simplicity throughout the rest of the paper we use

$$V_n, r, s, V_0, V_1, \alpha, \beta, M_n, N_n, M_0, M_1, N_0, N_1$$

instead of

$$V_n(x), r(x), s(x), V_0(x), V_1(x), \alpha(x), \beta(x), M_n(x), N_n(x), M_0(x), M_1(x), N_0(x), N_1(x),$$

respectively. For example, we write

$$V_n = rV_{n-1} + sV_{n-2}, \qquad V_0 = a, V_1 = b, \quad n \geq 2$$

for the equation (1.1).

Using the roots α, β and recurrence relation (1.1), Binet's formula of V_n can be given as follows:

Theorem 1.

(a): *(Distinct Roots Case: $\alpha \neq \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$V_n = \frac{r_1\alpha^n}{\alpha-\beta} + \frac{r_2\beta^n}{\beta-\alpha} = \frac{r_1\alpha^n - r_2\beta^n}{\alpha-\beta} \tag{1.6}$$

where

$$r_1 = V_1 - \beta V_0, \;\; r_2 = V_1 - \alpha V_0.$$

(b): *(Single Root Case: $\alpha = \beta$) Binet's formula of generalized Fibonacci (Horadam) polynomials is*

$$V_n = (D_1 + D_2 n)\alpha^n \tag{1.7}$$

where

$$\begin{aligned} D_1 &= V_0, \\ D_2 &= \frac{1}{\alpha}\left(V_1 - \alpha V_0\right). \end{aligned}$$

Note that Binet's formulas of M_n and N_n can be given, respectively, as follows:

$$M_n = \begin{cases} \dfrac{\alpha^n - \beta^n}{\alpha - \beta} & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ n\alpha^{n-1} & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases},$$

$$N_n = \begin{cases} \alpha^n + \beta^n & , \text{ if } \alpha \neq \beta \text{ (Distinct Roots Case)} \\ 2\alpha^n & , \text{ if } \alpha = \beta \text{ (Single Root Case)} \end{cases}.$$

Now, we define two sequences related to (r,s)-Fibonacci polynomials and (r,s)-Fibonacci-Lucas polynomials. For r,s satisfying Eq. (1.4) and (1.5), (r,s)-Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials are defined as

$$G_n(x) = rG_{n-1}(x) + sG_{n-2}(x) + 1 \quad \text{with} \quad G_0(x) = 0, G_1(x) = 1, \; n \geq 2, \tag{1.8}$$

and

$$H_n(x) = rH_{n-1}(x) + sH_{n-2}(x) + (1-s-r) \quad \text{with } H_0(x) = 3, H_1(x) = r+1, \; n \geq 2, \tag{1.9}$$

respectively.

Note that $G_2(x) = r+1$ and $H_2(x) = r^2+2s+1$. The first few values of Horadam-Leonardo polynomials and Horadam-Leonardo-Lucas polynomials are

$$0, 1, r+1, r^2+r+s+1, r^3+r^2+r+2rs+s+1, ...$$

and

$$3, r+1, r^2+2s+1, r^3+3sr+1, r^4+4r^2s+2s^2+1, ...$$

respectively. Note also that from the equations (1.8) and (1.9), we get

$$\begin{aligned} sG_{n-3}(x) &= G_{n-1}(x) - rG_{n-2}(x) - 1, \\ sH_{n-3}(x) &= H_{n-1}(x) - rH_{n-2}(x) - (1-s-r), \end{aligned}$$

and so the sequences $\{G_n(x)\}$ and $\{H_n(x)\}$ satisfy the following third order linear recurrences:

$$G_n(x) = (r+1)G_{n-1}(x) + (s-r)G_{n-2}(x) - sG_{n-3}(x), \tag{1.10}$$

$$H_n(x) = (r+1)H_{n-1}(x) + (s-r)H_{n-2}(x) - sH_{n-3}(x). \tag{1.11}$$

Remark 2. *Note that if $1-s-r=0$, i.e., $s=1-r$, $r=1-s$, then we see from (1.9) and (1.11) that the sequence $\{H_n(x)\}$ both have second order and third order linear relations. In this case, we get*

$$H_2(x) = r^2+2s+1 = r^2-2r+3 = s^2+2.$$

In this paper, we also consider and investigate the case $1-s-r=0$ by considering it in the general case, i.e. in (1.9) and (1.11) that is (2.3).

Note that if we define a sequence of polynomials as

$$Y_n(x) = rY_{n-1}(x) + sY_{n-2}(x) + c(x), \qquad \text{with} \quad Y_0(x) = d_1(x), Y_1(x) = d_2(x), \quad n \geq 2$$

where r,s satisfying Eq. (1.1) and $Y_0(x), Y_1(x)$ are arbitrary complex (or real) polynomials with real coefficients and $c(x)$ is a polynomial with real coefficients, then since

$$sY_{n-3}(x) = Y_{n-1}(x) - rY_{n-2}(x) - c(x),$$

we get

$$Y_n(x) = (r+1)Y_{n-1}(x) + (s-r)Y_{n-2}(x) - sY_{n-3}(x).$$

2. Generalized Horadam-Leonardo Polynomials

In this section, for r, s satisfying Eq. (1.1), we define and investigate a new sequence and its two special cases, namely the generalized Horadam-Leonardo, (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials.

For r, s satisfying Eq. (1.1), generalized Horadam-Leonardo polynomials $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2; r+1, s-r, -s)\}_{n\geq 0}$ (or shortly $\{W_n(x)\}_{n\geq 0}$) is defined by the third-order recurrence relation

$$W_n(x) = (r+1)W_{n-1}(x) + (s-r)W_{n-2}(x) - sW_{n-3}(x) \tag{2.1}$$

with the initial values $W_0(x) = c_0(x), W_1(x) = c_1(x), W_2(x) = c_2(x)$ not all being zero and $W_0(x), W_1(x), W_2(x)$ are arbitrary complex (or real) polynomials with real coefficients.

The sequence $\{W_n(x)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}(x) = \frac{s-r}{s}W_{-(n-1)}(x) + \frac{r+1}{s}W_{-(n-2)}(x) - \frac{1}{s}W_{-(n-3)}(x)$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n. Note that for $n \geq 1$, $W_{-n}(x)$ need not to be a polynomial in the ordinary sense.

Generalized Horadam-Leonardo polynomial are special cases of generalized Tribonacci polynomials, for some references on generalized Tribonacci polynomials and its special cases, see for example [2,10,11,12].

Note that the sequences $\{G_n(x)\}$ and $\{H_n(x)\}$ which are defined in the section Introduction, are the special cases of the generalized Horadam-Leonardo (sequence of) polynomials $\{W_n(x)\}$. For convenience, we can give the definition of these two special cases of the sequence $\{W_n(x)\}$ in this section as well. (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} G_n(x) &= (r+1)G_{n-1}(x) + (s-r)G_{n-2}(x) - sG_{n-3}(x), \\ G_0(x) &= 0, G_1(x) = 1, G_2(x) = r+1, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} H_n(x) &= (r+1)H_{n-1}(x) + (s-r)H_{n-2}(x) - sH_{n-3}(x), \\ H_0(x) &= 3, H_1(x) = r+1, H_2(x) = r^2+2s+1. \end{aligned} \tag{2.3}$$

The sequences $\{G_n(0, 1, r+1; r+1, s-r, -s)\}_{n\geq 0}$ and $\{H_n(3, r+1, r^2+2s+1; r+1, s-r, -s)\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n}(x) &= \frac{s-r}{s}G_{-(n-1)}(x) + \frac{r+1}{s}G_{-(n-2)}(x) - \frac{1}{s}G_{-(n-3)}(x), \\ H_{-n}(x) &= \frac{s-r}{s}H_{-(n-1)}(x) + \frac{r+1}{s}H_{-(n-2)}(x) - \frac{1}{s}H_{-(n-3)}(x), \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively.

Remark 3. *For the sake of simplicity throughout the rest of the paper, we use*

$$W_n, W_0, W_1, W_2, G_n, G_0, G_1, G_2, H_n, H_0, H_1, H_2,$$

instead of

$$W_n(x), W_0(x), W_1(x), W_2(x), G(x), G_0(x), G_1(x), G_2(x), H(x), H_0(x), H_1(x), H_2(x)$$

respectively, unless otherwise stated. For example, we write

$$W_n = (r+1)W_{n-1} + (s-r)W_{n-2} - sW_{n-3},\ W_0 = c_0, W_1 = c_1, W_2 = c_2,\ n \geq 3$$

for the equation (2.1). Also we write U_n, U_0, U_1, U_2 *instead of* $U_n(x)$ *with initial conditions* $U_0(x), U_1(x), U_2(x)$ *for any subsequence* $\{U_n(x)\}$ *of* $\{W_n\}$.

When r, s, W_0, W_1, W_2 are real numbers we call generalized Horadam-Leonardo, (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials as generalized Horadam-Leonardo, (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas numbers (sequences).

Some special cases of generalized Horadam-Leonardo sequence are given as follows (Table 1):

Table 1 A few special cases of generalized Horadam-Leonardo sequence.

No	Sequences (Numbers)	r, s	Notation	References
1	Generalized Leonardo	$r = 1, s = 1$	$\{W_n(W_0, W_1, W_2; 2, 0, -1)\}$	[17]
2	Generalized John	$r = 2, s = 1$	$\{W_n(W_0, W_1, W_2; 3, -1, -1)\}$	[18]
3	Generalized Ernst	$r = 1, s = 2$	$\{W_n(W_0, W_1, W_2; 2, 1, -2)\}$	[19]
4	Generalized Pisano	$r = 1, s = -\frac{1}{4}$	$\{W_n(W_0, W_1, W_2; 2, -\frac{5}{4}, \frac{1}{4})\}$	[20]
5	Generalized Edouard	$r = 6, s = -1$	$\{W_n(W_0, W_1, W_2; 7, -7, 1)\}$	[21]
6	Generalized Bigollo	$r = 3, s = -2$	$\{W_n(W_0, W_1, W_2; 4, -5, 2)\}$	[22]

Next, we present the first few values of the (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials with positive and negative subscripts (Table 2):

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	$r+1$	$r^2+r+s+1$	$r^3+r^2+r+2rs+s+1$
G_{-n}		0	$-\frac{1}{s}$	$\frac{1}{s^2}(r-s)$	$-\frac{1}{s^3}\left(s-rs+r^2+s^2\right)$
H_n	3	$r+1$	r^2+2s+1	$r^3+3sr+1$	$r^4+4r^2s+2s^2+1$
H_{-n}		$-\frac{1}{s}(r-s)$	$\frac{1}{s^2}\left(r^2+s^2+2s\right)$	$-\frac{1}{s^3}\left(r^3+3rs-s^3\right)$	$\frac{1}{s^4}\left(r^4+4r^2s+s^4+2s^2\right)$

Some special cases of (r, s)-Horadam-Leonardo sequence $\{G_n(0, 1, r+1; r+1, s-r, -s)\}$ and (r, s)-Horadam-Leonardo-Lucas sequence $\{H_n(3, r+1, r^2+2s+1; r+1, s-r, -s)\}$ are given as follows:

(1) $G_n(0, 1, 2; 2, 0, -1) = G_n$, modified Leonardo sequence, see [17].

(2) $H_n(3, 2, 4; 2, 0, -1) = H_n$, Leonardo-Lucas sequence, see [17].

(3) $G_n(0, 1, 3; 3, -1, -1) = J_n$, John sequence, see [18].

(4) $H_n(3, 3, 7; 3, -1, -1) = H_n$, John-Lucas sequence, see [18].

UNDER PEER REVIEW

(5) $G_n(0,1,2;2,1,-2) = E_n$, Ernst sequence, see [19].

(6) $H_n(3,2,6;2,1,-2) = H_n$, Ernst-Lucas sequence, see [19].

(7) $G_n(0,1,2;2,-\frac{5}{4},\frac{1}{4}) = P_n$, Pisano sequence, see [20].

(8) $H_n(3,2,\frac{3}{2};2,-\frac{5}{4},\frac{1}{4}) = R_n$, Pisano-Lucas sequence, see [20].

(9) $G_n(0,1,7;7,-7,1) = E_n$, Edouard sequence, see [21].

(10) $H_n(3,7,35;7;7,-7,1) = K_n$, Edouard-Lucas sequence, see [21].

(11) $G_n(0,1,4;4,-5,2) = B_n$, Bigollo sequence, see [22].

(12) $H_n(3,4,6;1,4,-5,2) = C_n$, Bigollo-Lucas sequence, see [22].

The characteristic equation (the cubic equation, auxiliary equation, polynomial) of W_n is given as

$$y^3 - (r+1)y^2 - (s-r)y + s = (y^2 - ry - s)(y-1) = 0. \tag{2.4}$$

The roots of characteristic equation are

$$\alpha = \frac{r+\sqrt{r^2+4s}}{2}, \quad \beta = \frac{r-\sqrt{r^2+4s}}{2}, \ \gamma = 1,$$

where α and β are as in (1.3).

We next present Binet's formula of generalized Horadam-Leonardo polynomials

Corollary 4. *Binet's formula of generalized Horadam-Leonardo polynomials is given as follows according to the roots of characteristic equation (2.4):*

(a): *(Three Distinct Roots Case: $\alpha \neq \beta \neq \gamma = 1$)*

$$\begin{aligned} W_n &= \frac{W_2-(\beta+1)W_1+\beta W_0}{(\alpha-\beta)(\alpha-1)}\alpha^n + \frac{W_2-(\alpha+1)W_1+\alpha W_0}{(\beta-\alpha)(\beta-1)}\beta^n + \frac{W_2+(-(r+1)+1)W_1+(-s)W_0}{(-s)-(r+1)+2} \\ &= \frac{(\alpha W_2+\alpha(-(r+1)+\alpha)W_1+(-s)W_0)}{(r+1)\alpha^2+2((s-r)-r)\alpha+3(-s)}\alpha^n + \frac{(\beta W_2+\beta(-(r+1)+\beta)W_1+(-s)W_0)}{(r+1)\beta^2+2((s-r)-r)\beta+3(-s)}\beta^n \\ &\quad + \frac{W_2+(-(r+1)+1)W_1+(-s)W_0}{(r+1)+2((s-r)-r)+3(-s)}. \end{aligned}$$

(b): *(Two Distinct Roots Case: $\alpha \neq \beta = \gamma = 1$)*

$W_n = \frac{1}{(1-(-s))^2}((W_2-2W_1+W_0)\alpha^n + (-W_2+2W_1+(-s)((-s)-2)W_0) + (1-(-s))(W_2-(1+(-s))W_1+(-s)W_0)n)$.

(c): *(Single Root Case: $\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$)*

$$W_n = \frac{1}{2}(n(n-1)W_2 - 2n(n-2)W_1 + (n-1)(n-2)W_0).$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 7.]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n z^n$ of the sequence W_n.

Lemma 5. *Suppose that* $f_{W_n}(z) = \sum_{n=0}^{\infty} W_n z^n$ *is the ordinary generating function of the generalized Horadam-Leonardo (sequence of) polynomials* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n z^n = \frac{W_0 + (W_1 - (r+1)W_0)z + (W_2 - (r+1)W_1 - (s-r)W_0)z^2}{1-(r+1)z-(s-r)z^2-(-s)z^3}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Lemma 9.]. □

The last Lemma gives the following results as particular examples (generating functions of (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials).

Corollary 6. *Generating functions of* (r, s)*-Horadam-Leonardo and* (r, s)*-Horadam-Leonardo-Lucas polynomials are*

$$\begin{aligned}
\sum_{n=0}^{\infty} G_n z^n &= \frac{z}{1-(r+1)z-(s-r)z^2-(-s)z^3}, \\
\sum_{n=0}^{\infty} H_n z^n &= \frac{3-2(r+1)z-(s-r)z^2}{1-(r+1)z-(s-r)z^2-(-s)z^3},
\end{aligned}$$

respectively.

Proof. In the last Lemma, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$ and $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$, respectively. □

Next, we present Binet's formulas of (r, s)-Horadam-Leonardo and (r, s)-Horadam-Leonardo-Lucas polynomials as a special case of Corollary 4.

Corollary 7. *For all integers* n*, Binet's formulas of* (r, s)*-Horadam-Leonardo and* (r, s)*-Horadam-Leonardo-Lucas polynomials are given as follows:*

(a): *(Three Distinct Roots Case:* $\alpha \neq \beta \neq \gamma = 1$*)*

$$\begin{aligned}
G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-1)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-1)} + \frac{1}{(1-\alpha)(1-\beta)} \\
&= \frac{\alpha^{n+2}}{(r+1)\alpha^2 + 2(s-r)\alpha + 3(-s)} + \frac{\beta^{n+2}}{(r+1)\beta^2 + 2(s-r)\beta + 3(-s)} + \frac{1}{(r+1) + 2(s-r) + 3(-s)}, \\
H_n &= \alpha^n + \beta^n + 1.
\end{aligned}$$

(b): *(Two Distinct Roots Case:* $\alpha \neq \beta = \gamma = 1$*)*

$$G_n = \frac{\alpha^{n+1} + ((1-\alpha)n - \alpha)}{(1-\alpha)^2}, \;\; H_n = \alpha^n + 2.$$

(c): *(Single Root Case:* $\alpha = \beta = \gamma = 1 = \frac{(r+1)}{3}$*)*

$$G_n = \frac{n(n+1)}{2}, \;\; H_n = 3.$$

Proof. In Corollary 4, take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$ and $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$, respectively. □

3. Simson's Formulas of Horadam-Leonardo Polynomials

The following theorem gives Simson's formula of the generalized Horadam-Leonardo polynomials $\{W_n\}$.

THEOREM 8 (Simson's Formula of Generalized Horadam-Leonardo Polynomials). *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = (-s)^n \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}. \tag{3.1}$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 33.]. □

The previous theorem gives the following results as particular examples.

COROLLARY 9. *For all integers n, Simson's formula of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials are given as*

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -(-s)^{n-1}, \quad \begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = (r^2+4s)(r+s-1)^2(-s)^{n-2}$$

respectively.

Note also that (3.1) can be written as

$$\begin{vmatrix} W_{n+m+2} & W_{n+m+1} & W_{n+m} \\ W_{n+m+1} & W_{n+m} & W_{n+m-1} \\ W_{n+m} & W_{n+m-1} & W_{n+m-2} \end{vmatrix} = (-s)^{n+m} \begin{vmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{vmatrix}$$

for all integers n, m.

We define

$$\begin{aligned} \Lambda_W(n) &= W_{n+2}^3 - r(r-s+1)W_{n+1}^3 + (-s)^2 W_n^3 - 2(r+1)W_{n+1}W_{n+2}^2 - (s-r)W_n W_{n+2}^2 \\ &\quad + (r^2+3r-s+1)W_{n+2}W_{n+1}^2 + ((r+1)(-s)+(s-r)^2)W_n W_{n+1}^2 \\ &\quad + (r+1)(-s)W_n^2 W_{n+2} + 2(s-r)(-s)W_n^2 W_{n+1} - (r^2+r-4s-rs)W_{n+2}W_{n+1}W_n \end{aligned}$$

Then

$$\begin{aligned} \Lambda_W(0) &= W_2^3 - r(r-s+1)W_1^3 + (-s)^2 W_0^3 - 2(r+1)W_1 W_2^2 - (s-r)W_0 W_2^2 \\ &\quad + (r^2+3r-s+1)W_2 W_1^2 + ((r+1)(-s)+(s-r)^2)W_0 W_1^2 \\ &\quad + (r+1)(-s)W_0^2 W_2 + 2(s-r)(-s)W_0^2 W_1 - (r^2+r-4s-rs)W_2 W_1 W_0. \end{aligned} \tag{3.2}$$

Simson's formulas of W_n, G_n H_n can be given in the following forms.

LEMMA 10. *For all integers n, we have*

(a): $\Lambda_W(n) = (-s)^n \Lambda_W(0)$, *i.e.,*

$$\begin{aligned}
& W_{n+2}^3 - r(r-s+1)W_{n+1}^3 + (-s)^2 W_n^3 - 2(r+1)W_{n+1}W_{n+2}^2 - (s-r)W_n W_{n+2}^2 \\
& +(r^2+3r-s+1)W_{n+2}W_{n+1}^2 + ((r+1)(-s)+(s-r)^2)W_n W_{n+1}^2 \\
& +(r+1)(-s)W_n^2 W_{n+2} + 2(s-r)(-s)W_n^2 W_{n+1} - (r^2+r-4s-rs)W_{n+2}W_{n+1}W_n \\
= \; & (-s)^n (W_2^3 - r(r-s+1)W_1^3 + (-s)^2 W_0^3 - 2(r+1)W_1 W_2^2 - (s-r)W_0 W_2^2 \\
& +(r^2+3r-s+1)W_2 W_1^2 + (r^2+s^2-s-3rs)W_0 W_1^2 + (r+1)(-s)W_0^2 W_2 \\
& +2(s-r)(-s)W_0^2 W_1 - (r^2+r-4s-rs)W_2 W_1 W_0).
\end{aligned}$$

(b): $\Lambda_G(n) = (-s)^n \Lambda_G(0) = (-s)^{n+1}$, *i.e.,*

$$\begin{aligned}
& G_{n+2}^3 - r(r-s+1)G_{n+1}^3 + (-s)^2 G_n^3 - 2(r+1)G_{n+1}G_{n+2}^2 - (s-r)G_n G_{n+2}^2 \\
& +(r^2+3r-s+1)G_{n+2}G_{n+1}^2 + (r^2+s^2-s-3rs)G_n G_{n+1}^2 \\
& +(r+1)(-s)G_n^2 G_{n+2} + 2(s-r)(-s)G_n^2 G_{n+1} - (r^2+r-4s-rs)G_{n+2}G_{n+1}G_n \\
= \; & (-s)^{n+1}.
\end{aligned}$$

(c): $\Lambda_H(n) = (-s)^n \Lambda_H(0) = -(r^2+4s)(r+s-1)^2(-s)^n$, *i.e.,*

$$\begin{aligned}
& H_{n+2}^3 - r(r-s+1)H_{n+1}^3 + (-s)^2 H_n^3 - 2(r+1)H_{n+1}H_{n+2}^2 - (s-r)H_n H_{n+2}^2 \\
& +((r+1)^2-(s-r))H_{n+2}H_{n+1}^2 + ((r+1)(-s)+(s-r)^2)H_n H_{n+1}^2 \\
& +(r+1)(-s)H_n^2 H_{n+2} + 2(s-r)(-s)H_n^2 H_{n+1} - (r^2+r-4s-rs)H_{n+2}H_{n+1}H_n \\
= \; & -(r^2+4s)(r+s-1)^2(-s)^n.
\end{aligned}$$

Proof. Replace r, s and t with $r+1,\ s-r,\ -s$, respectively, in [16, Lemma 35.]. □

4. Some Identities of Generalized Horadam-Leonardo Polynomials

In this section, we obtain some identities of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 11. *The following equalities are true:*

(a): $(-s)^3 H_n = (3rs+r^3-s^3)G_{n+4} - (4r^2s-rs^3+3rs+r^3+r^4+2s^2)G_{n+3} + (4r^2s+r^4+2s^2+s^4)G_{n+2}$.

(b): $(-s)^2 H_n = (2s+r^2+s^2)G_{n+3} - (2s+rs^2+3rs+r^2+r^3)G_{n+2} + (3rs+r^3-s^3)G_{n+1}$.

(c): $(-s)H_n = -(s-r)G_{n+2} + (rs-2s-r-r^2)G_{n+1} + (r^2+s^2+2s)G_n$.

(d): $H_n = 3G_{n+1} - 2(r+1)G_n - (s-r)G_{n-1}$.

(e): $H_n = (r+1)G_n + 2(s-r)G_{n-1} + 3(-s)G_{n-2}$.

(f): $s(r^2+4s)\left(r+s-1\right)^2 G_n = (-r^3+r^2s+2r^2-3rs-r+4s^2+4s)H_{n+4} + (r^4-r^3s-r^3+4r^2s-r^2-4rs^2-3rs+r+2s^2-2s)H_{n+3} - (2s+4r^2s+r^2s^2-6rs+r^2-2r^3+r^4+2s^2+4s^3)H_{n+2}$.

(g): $-(r^2+4s)(r+s-1)^2G_n = -2(r^2-r+3s+1)H_{n+3} + (-r-2s+7rs-r^2+2r^3+2)H_{n+2} + (-r+4s+r^2s-3rs+2r^2-r^3+4s^2)H_{n+1}$.

(h): $-(r^2+4s)(r+s-1)^2G_n = (rs-8s-r-r^2)H_{n+2} + (r+2s-r^2s+5rs+r^3-2s^2)H_{n+1} + 2s(-r+3s+r^2+1)H_n$.

(i): $-(r^2+4s)(r+s-1)^2G_n = -2(3s+rs+r^2+s^2)H_{n+1} + (2s+rs^2+5rs+r^2+r^3-2s^2)H_n + s(r+8s-rs+r^2)H_{n-1}$.

(j): $-(r^2+4s)(r+s-1)^2G_n = -(4s+rs^2+2r^2s+3rs+r^2+r^3+4s^2)H_n + (-rs^2+r^2s+7rs+2r^3+2s^2-2s^3)H_{n-1} + 2s(3s+rs+r^2+s^2)H_{n-2}$.

Proof. Replace r, s and t with $r+1,\ s-r,\ -s$, respectively, in [16, Lemma 36.]. □

Next, we give a few basic relations between $\{G_n\}$ and $\{W_n\}$.

LEMMA 12. *The following equalities are true:*

(a): $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)G_n = ((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n+2} + (W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n+1} + ((-s)W_1^2-(-s)W_0W_2)W_n$.

(b): $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)G_n = (W_2^2+(r+1)^2W_1^2+(r+1)(-s)W_0^2-2(r+1)W_1W_2-(s-r)W_0W_2+((r+1)(s-r)-(-s))W_0W_1)W_{n+1} + (((-s)+(r+1)(s-r))W_1^2+(s-r)(-s)W_0^2-(s-r)W_1W_2-(-s)W_0W_2+(s-r)^2W_0W_1)W_n + ((r+1)(-s)W_1^2+(-s)^2W_0^2-(-s)W_1W_2+(s-r)(-s)W_0W_1)W_{n-1}$.

(c): $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)G_n = ((r+1)W_2^2+((r+1)^3+(-s)+(r+1)(s-r))W_1^2+(-s)((s-r)+(r+1)^2)W_0^2-((s-r)+2(r+1)^2)W_1W_2-((-s)+(r+1)(s-r))W_0W_2+((r+1)^2(s-r)+(s-r)^2-(r+1)(-s))W_0W_1)W_n + ((s-r)W_2^2+(r+1)((r+1)(s-r)+(-s))W_1^2+(-s)((-s)+(r+1)(s-r))W_0^2-((-s)+2(r+1)(s-r))W_1W_2-(s-r)^2W_0W_2+(r+1)(s-r)^2W_0W_1)W_{n-1} + ((-s)W_2^2+(r+1)^2(-s)W_1^2+(r+1)(-s)^2W_0^2-2(r+1)(-s)W_1W_2-(s-r)(-s)W_0W_2+(-s)((r+1)(s-r)-(-s))W_0W_1)W_{n-2}$.

(d): $(-s)W_n = (W_2-(r+1)W_1-(s-r)W_0)G_{n+2} + (-(r+1)W_2+(r+1)^2W_1+((-s)+(r+1)(s-r))W_0)G_{n+1} + (-(s-r)W_2+((-s)+(r+1)(s-r))W_1+((s-r)^2-(r+1)(-s))W_0)G_n$.

(e): $W_n = W_0G_{n+1} + (W_1-(r+1)W_0)G_n + (W_2-(r+1)W_1-(s-r)W_0)G_{n-1}$.

(f): $W_n = W_1G_n + (W_2-(r+1)W_1)G_{n-1} + (-s)W_0G_{n-2}$.

Proof. Replace r, s and t with $r+1,\ s-r,\ -s$, respectively, in [16, Lemma 37.]. □

Now, we present a few basic relations between $\{H_n\}$ and $\{W_n\}$.

LEMMA 13. *The following equalities are true:*

(a): $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)H_n = (3W_2^2+((r+1)^2-(s-r))W_1^2+(r+1)(-s)W_0^2-4(r+1)W_1W_2-2(s-r)W_0W_2+((r+1)(s-r)-3(-s))W_0W_1)W_{n+2} + (-2(r+1)W_2^2+3(-s)W_1^2-2(s-r)W_1W_2-3(-s)W_0W_2+3(r+1)(s-r)W_1^2+$

$2(s-r)(-s)W_0^2+2(r+1)^2W_1W_2+2(s-r)^2W_0W_1+(r+1)(s-r)W_0W_2+2(r+1)(-s)W_0W_1)W_{n+1}+(-(s-r)W_2^2+((s-r)^2+(r+1)(-s))W_1^2+3(-s)^2W_0^2+((r+1)(s-r)-3(-s))W_1W_2+2(r+1)(-s)W_0W_2+4(s-r)(-s)W_0W_1)W_n.$

(b): $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)H_n=((r+1)^3W_1^2+(r+1)W_2^2+3(-s)W_1^2+(r+1)^2(-s)W_0^2-2(s-r)W_1W_2-3(-s)W_0W_2+2(r+1)(s-r)W_1^2+2(s-r)(-s)W_0^2-2(r+1)^2W_1W_2+2(s-r)^2W_0W_1-(r+1)(s-r)W_0W_2-(r+1)(-s)W_0W_1+(r+1)^2(s-r)W_0W_1)W_{n+1}+(3(-s)^2W_0^2+2(s-r)W_2^2+(r+1)^2(s-r)W_1^2-3(-s)W_1W_2+(r+1)(-s)W_1^2-2(s-r)^2W_0W_2-3(r+1)(s-r)W_1W_2+2(r+1)(-s)W_0W_2+(s-r)(-s)W_0W_1+(r+1)(s-r)(-s)W_0^2+(r+1)(s-r)^2W_0W_1)W_n+(3(-s)W_2^2+(-s)((r+1)^2-(s-r))W_1^2+(r+1)(-s)^2W_0^2-4(r+1)(-s)W_1W_2-2(s-r)(-s)W_0W_2+(-s)((r+1)(s-r)-3(-s))W_0W_1)W_{n-1}.$

(c): $(-W_2^2-(r-s+1)W_1^2+sW_0^2+(r+2)W_1W_2-rW_0W_2+(r-2s)W_0W_1)(-W_2+rW_1+sW_0)H_n=(((r+1)^2+2(s-r))W_2^2+(r+1)((r+1)^3+3(r+1)(s-r)+4(-s))W_1^2+(-s)((r+1)^3+3(r+1)(s-r)+3(-s))W_0^2-(3(-s)+2(r+1)^3+5(r+1)(s-r))W_1W_2-((r+1)^2(s-r)+(r+1)(-s)+2(s-r)^2)W_0W_2+((r+1)^3(s-r)-(r+1)^2(-s)+3(r+1)(s-r)^2+(s-r)(-s))W_0W_1)W_n+((3(-s)+(r+1)(s-r))W_2^2+((r+1)^3(s-r)+2(r+1)(s-r)^2+2(s-r)(-s)+(r+1)^2(-s))W_1^2+(-s)((r+1)(-s)+(r+1)^2(s-r)+2(s-r)^2)W_0^2-2(2(r+1)(-s)+(s-r)^2+(r+1)^2(s-r))W_1W_2-(s-r)((r+1)(s-r)+5(-s))W_0W_2+(2(s-r)^3+(r+1)^2(s-r)^2-3(-s)^2)W_0W_1)W_{n-1}+((r+1)(-s)W_2^2+(-s)((r+1)^3+3(-s)+2(r+1)(s-r))W_1^2+(-s)^2(r^2+2s+1)W_0^2-2(-s)((s-r)+(r+1)^2)W_1W_2-(-s)(3(-s)+(r+1)(s-r))W_0W_2+(-s)(2(s-r)^2+(r+1)^2(s-r)-(r+1)(-s))W_0W_1)W_{n-2}.$

(d): $-(r^2+4s)(r+s-1)^2W_n=(-2((r+1)^2+3(s-r))W_2+(2(r+1)^3+9(-s)+7(r+1)(s-r))W_1+(4(s-r)^2-3(r+1)(-s)+(r+1)^2(s-r))W_0)H_{n+2}+((2(r+1)^3+9(-s)+7(r+1)(s-r))W_2-2((r+1)^4+4(r+1)^2(s-r)+6(-s)(r+1)+(s-r)^2)W_1-(4(r+1)(s-r)^2+6(-s)(s-r)-(-s)(r+1)^2+(r+1)^3(s-r))W_0)H_{n+1}+((-3(r+1)(-s)+4(s-r)^2+(r+1)^2(s-r))W_2-(4(r+1)(s-r)^2+(r+1)^3(s-r)-(r+1)^2(-s)+6(s-r)(-s))W_1+(-(r+1)^2(s-r)^2+2(r+1)^3(-s)+9(-s)^2-4(s-r)^3+10(r+1)(s-r)(-s))W_0)H_n.$

(e): $-(r^2+4s)(r+s-1)^2W_n=((9(-s)+(r+1)(s-r))W_2-((r+1)^2(s-r)+3(-s)(r+1)+2(s-r)^2)W_1-2(-s)(3(s-r)+(r+1)^2)W_0)H_{n+1}+(-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))W_2+((r+1)^3(s-r)+3(r+1)(s-r)^2+(r+1)^2(-s)+3(s-r)(-s))W_1+(-s)(9(-s)+2(r+1)^3+7(r+1)(s-r))W_0)H_n+(-2(-s)((r+1)^2+3(s-r))W_2+(-s)(2(r+1)^3+9(-s)+7(r+1)(s-r))W_1+(-s)(4(s-r)^2+(r+1)^2(s-r)-3(r+1)(-s))W_0)H_{n-1}.$

(f): $-(r^2+4s)(r+s-1)^2W_n=(2(3(r+1)(-s)-(s-r)^2)W_2+(3(s-r)(-s)+(r+1)(s-r)^2-2(r+1)^2(-s))W_1+(-s)(9(-s)+(r+1)(s-r))W_0)H_n+(((r+1)(s-r)^2-2(r+1)^2(-s)+3(s-r)(-s))W_2+(2(r+1)^3(-s)-(r+1)^2(s-r)^2+4(r+1)(s-r)(-s)-2(s-r)^3+9(-s)^2)W_1-(-s)(2(s-r)^2+3(r+1)(-s)+(r+1)^2(s-r))W_0)H_{n-1}+((-s)(9(-s)+(r+1)(s-r))W_2-(-s)((r+1)^2(s-r)+3(r+1)(-s)+2(s-r)^2)W_1-2(-s)^2(3(s-r)+(r+1)^2)W_0)H_{n-2}.$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Lemma 38.]. □

5. Recurrence Properties of Generalized Horadam-Leonardo Polynomials

Next, we present a formula for W_{-n}.

THEOREM 14. *For $n \in \mathbb{Z}$, we have*

$$W_{-n} = (-s)^{-n}(W_{2n} - H_n W_n + \frac{1}{2}(H_n^2 - H_{2n})W_0).$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 39.]. □

Now, we have the following corollary.

COROLLARY 15. *For $n \in \mathbb{Z}$, we have*

(a): $G_{-n} = \frac{1}{(-s)^{n+1}}((2(r+1)(-s) - (s-r)^2)G_n^2 + (-s)G_{2n} + (s-r)G_{n+2}G_n - (3(-s) + (r+1)(s-r))G_{n+1}G_n)$.

(b): $H_{-n} = \frac{1}{2(-s)^n}(H_n^2 - H_{2n})$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 42] or take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$ and $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2 + 2s + 1$, respectively, in the last Theorem. □

6. Generalized Horadam-Leonardo Polynomials by Matrix Methods

In this section, we present matrix representations of the sequences W_n, G_n and H_n. We also introduce Simson matrix and investigate its properties.

6.1. Matrix Representations of the Sequences W_n, G_n and H_n. We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = (-s)$. We also define

$$B_n = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} W_{n+1} & (s-r)W_n + (-s)W_{n-1} & (-s)W_n \\ W_n & (s-r)W_{n-1} + (-s)W_{n-2} & (-s)W_{n-1} \\ W_{n-1} & (s-r)W_{n-2} + (-s)W_{n-3} & (-s)W_{n-2} \end{pmatrix}.$$

THEOREM 16. *For all integers m,n, we have the following properties:*

(a): $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix}.$$

(b): $D_1 A^n = A^n D_1$.

(c): $D_{n+m} = D_n B_m = B_m D_n$, *i.e.,*

(d):

$$A^n = G_{n-1}A^2 + ((s-r)G_{n-2} + (-s)G_{n-3})A + (-s)G_{n-2}I,$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 51.]. □

Next, we present matrix formulas for the generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials.

In the next Corollary, we use $\Lambda_W(0)$ given in (3.2), i.e.,

$$\begin{aligned} \Lambda_W(0) &= W_2^3 - r(r-s+1)W_1^3 + (-s)^2 W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 \\ &+ (r^2+3r-s+1)W_2W_1^2 + ((r+1)(-s) + (s-r)^2)W_0W_1^2 \\ &+ (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 - (r^2+r-4s-rs)W_2W_1W_0. \end{aligned}$$

COROLLARY 17. *For all integers n, we have the following formulas for generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials.*

(a): *Generalized Horadam-Leonardo polynomials.*

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$a_{11} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+3} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+2} + ((-s)W_1^2 - (-s)W_0W_2)W_{n+1}$,

$a_{21} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+2} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_{n+1} + ((-s)W_1^2 - (-s)W_0W_2)W_n$,

$a_{31} = ((r+1)W_1^2 + (-s)W_0^2 - W_1W_2 + (s-r)W_0W_1)W_{n+1} + (W_2^2 - (r+1)W_1W_2 - (s-r)W_0W_2 - (-s)W_0W_1)W_n + ((-s)W_1^2 - (-s)W_0W_2)W_{n-1}$,

$a_{12} = (s-r)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n+2}+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n+1}+((-s)W_1^2-(-s)W_0W_2)W_n)+(-s)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n+1}+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_n+((-s)W_1^2-(-s)W_0W_2)W_{n-1})$,

$a_{22} = (s-r)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n+1}+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_n+((-s)W_1^2-(-s)W_0W_2)W_{n-1})+(-s)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_n+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n-1}+((-s)W_1^2-(-s)W_0W_2)W_{n-2})$,

$a_{32} = (s-r)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_n+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n-1}+((-s)W_1^2-(-s)W_0W_2)W_{n-2})+(-s)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n-1}+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n-2}+((-s)W_1^2-(-s)W_0W_2)W_{n-3})$,

$a_{13} = (-s)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n+2}+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n+1}+((-s)W_1^2-(-s)W_0W_2)W_n)$,

$a_{23} = (-s)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_{n+1}+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_n+((-s)W_1^2-(-s)W_0W_2)W_{n-1})$,

$a_{33} = (-s)(((r+1)W_1^2+(-s)W_0^2-W_1W_2+(s-r)W_0W_1)W_n+(W_2^2-(r+1)W_1W_2-(s-r)W_0W_2-(-s)W_0W_1)W_{n-1}+((-s)W_1^2-(-s)W_0W_2)W_{n-2})$,

(b): (r,s)-*Horadam-Leonardo-Lucas polynomials.*

$$\begin{pmatrix} r+1 & s-r & -s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n$$

$$= \frac{1}{4(r+1)^3(-s)-(r+1)^2(s-r)^2-4(s-r)^3+27(-s)^2+18(r+1)(s-r)(-s)} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

where

$b_{11} = (9(-s)+(r+1)(s-r))H_{n+3}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n+2}-2(-s)((r+1)^2+3(s-r))H_{n+1}$,

$b_{21} = (9(-s)+(r+1)(s-r))H_{n+2}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n+1}-2(-s)((r+1)^2+3(s-r))H_n$,

$b_{31} = (9(-s)+(r+1)(s-r))H_{n+1}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_n-2(-s)((r+1)^2+3(s-r))H_{n-1}$,

$b_{12} = (s-r)((9(-s)+(r+1)(s-r))H_{n+2}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n+1}-2(-s)((r+1)^2+3(s-r))H_n)+(-s)((9(-s)+(r+1)(s-r))H_{n+1}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_n-2(-s)((r+1)^2+3(s-r))H_{n-1})$,

$b_{22}=(s-r)((9(-s)+(r+1)(s-r))H_{n+1}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_n-2(-s)((r+1)^2+3(s-r))H_{n-1})+(-s)((9(-s)+(r+1)(s-r))H_n-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n-1}-2(-s)((r+1)^2+3(s-r))H_{n-2}),$

$b_{32}=(s-r)((9(-s)+(r+1)(s-r))H_n-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n-1}-2(-s)((r+1)^2+3(s-r))H_{n-2})+(-s)((9(-s)+(r+1)(s-r))H_{n-1}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n-2}-2(-s)((r+1)^2+3(s-r))H_{n-3}),$

$b_{13}=(-s)((9(-s)+(r+1)(s-r))H_{n+2}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n+1}-2(-s)((r+1)^2+3(s-r))H_n),$

$b_{23}=(-s)((9(-s)+(r+1)(s-r))H_{n+1}-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_n-2(-s)((r+1)^2+3(s-r))H_{n-1}),$

$b_{33}=(-s)((9(-s)+(r+1)(s-r))H_n-((r+1)^2(s-r)+2(s-r)^2+3(r+1)(-s))H_{n-1}-2(-s)((r+1)^2+3(s-r))H_{n-2}).$

Proof. Replace r,s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 52]. □

Now, we present an identity for W_{n+m}.

THEOREM 18. *(Honsberger's Identity) For all integers m and n, we have*

$$\begin{aligned} W_{n+m} &= W_nG_{m+1}+W_{n-1}((s-r)G_m+(-s)G_{m-1})+(-s)W_{n-2}G_m \\ &= W_nG_{m+1}+\left((s-r)W_{n-1}+(-s)W_{n-2}\right)G_m+(-s)W_{n-1}G_{m-1}. \end{aligned}$$

Proof. Replace r,s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 53]. □

COROLLARY 19. *For all integers m,n, we have the following properties:*

$$\begin{aligned} G_{n+m} &= G_nG_{m+1}+G_{n-1}((s-r)G_m+(-s)G_{m-1})+(-s)G_{n-2}G_m, \\ H_{n+m} &= H_nG_{m+1}+H_{n-1}((s-r)G_m+(-s)G_{m-1})+(-s)H_{n-2}G_m. \end{aligned}$$

Proof. Replace r,s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 54]. □

COROLLARY 20. *For all integers m,n,j, we have the following properties:*

$$\begin{aligned} W_{mn+j} &= G_{mn-1}W_{j+2}+\left((s-r)G_{mn-2}+(-s)G_{mn-3}\right)W_{j+1}+(-s)G_{mn-2}W_j, \\ G_{mn+j} &= G_{mn-1}G_{j+2}+\left((s-r)G_{mn-2}+(-s)G_{mn-3}\right)G_{j+1}+(-s)G_{mn-2}G_j, \\ H_{mn+j} &= G_{mn-1}H_{j+2}+\left((s-r)G_{mn-2}+(-s)G_{mn-3}\right)H_{j+1}+(-s)G_{mn-2}H_j. \end{aligned}$$

Proof. Replace r,s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 55.]. □

6.2. Simson Matrix and its Properties. For $n \in \mathbb{Z}$, we define

$$f_W(n) = \begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence W_n. Similarly, as special cases of W_n, Simson matrices of the sequences G_n and H_n are

$$f_G(n) = \begin{pmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{pmatrix}, \; f_H(n) = \begin{pmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{pmatrix},$$

respectively.

Lemma 21. *For all integers n, m and j, the followings hold.*

(a): $f_W(n) = (r+1)f_W(n-1) + (s-r)f_W(n-2) + (-s)f_W(n-3)$.

(b): $f_W(n) = Af_W(n-1)$ *and* $f_W(n) = A^n f_W(0)$, *i.e.,*

(c): $f_W(n+m) = A^n f_W(m)$ *and* $f_W(n+m) = A^m f_W(n)$ *i.e., and* $f_W(n) = A^m f_W(n-m)$, *i.e.,*

(d):

$$f_W(mn+j) = A^{mn} f_W(j)$$

and

$$f_W(mn+j) = (G_{n-1}A^2 + ((s-r)G_{n-2} + (-s)G_{n-3})A + (-s)G_{n-2}I)^m f_W(j).$$

(e):

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \begin{pmatrix} G_{n+1} & (s-r)G_n + (-s)G_{n-1} & (-s)G_n \\ G_n & (s-r)G_{n-1} + (-s)G_{n-2} & (-s)G_{n-1} \\ G_{n-1} & (s-r)G_{n-2} + (-s)G_{n-3} & (-s)G_{n-2} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}.$$

(f):

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{1}{\Lambda_W(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}$$

where $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}$ and $\Lambda_W(0)$ are as in Corollary 17 (a) (in the last identity above, we replace n with m in $a_{11}, a_{21}, a_{31}, a_{12}, a_{22}, a_{32}, a_{13}, a_{23}, a_{33}$).

(g):

$$\begin{pmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{pmatrix} = \frac{\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} W_2 & W_1 & W_0 \\ W_1 & W_0 & W_{-1} \\ W_0 & W_{-1} & W_{-2} \end{pmatrix}}{-(r^2+4s)(r+s-1)^2}$$

where $b_{11}, b_{21}, b_{31}, b_{12}, b_{22}, b_{32}, b_{13}, b_{23}, b_{33}$ *are as in Corollary 17 (b) (in the last identity above, we replace* n *with* m *in* $b_{11}, b_{21}, b_{31}, b_{12}, b_{22}, b_{32}, b_{13}, b_{23}, b_{33}$*).*

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Lemma 56]. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

THEOREM 22. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_W(n+m)) = (-s)^n \det(f_W(m)),$$

and

$$\det(f_W(n)) = (-s)^m \det(f_W(n-m)).$$

(b): *(see Theorem 8) Simson's (or Cassini's) Identity:*

$$\det(f_W(n)) = (-s)^n \det(f_W(0)).$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 57]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of (r, s)-Horadam-Leonardo polynomials (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = (r+1)$).

COROLLARY 23. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_G(n+m)) = (-s)^n \det(f_G(m)),$$

and

$$\det(f_G(n)) = (-s)^m \det(f_G(n-m)).$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_G(n)) = (-s)^n \det(f_G(0)),$$

i.e.,

$$\begin{vmatrix} G_{n+2} & G_{n+1} & G_n \\ G_{n+1} & G_n & G_{n-1} \\ G_n & G_{n-1} & G_{n-2} \end{vmatrix} = -(-s)^{n-1}.$$

Taking $W_n = H_n$ with $H_0 = 3, H_1 = (r+1), H_2 = r^2+2s+1$ in the last Theorem, we have the following Corollary which gives determinantal formulas of (r,s)-Horadam-Leonardo-Lucas polynomials.

Corollary 24. *For all integers n and m, the following identities hold.*

(a): *Catalan's Identity:*

$$\det(f_H(n+m)) = (-s)^n \det(f_H(m)),$$

and

$$\det(f_H(n)) = (-s)^m \det(f_H(n-m)), .$$

(b): *Simson's (or Cassini's) Identity:*

$$\det(f_H(n)) = (-s)^n \det(f_H(0)).$$

7. The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials

In this section, we give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials.

7.1. The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials in Terms of Generalized Horadam-Leonardo Polynomials. We can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials in terms of elements of the sequence of generalized Horadam-Leonardo polynomials.

Theorem 25. *For all integers m and j, we have the following sum formulas.*

(a): *If $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_W(z)}{\Gamma_W(z)}$$

where

$\Theta_W(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6,$

$z^{n+3}\Theta_1 = z^{n+3}(-W_jW_{m+2}^2W_{m+mn+2} + (-W_{j+1} + (r+1)W_j)W_{m+2}^2W_{m+mn+1} + (-W_{j+2} + (r+1)W_{j+1} + (s-r)W_j)W_{m+2}^2W_{m+mn} + (-W_{j+2} + (s-r)W_j)W_{m+1}^2W_{m+mn+2} - ((-s) + (r+1)(s-r))W_jW_{m+1}^2W_{m+mn+1} + ((s-r)W_{j+2} - ((-s) + (r+1)(s-r))W_{j+1} - (s-r)^2W_j)W_{m+1}^2W_{m+mn} - (-s)W_m^2W_{j+1}W_{m+mn+2} + (-s)(-W_{j+2} + (r+1)W_{j+1})W_m^2W_{m+mn+1} - (-s)^2W_jW_m^2W_{m+mn} + (W_{j+1} + (r+1)W_j)W_{m+2}W_{m+1}W_{m+mn+2} + (W_{j+2} - (r+1)W_{j+1})W_{m+2}W_mW_{m+mn+2} + (-(s-r)W_{j+1} + (-s)W_j)W_{m+1}W_mW_{m+mn+2} + (W_{j+2} - (r+1)^2W_j)W_{m+2}W_{m+1}W_{m+mn+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} + (-s)W_j)W_{m+2}W_mW_{m+mn+1} + (-(s-r)W_{j+2} + ((-s) + (r+1)(s-r))W_{j+1} - (r+1)(-s)W_j)W_{m+1}W_mW_{m+mn+1} + (r+1)W_{j+2}W_{m+1}W_{m+2}W_{m+mn} + (-s)W_jW_{m+1}W_{m+2}W_{m+mn} - (r+1)(s-r)W_jW_{m+1}W_{m+2}W_{m+mn} - (r+1)^2W_{j+1}W_{m+1}W_{m+2} W_{m+mn} + (-s)(W_{j+1} - (r+1)W_j)W_{m+2}W_mW_{m+mn} + (-s)(W_{j+2} - (r+1)W_{j+1} - 2(s-r)W_j) W_{m+1}W_mW_{m+mn}),$

$z^{n+2}\Theta_2 = z^{n+2}((-W_0W_{j+2}+((r+1)W_0-W_1)W_{j+1}+(2W_2-(r+1)W_1)W_j)W_{m+2}W_{m+mn+2}+ (2W_1W_{j+2}+((s-r)W_0-W_2)W_{j+1}-((r+1)W_2+2(s-r)W_1+(-s)W_0)W_j)W_{m+1}W_{m+mn+2}+ (-W_2W_{j+2}+((r+1)W_2+(s-r)W_1+2(-s)W_0)W_{j+1}-(-s)W_1W_j)W_mW_{m+mn+2}+(((r+1)W_0- W_1)W_{j+2}+(2W_2-(r+1)^2W_0-(s-r)W_0)W_{j+1}+(-2(r+1)W_2+(r+1)^2W_1-(-s)W_0)W_j)W_{m+2} W_{m+mn+1}+((-W_2+(s-r)W_0)W_{j+2}-((r+1)(s-r)W_0+(-s)W_0)W_{j+1}+((r+1)^2W_2+2((-s)+(r+ 1)(s-r))W_1+(r+1)(-s)W_0)W_j)W_{m+1}W_{m+mn+1}+(((r+1)W_2+(s-r)W_1+2(-s)W_0)W_{j+2}-((r+ 1)^2W_2+(r+1)(s-r)W_1+2(r+1)(-s)W_0+(s-r)W_2+(-s)W_1)W_{j+1}+(-s)((r+1)W_1-W_2)W_j) W_mW_{m+mn+1}+((2W_2-(r+1)W_1)W_{j+2}+((r+1)^2W_1-2(r+1)W_2-(-s)W_0)W_{j+1}+((r+1)(s- r)W_1-2(s-r)W_2-(-s)W_1+(r+1)(-s)W_0)W_j)W_{m+2}W_{m+mn}+(-((r+1)W_2+2(s-r)W_1+ (-s)W_0)W_{j+2}+((r+1)^2W_2+2((-s)+(r+1)(s-r))W_1+(r+1)(-s)W_0)W_{j+1}+(((r+1)(s- r)-(-s))W_2+2(s-r)^2W_1+2(s-r)(-s)W_0)W_j)W_{m+1}W_{m+mn}+(-s)(-W_1W_{j+2}+((r+1)W_1- W_2)W_{j+1}+((r+1)W_2+2(s-r)W_1+2(-s)W_0)W_j)W_mW_{m+mn}),$

$z^{n+1}\Theta_3 = z^{n+1}(((W_0W_2-W_1^2)W_{j+2}+(-(-s)W_0^2+W_1W_2-(r+1)W_0W_2-(s-r)W_0W_1)W_{j+1}+(-W_2^2+(s-r)W_1^2+(r+1)W_1W_2+(-s)W_0W_1)W_j)W_{m+mn+2}+((-(-s)W_0^2+W_1W_2- (r+1)W_0W_2-(s-r)W_0W_1)W_{j+2}+(-W_2^2+(r+1)(-s)W_0^2+((r+1)^2+(s-r))W_0W_2+((-s)+ (r+1)(s-r))W_0W_1)W_{j+1}+((r+1)W_2^2-((r+1)(s-r)+(-s))W_1^2-(r+1)^2W_1W_2+(-s)W_0W_2- (r+1)(-s)W_0W_1)W_j)W_{m+mn+1}+((-W_2^2+(s-r)W_1^2+(r+1)W_1W_2+(-s)W_0W_1)W_{j+2}+((r+1)W_2^2-((r+1)(s-r)+(-s))W_1^2-(r+1)^2W_1W_2+(-s)W_0W_2-(r+1)(-s)W_0W_1)W_{j+1}+ ((s-r)W_2^2-(s-r)^2W_1^2-(-s)^2W_0^2+((-s)-(r+1)(s-r))W_1W_2-(r+1)(-s)W_0W_2-2(s- r)(-s)W_0W_1)W_j)W_{m+mn}),$

$z^2\Theta_4 = z^2((W_0W_{j+2}+(W_1-(r+1)W_0)W_{j+1}+(W_2-(r+1)W_1-(s-r)W_0)W_j)W_{m+2}^2+((W_2- (s-r)W_0)W_{j+2}+((-s)W_0+(r+1)(s-r)W_0)W_{j+1}+((s-r)^2W_0+((r+1)(s-r)+(-s))W_1- (s-r)W_2)W_j)W_{m+1}^2+(-s)(W_1W_{j+2}+(W_2-(r+1)W_1)W_{j+1}+(-s)W_0W_j)W_m^2+(-(W_1+(r+ 1)W_0)W_{j+2}+((r+1)^2W_0-W_2)W_{j+1}+(-(r+1)W_2+(r+1)^2W_1+((r+1)(s-r)-(-s))W_0)W_j) W_{m+1}W_{m+2}+(((r+1)W_1-W_2)W_{j+2}+((r+1)W_2-((s-r)+(r+1)^2)W_1-(-s)W_0)W_{j+1}+ (-s)((r+1)W_0-W_1)W_j)W_{m+2}W_m+(((s-r)W_1-(-s)W_0)W_{j+2}+((s-r)W_2-((r+1)(s-r)+ (-s))W_1+(r+1)(-s)W_0)W_{j+1}+(-s)(-W_2+(r+1)W_1+2(s-r)W_0)W_j)W_{m+1}W_m),$

$z\Theta_5 = z(((W_1^2-W_0W_2)W_{j+2}+((-s)W_0^2-W_1W_2+(r+1)W_0W_2+(s-r)W_0W_1)W_{j+1}+ (-2W_2^2-(r+1)^2W_1^2-(-s)(r+1)W_0^2+3(r+1)W_1W_2+2(s-r)W_0W_2+(2(-s)-(s-r)(r+ 1))W_0W_1)W_j)W_{m+2}+(((-s)W_0^2-W_1W_2+(r+1)W_0W_2+(s-r)W_0W_1)W_{j+2}+(W_2^2-((r+1)^2+ (s-r))W_0W_2-((r+1)(s-r)+(-s))W_0W_1)W_{j+1}+((r+1)W_2^2-2((-s)+(r+1)(s-r))W_1^2-2(s- r)(-s)W_0^2+(2(s-r)-(r+1)^2)W_1W_2+(2(-s)-(r+1)(s-r))W_0W_2-(2(s-r)^2+(r+1)(-s))W_0W_1) W_j)W_{m+1}+((W_2^2-(s-r)W_1^2-(r+1)W_1W_2-(-s)W_0W_1)W_{j+2}+(-(r+1)W_2^2+((r+1)(s-r)+ (-s))W_1^2+(r+1)^2W_1W_2-(-s)W_0W_2+(r+1)(-s)W_0W_1)W_{j+1}+(-s)(-(r+1)W_1^2-2(-s)W_0^2+ 2W_1W_2-(r+1)W_0W_2-2(s-r)W_0W_1)W_j)W_m-(r+1)(-s)W_0^2W_{j+1}W_{m+1}),$

UNDER PEER REVIEW

$\Theta_6 = (W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 + ((r+1)^2 - (s-r))W_1^2W_2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2)W_j$,

and

$\Gamma_W(z) = z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4$,

$z^3\Gamma_1 = z^3(-(-s)^m(W_2^3 + ((-s)+(r+1)(s-r))W_1^3 + (-s)^2W_0^3 + ((r+1)^2 - (s-r))W_1^2W_2 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 + (r+1)(-s)W_0^2W_2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2))$,

$z^2\Gamma_2 = z^2((3W_2 - 2(r+1)W_1 - (s-r)W_0)W_{m+2}^2 + (((r+1)^2 - (s-r))W_2 + (3(r+1)(s-r) + 3(-s))W_1 + ((s-r)^2 + (r+1)(-s))W_0)W_{m+1}^2 + (-s)((r+1)W_2 + 2(s-r)W_1 + 3(-s)W_0)W_m^2 + (-4(r+1)W_2 + 2((r+1)^2 - (s-r))W_1 + ((r+1)(s-r) - 3(-s))W_0)W_{m+2}W_{m+1} + (-2(s-r)W_2 + ((r+1)(s-r) - 3(-s))W_1 + 2(r+1)(-s)W_0)W_{m+2}W_m + (((r+1)(s-r) - 3(-s))W_2 + 2((s-r)^2 + (r+1)(-s))W_1 + 4(s-r)(-s)W_0)W_{m+1}W_m)$,

$z\Gamma_3 = z((-3W_2^2 + ((s-r) - (r+1)^2)W_1^2 - (-s)(r+1)W_0^2 + 4(r+1)W_1W_2 + 2(s-r)W_0W_2 + (3(-s) - (s-r)(r+1))W_0W_1)W_{m+2} + (2(r+1)W_2^2 - (3(r+1)(s-r) + 3(-s))W_1^2 - 2(s-r)(-s)W_0^2 + (2(s-r) - 2(r+1)^2)W_1W_2 + (3(-s) - (r+1)(s-r))W_0W_2 - 2((s-r)^2 + (r+1)(-s))W_0W_1) W_{m+1} + ((s-r)W_2^2 - ((s-r)^2 + (r+1)(-s))W_1^2 - 3(-s)^2W_0^2 + (3(-s) - (r+1)(s-r))W_1W_2 - 2(r+1)(-s)W_0W_2 - 4(s-r)(-s)W_0W_1)W_m)$,

$\Gamma_4 = W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 + ((r+1)^2 - (s-r))W_1^2W_2 - (s-r)W_0W_2^2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2$.

(b): *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c): *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d): *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 61]. $\square$

Now, we consider special cases of the last Theorem.

THEOREM 26. *We have the following sum formulas.*

(a): $(m=1, j=0)$.

(i): *If* $z^3(-(-s))+z^2(-1)(s-r)+z(-1)(r+1)+1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_1}{z^3(-(-s))+z^2(-1)(s-r)+z(-1)(r+1)+1}$$

where

$\Omega_1 = z^{n+3}(-1)(-s)W_n + z^{n+2}((r+1)W_{n+1} - W_{n+2}) + z^{n+1}(-1)W_{n+1} + z^2(W_2 - (r+1)W_1 - (s-r)W_0) + z(W_1 - (r+1)W_0) + W_0.$

(ii): *If* $z^3(-(-s))+z^2(-1)(s-r)+z(-1)(r+1)+1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_2}{3z^2(-(-s))+2z(-1)(s-r)+(-1)(r+1)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-1)(-s)W_n + (n+2)z^{n+1}((r+1)W_{n+1} - W_{n+2}) + (n+1)z^n(-1)W_{n+1} + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0).$

(iii): *If* $z^3(-(-s))+z^2(-1)(s-r)+z(-1)(r+1)+1 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_3}{6z(-(-s))+2(-1)(s-r)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-1)(-s)W_n + (n+2)(n+1)z^n((r+1)W_{n+1} - W_{n+2}) + (n+1)nz^{n-1}(-1)W_{n+1} + 2(W_2 - (r+1)W_1 - (s-r)W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_4}{3z^2(-(-s))+2z(-1)(s-r)+(-1)(r+1)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-1)(-s)W_n + (n+2)z^{n+1}((r+1)W_{n+1} - W_{n+2}) + (n+1)z^n(-1)W_{n+1} + 2z(W_2 - (r+1)W_1 - (s-r)W_0) + (W_1 - (r+1)W_0).$

(iv): *If* $z^3(-(-s))+z^2(-1)(s-r)+z(-1)(r+1)+1 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_5}{6(-(-s))}$$

where

UNDER PEER REVIEW

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)(-s)W_n + (n+2)(n+1)nz^{n-1}((r+1)W_{n+1} - W_{n+2}) + (n+1)n(n-1)z^{n-2}(-1)W_{n+1}$.

(b): $(m = 2, j = 0)$.

(i): *If* $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2+2s+1) + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_1}{z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2+2s+1) + 1}$$

where

$\Omega_1 = z^{n+3}(-1)(-s)^2W_{2n} + z^{n+2}((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + z^{n+1}(-1)W_{2n+2} + z^2(-(s-r)W_2 + ((-s) + (r+1)(s-r))W_1 + ((s-r)^2 - (r+1)(-s))W_0) + z(W_2 - ((r+1)^2 + 2(s-r))W_0) + W_0$.

(ii): *If* $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2+2s+1) + 1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_2}{3z^2(-(-s)^2) + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)(r^2+2s+1)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-1)(-s)^2W_{2n} + (n+2)z^{n+1}((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + (n+1)z^n(-1)W_{2n+2} + 2z(-(s-r)W_2 + ((-s) + (r+1)(s-r))W_1 + ((s-r)^2 - (r+1)(-s))W_0) + (W_2 - ((r+1)^2 + 2(s-r))W_0)$.

(iii): *If* $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2+2s+1) + 1 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_3}{6z(-(-s)^2) + 2(-2(r+1)(-s) + (s-r)^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-1)(-s)^2W_{2n} + (n+2)(n+1)z^n((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + (n+1)nz^{n-1}(-1)W_{2n+2} + 2(-(s-r)W_2 + ((-s) + (r+1)(s-r))W_1 + ((s-r)^2 - (r+1)(-s))W_0)$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_4}{3z^2(-(-s)^2) + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)(r^2+2s+1)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-1)(-s)^2W_{2n} + (n+2)z^{n+1}((s-r)W_{2n+2} - ((r+1)(s-r) + (-s))W_{2n+1} - (r+1)(-s)W_{2n}) + (n+1)z^n(-1)W_{2n+2} + 2z(-(s-r)W_2 + ((-s) + (r+1)(s-r))W_1 + ((s-r)^2 - (r+1)(-s))W_0) + (W_2 - ((r+1)^2 + 2(s-r))W_0)$.

(iv): *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_5}{6(-(-s)^2)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-1)(-s)^2W_{2n} + (n+2)(n+1)nz^{n-1}((s-r)W_{2n+2} - ((r+1)(s-r)+(-s))W_{2n+1} - (r+1)(-s)W_{2n}) + (n+1)n(n-1)z^{n-2}(-1)W_{2n+2}.$

(c): $(m=2, j=1)$.

(i): *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_1}{z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1}$$

where

$\Omega_1 = z^{n+3}(-(-s)^2W_{2n+1}) + z^{n+2}(-(-s)W_{2n+2} + ((s-r)^2-(r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + z^{n+1}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + z^2(-s)(W_2-(r+1)W_1-(s-r)W_0) + z((r+1)W_2 - ((r+1)^2+(s-r))W_1 + (-s)W_0) + W_1.$

(ii): *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1=u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-(-s)^2)+2z(-2(r+1)(-s)+(s-r)^2)+(-1)(r^2+2s+1)}$$

where

$\Omega_2 = (n+3)z^{n+2}(-(-s)^2W_{2n+1}) + (n+2)z^{n+1}(-(-s)W_{2n+2} + ((s-r)^2-(r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)z^n(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2z(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2+(s-r))W_1 + (-s)W_0).$

(iii): *If* $z^3(-(-s)^2)+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)(r^2+2s+1)+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_3}{6z(-(-s)^2)+2(-2(r+1)(-s)+(s-r)^2)}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}(-(-s)^2W_{2n+1}) + (n+2)(n+1)z^n(-(-s)W_{2n+2} + ((s-r)^2-(r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)nz^{n-1}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2(-s)(W_2-(r+1)W_1-(s-r)W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-(-s)^2)+2z(-2(r+1)(-s)+(s-r)^2)+(-1)(r^2+2s+1)}$$

where

$\Omega_4 = (n+3)z^{n+2}(-(-s)^2W_{2n+1}) + (n+2)z^{n+1}(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)z^n(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n}) + 2z(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0)$.

(iv): *If* $z^3(-(-s)^2) + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)(r^2+2s+1) + 1 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_5}{6(-(-s)^2)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n(-(-s)^2W_{2n+1}) + (n+2)(n+1)nz^{n-1}(-(-s)W_{2n+2} + ((s-r)^2 - (r+1)(-s))W_{2n+1} + (s-r)(-s)W_{2n}) + (n+1)n(n-1)z^{n-2}(-1)((r+1)W_{2n+2} + (s-r)W_{2n+1} + (-s)W_{2n})$.

7.2. The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials in Terms of Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo Polynomials. We can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo polynomials.

Theorem 27. *For all integers* m *and* j, *we have the following sum formulas.*

(a): *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4} = \frac{\Theta_G(z)}{\Gamma_G(z)}$$

where

$\Theta_G(z) = z^{n+3}\Theta_1 + z^{n+2}\Theta_2 + z^{n+1}\Theta_3 + z^2\Theta_4 + z\Theta_5 + \Theta_6$,

$z^{n+3}\Theta_1 = z^{n+3}((-W_jG_{m+2}^2 + (-W_{j+2} + (s-r)W_j)G_{m+1}^2 - (-s)W_{j+1}G_m^2 + (W_{j+1} + (r+1)W_j) G_{m+2}G_{m+1} + (W_{j+2} - (r+1)W_{j+1})G_{m+2}G_m + (-(s-r)W_{j+1} + (-s)W_j)G_{m+1}G_m)G_{m+mn+2} + ((-W_{j+1} + (r+1)W_j)G_{m+2}^2 - ((-s) + (r+1)(s-r))W_jG_{m+1}^2 + (-s)(-W_{j+2} + (r+1)W_{j+1})G_m^2 + (W_{j+2} - (r+1)^2W_j)G_{m+2}G_{m+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} + (-s)W_j)G_{m+2}G_m + (-(s-r)W_{j+2} + ((-s) + (r+1)(s-r))W_{j+1} - (r+1)(-s)W_j)G_{m+1}G_m)G_{m+mn+1} + ((-W_{j+2} + (r+1)W_{j+1} + (s-r)W_j)G_{m+2}^2 + ((s-r)W_{j+2} - ((-s) + (r+1)(s-r))W_{j+1} - (s-r)^2W_j)G_{m+1}^2 - W_j(-s)^2G_m^2 + (W_{j+2}(r+1) - (r+1)^2W_{j+1} + ((-s) - (r+1)(s-r))W_j)G_{m+2}G_{m+1} + (-s)(W_{j+1} - (r+1)W_j)G_{m+2}G_m + (-s)(W_{j+2} - (r+1)W_{j+1} - 2(s-r)W_j)G_{m+1}G_m)G_{m+mn})$,

$z^{n+2}\Theta_2 = z^{n+2}(((((r+1)W_j - W_{j+1})G_{m+2} + (2W_{j+2} - (r+1)W_{j+1} - ((r+1)^2 + 2(s-r))W_j)G_{m+1} + (-(r+1)W_{j+2} + ((r+1)^2 + (s-r))W_{j+1} - (-s)W_j)G_m)G_{m+mn+2} + ((2(r+1)W_{j+1} - W_{j+2} - (r+1)^2W_j)G_{m+2} + (-(r+1)W_{j+2} + ((r+1)^3 + 2(-s) + 2(r+1)(s-r))W_j)G_{m+1} + (((r+1)^2 + (s-r))W_{j+2} -$

$((r+1)^3+2(r+1)(s-r)+(-s))W_{j+1})G_m)G_{m+mn+1}+(((r+1)W_{j+2}-(r+1)^2W_{j+1}-((r+1)(s-r)+(-s))W_j)G_{m+2}+(-((r+1)^2+2(s-r))W_{j+2}+((r+1)^3+2(r+1)(s-r)+2(-s))W_{j+1}+((r+1)^2(s-r)+2(s-r)^2-(r+1)(-s))W_j)G_{m+1}+2(s-r)(-s)W_jG_m-(-s)G_mW_{j+2}+(r+1)^2(-s)W_jG_m)G_{m+mn}),$

$z^{n+1}\Theta_3 = z^{n+1}((-W_{j+2}+(r+1)W_{j+1}+(s-r)W_j)G_{m+mn+2}+((r+1)W_{j+2}-(r+1)^2W_{j+1}-((r+1)(s-r)+(-s))W_j)G_{m+mn+1}+((s-r)W_{j+2}-((r+1)(s-r)+(-s))W_{j+1}-((s-r)^2-(r+1)(-s))W_j)G_{m+mn}),$

$z^2\Theta_4 = z^2(G_{m+2}^2W_{j+1}+((r+1)W_{j+2}+(-s)W_j)G_{m+1}^2+(-s)G_m^2W_{j+2}-((r+1)W_{j+1}+W_{j+2})G_{m+1}G_{m+2}-((s-r)W_{j+1}+(-s)W_j)G_mG_{m+2}+((s-r)W_{j+2}-(-s)W_{j+1})G_mG_{m+1}),$

$z\Theta_5 = z((W_{j+2}-(r+1)W_{j+1})G_{m+2}+(-(r+1)W_{j+2}+(r+1)^2W_{j+1}-2(-s)W_j)G_{m+1}+(-(s-r)W_{j+2}+((-s)W_{j+1}+(r+1)(s-r)W_{j+1})+(r+1)(-s)W_j)G_m),$

$\Theta_6 = (-s)W_j,$

and

$\Gamma_G(z) = \Gamma_1(z)+\Gamma_2(z)+\Gamma_3(z)+\Gamma_4(z),$

$z^3\Gamma_1 = z^3(-(-s)^{m+1}),$

$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2+((r+1)^3+2(r+1)(s-r)+3(-s))G_{m+1}^2+((r+1)^2(-s)+2(s-r)(-s))G_m^2-2((r+1)^2+(s-r))G_{m+1}G_{m+2}-((r+1)(s-r)+3(-s))G_mG_{m+2}+((r+1)^2(s-r)+2(s-r)^2-(r+1)(-s))G_mG_{m+1}),$

$z\Gamma_3 = z((s-r)G_{m+2}-((r+1)(s-r)+3(-s))G_{m+1}+(2(r+1)(-s)-(s-r)^2)G_m),$

$\Gamma_4 = (-s).$

(b): *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4 = u(z-a)(z-b)(z-c) = 0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^kW_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}.$$

(c): *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$, *i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^kW_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1+(n+2)(n+1)z^n\Theta_2+(n+1)nz^{n-1}\Theta_3+2\Theta_4}{6z\Gamma_1+2\Gamma_2},$$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^kW_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1+(n+2)z^{n+1}\Theta_2+(n+1)z^n\Theta_3+2z\Theta_4+\Theta_5}{3z^2\Gamma_1+2z\Gamma_2+\Gamma_3}.$$

(d): *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4 = u(z-a)^3 = 0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^kW_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1+(n+2)(n+1)nz^{n-1}\Theta_2+(n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace r,s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 63]. □

Now, we consider special cases of the last Theorem.

THEOREM 28. *We have the following sum formulas.*

(a): $(m=1, j=0)$.

(i): *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_1}{z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1}$$

where

$\Omega_1 = z^{n+3}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+z^{n+2}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+z^{n+1}(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+z^2(W_2-(r+1)W_1-(s-r)W_0)+z(W_1-(r+1)W_0)+W_0.$

(ii): *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1 = u(z-a)(z-b)(z-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_2}{3z^2(-1)(-s)+2z(-(s-r))+(-(r+1))}$$

where

$\Omega_2 = (n+3)z^{n+2}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)z^{n+1}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)z^n(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+2z(W_2-(r+1)W_1-(s-r)W_0)+(W_1-(r+1)W_0).$

(iii): *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_3}{6z(-1)(-s)+2(-(s-r))}$$

where

$\Omega_3 = (n+3)(n+2)z^{n+1}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)(n+1)z^n((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)nz^{n-1}(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+2(W_2-(r+1)W_1-(s-r)W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_4}{3z^2(-1)(-s)+2z(-(s-r))+(-(r+1))}$$

where

$\Omega_4 = (n+3)z^{n+2}((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)z^{n+1}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)z^n(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)+2z(W_2-(r+1)W_1-(s-r)W_0)+(W_1-(r+1)W_0)$

(iv): *If* $z^3(-1)(-s)+z^2(-(s-r))+z(-(r+1))+1 = u(z-a)^3 = 0$ *for some* $u,a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_k = \frac{\Omega_5}{6(-1)(-s)}$$

where

$\Omega_5 = (n+3)(n+2)(n+1)z^n((-W_2+(r+1)W_1+(s-r)W_0)G_{n+2}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_{n+1}+((s-r)W_2-((r+1)(s-r)+(-s))W_1+((r+1)(-s)-(s-r)^2)W_0)G_n)+(n+2)(n+1)nz^{n-1}((-W_1+(r+1)W_0)G_{n+2}-(W_2-2(r+1)W_1+(r+1)^2W_0)G_{n+1}+((r+1)W_2-(r+1)^2W_1-((r+1)(s-r)+(-s))W_0)G_n)+(n+1)n(n-1)z^{n-2}(-W_0G_{n+2}+(-W_1+(r+1)W_0)G_{n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_n)$

(b): $(m=2, j=0)$.

(i): *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_1}{z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1}$$

where

$\Omega_1 = z^{n+3}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+z^{n+2}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+z^{n+1}(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+z^2(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)+z(W_2-((r+1)^2+2(s-r))W_0)+W_0.$

(ii): *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u,a,b,c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z=a$ *or* $z=b$ *or* $z=c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_2}{3z^2(-1)(-s)^2+2z((s-r)^2-2(r+1)(-s))+(-((r+1)^2+2(s-r)))}$$

where

$\Omega_2 = (n+3)z^{n+2}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)z^{n+1}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+$

$1)^2W_0)G_{2n})+(n+1)z^n(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+2z(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)+(W_2-((r+1)^2+2(s-r))W_0).$

(iii): *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1=u(z-a)^2(z-b)=0$ *for some* $u,a,b\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b$*, i.e.,* $z=a$ *or* $z=b$ *then for* $z=a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_3}{6z(-1)(-s)^2+2((s-r)^2-2(r+1)(-s))}$$

where

$\Omega_3=(n+3)(n+2)z^{n+1}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)(n+1)z^n((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+(n+1)nz^{n-1}(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+2(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)$

and for $z=b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_4}{3z^2(-1)(-s)^2+2z((s-r)^2-2(r+1)(-s))+(-((r+1)^2+2(s-r)))}$$

where

$\Omega_4=(n+3)z^{n+2}(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)z^{n+1}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+(n+1)z^n(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n})+2z(-(s-r)W_2+((r+1)(s-r)+(-s))W_1+((s-r)^2-(r+1)(-s))W_0)+(W_2-((r+1)^2+2(s-r))W_0).$

(iv): *If* $z^3(-1)(-s)^2+z^2((s-r)^2-2(r+1)(-s))+z(-((r+1)^2+2(s-r)))+1=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $z=a$*, then*

$$\sum_{k=0}^{n} z^k W_{2k} = \frac{\Omega_5}{6(-1)(-s)^2}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-s)((-W_2+(r+1)W_1+(s-r)W_0)G_{2n+2}+((r+1)W_2-(r+1)^2W_1-((-s)+(r+1)(s-r))W_0)G_{2n+1}+((s-r)W_2-((-s)+(r+1)(s-r))W_1+((r+1)(-s)-(s-r)^2)W_0)G_{2n})+(n+2)(n+1)nz^{n-1}((-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n+2}+(((r+1)^2+(s-r))W_2-((r+1)^3+2(r+1)(s-r)+(-s))W_1)G_{2n+1}+(-s)(-W_2+2(s-r)W_0+(r+1)^2W_0)G_{2n})+(n+1)n(n-1)z^{n-2}(-W_1G_{2n+2}+((r+1)W_1-W_2)G_{2n+1}-(-s)W_0G_{2n}).$

(c): $(m=2,j=1)$.

(i): *If* $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 \neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_1}{z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1}$$

where

$$\Omega_1 = z^{n+3}(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) + z^{n+2}(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0)G_{2n+2} + (-((r+1)(s-r) + (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0)G_{2n+1} + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) + z^{n+1}(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1 G_{2n}) + z^2(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + z((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0) + W_1.$$

(ii): *If* $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_2}{3z^2(-1)(-s)^2 + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)((r+1)^2 + 2(s-r))}$$

where

$$\Omega_2 = (n+3)z^{n+2}(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) + (n+2)z^{n+1}(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0)G_{2n+2} + (-((r+1)(s-r) + (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0)G_{2n+1} + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) + (n+1)z^n(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1 G_{2n}) + 2z(-s)(W_2 - (r+1)W_1 - (s-r)W_0) + ((r+1)W_2 - ((r+1)^2 + (s-r))W_1 + (-s)W_0).$$

(iii): *If* $z^3(-1)(-s)^2 + z^2(-2(r+1)(-s) + (s-r)^2) + z(-1)((r+1)^2 + 2(s-r)) + 1 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_3}{6z(-1)(-s)^2 + 2(-2(r+1)(-s) + (s-r)^2)}$$

where

$$\Omega_3 = (n+3)(n+2)z^{n+1}(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) + (n+2)(n+1)z^n(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0) G_{2n+2} + (-((r+1)(s-r) + (-s))W_2 + (s-r)((r+1)^2 + (s-r))W_1 + (-s)((s-r) + (r+1)^2)W_0) G_{2n+1} + (-s)(-(r+1)W_2 + ((r+1)^2 + (s-r))W_1 - (-s)W_0)G_{2n}) + (n+1)nz^{n-1}(-W_2G_{2n+2} - ((s-r)W_1 + (-s)W_0)G_{2n+1} - (-s)W_1G_{2n}) + 2(-s)(W_2 - (r+1)W_1 - (s-r)W_0)$$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{2k+1} = \frac{\Omega_4}{3z^2(-1)(-s)^2 + 2z(-2(r+1)(-s) + (s-r)^2) + (-1)((r+1)^2 + 2(s-r))}$$

where

$$\Omega_4 = (n+3)z^{n+2}(-s)^2(-W_0G_{2n+2} + ((r+1)W_0 - W_1)G_{2n+1} + (-W_2 + (r+1)W_1 + (s-r)W_0)G_{2n}) + (n+2)z^{n+1}(((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 - (r+1)(-s)W_0)G_{2n+2} + (-((r+1)(s-r) +$$

$(-s))W_2+(s-r)((r+1)^2+(s-r))W_1+(-s)((s-r)+(r+1)^2)W_0)G_{2n+1}+(-s)(-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n})+(n+1)z^n(-W_2G_{2n+2}-((s-r)W_1+(-s)W_0)G_{2n+1}-(-s)W_1 G_{2n})+2z(-s)(W_2-(r+1)W_1-(s-r)W_0)+((r+1)W_2-((r+1)^2+(s-r))W_1+(-s)W_0)$.

(iv): *If* $z^3(-1)(-s)^2+z^2(-2(r+1)(-s)+(s-r)^2)+z(-1)((r+1)^2+2(s-r))+1=u(z-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$, *i.e.,* $z=a$, *then*

$$\sum_{k=0}^{n} z^k W_{2k+1}=\frac{\Omega_5}{6(-1)(-s)^2}$$

where

$\Omega_5=(n+3)(n+2)(n+1)z^n(-s)^2(-W_0G_{2n+2}+((r+1)W_0-W_1)G_{2n+1}+(-W_2+(r+1)W_1+(s-r)W_0)G_{2n})+(n+2)(n+1)nz^{n-1}(((s-r)W_2-((r+1)(s-r)+(-s))W_1-(r+1)(-s)W_0) G_{2n+2}+(-((r+1)(s-r)+(-s))W_2+(s-r)((r+1)^2+(s-r))W_1+(-s)((s-r)+(r+1)^2)W_0)G_{2n+1}+(-s)(-(r+1)W_2+((r+1)^2+(s-r))W_1-(-s)W_0)G_{2n})+(n+1)n(n-1)z^{n-2}(-W_2G_{2n+2}-((s-r)W_1+(-s)W_0)G_{2n+1}-(-s)W_1G_{2n})$.

7.3. The Sum Formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of Generalized Horadam-Leonardo Polynomials in Terms of Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo-Lucas Polynomials. We can give the sum formula $\sum_{k=0}^{n} z^k W_{mk+j}$ of generalized Horadam-Leonardo polynomials in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials.

THEOREM 29. *For all integers m and j, we have the followings sum formulas.*

(a): *If* $z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4\neq 0$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4} = \frac{\Theta_W(z)}{\Gamma_W(z)}$$

where

$\Theta_W(z)=z^{n+3}\Theta_1+z^{n+2}\Theta_2+z^{n+1}\Theta_3+z^2\Theta_4+z\Theta_5+\Theta_6$,

$z^{n+3}\Theta_1 = z^{n+3}((-W_jH_{m+2}^2+(-W_{j+2}+(s-r)W_j)H_{m+1}^2-(-s)W_{j+1}H_m^2+(W_{j+1}+(r+1)W_j)H_{m+2}H_{m+1}+(W_{j+2}-(r+1)W_{j+1})H_{m+2}H_m+(-(s-r)W_{j+1}+(-s)W_j)H_{m+1}H_m)H_{m+mn+2}+((-W_{j+1}+(r+1)W_j)H_{m+2}^2-((-s)+(r+1)(s-r))W_jH_{m+1}^2+(-s)(-W_{j+2}+(r+1)W_{j+1})H_m^2+(W_{j+2}-(r+1)^2W_j)H_{m+2}H_{m+1}+(-(r+1)W_{j+2}+((r+1)^2+(s-r))W_{j+1}+(-s)W_j)H_{m+2}H_m+(-(s-r)W_{j+2}+((r+1)(s-r)+(-s))W_{j+1}-(r+1)(-s)W_j)H_{m+1}H_m)H_{m+mn+1}+((-W_{j+2}+(r+1)W_{j+1}+(s-r)W_j)H_{m+2}^2+((s-r)W_{j+2}-((r+1)(s-r)+(-s))W_{j+1}-(s-r)^2W_j)H_{m+1}^2-W_j(-s)^2H_m^2+((r+1)W_{j+2}-(r+1)^2W_{j+1}+((-s)-(r+1)(s-r))W_j)H_{m+2}H_{m+1}+(-s)(W_{j+1}-(r+1)W_j)H_{m+2}H_m+(-s)(W_{j+2}-(r+1)W_{j+1}-2(s-r)W_j)H_{m+1}H_m)H_{m+mn})$,

$z^{n+2}\Theta_2 = z^{n+2}(((-3W_{j+2}+2(r+1)W_{j+1}+((r+1)^2+4(s-r))W_j)H_{m+2}+(2(r+1)W_{j+2}+((s-r)-(r+1)^2)W_{j+1}-((r+1)^3+4(r+1)(s-r)+3(-s))W_j)H_{m+1}+(-((r+1)^2+2(s-r))W_{j+2}+((r+1)^3+3(r+1)(s-r)+6(-s))W_{j+1}-(r+1)(-s)W_j)H_m)H_{m+mn+2}+((2(r+1)W_{j+2}-((r+1)^2-(s-r))W_{j+1}-((r+1)^3+3(-s)+4(r+1)(s-r))W_j)H_{m+2}+(-((r+1)^2-(s-r))W_{j+2}-3((-s)+(r+1)(s-r))W_{j+1}+(r+1)((r+1)^3+4(r+1)(s-r)+5(-s))W_j)H_{m+1}+(((r+1)^3+3(r+1)(s-r)+6(-s))W_{j+2}-((r+1)^4+4(r+1)^2(s-r)+2(s-r)^2+7(r+1)(-s))W_{j+1}-2(s-r)(-s)W_j)H_m)H_{m+mn+1}+((((r+1)^2+4(s-r))W_{j+2}-((r+1)^3+4(r+1)(s-r)+3(-s))W_{j+1}-((r+1)^2(s-r)-2(r+1)(-s)+4(s-r)^2)W_j)H_{m+2}+(-((r+1)^3+4(r+1)(s-r)+3(-s))W_{j+2}+((r+1)^4+4(r+1)^2(s-r)+5(r+1)(-s))W_{j+1}+((r+1)^3(s-r)-(r+1)^2(-s)+4(r+1)(s-r)^2+4(s-r)(-s))W_j)H_{m+1}+(-s)(-(r+1)W_{j+2}-2(s-r)W_{j+1}+((r+1)^3+4(r+1)(s-r)+6(-s))W_j)H_m)H_{m+mn}),$

$z^{n+1}\Theta_3 = z^{n+1}((2(3(s-r)+(r+1)^2)W_{j+2}-(2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+1}-((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)W_j)H_{m+mn+2}+(-(2(r+1)^3+9(-s)+7(r+1)(s-r))W_{j+2}+2((r+1)^4+4(r+1)^2(s-r)+6(r+1)(-s)+(s-r)^2)W_{j+1}+((r+1)^3(s-r)-(r+1)^2(-s)+4(r+1)(s-r)^2+6(s-r)(-s))W_j)H_{m+mn+1}+(-((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)W_{j+2}+((r+1)^3(s-r)-(r+1)^2(-s)+4(r+1)(s-r)^2+6(s-r)(-s))W_{j+1}+(-2(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-9(-s)^2-10(r+1)(s-r)(-s))W_j)H_{m+mn}),$

$z^2\Theta_4 = z^2((3W_{j+2}-2(r+1)W_{j+1}-(s-r)W_j)H_{m+2}^2+(((r+1)^2-(s-r))W_{j+2}+3((r+1)(s-r)+(-s))W_{j+1}+((s-r)^2+(r+1)(-s))W_j)H_{m+1}^2+(-s)((r+1)W_{j+2}+2(s-r)W_{j+1}+3(-s)W_j)H_m^2+(-4(r+1)W_{j+2}+2((r+1)^2-(s-r))W_{j+1}+((r+1)(s-r)-3(-s))W_j)H_{m+2}H_{m+1}+(-2(s-r)W_{j+2}+((r+1)(s-r)-3(-s))W_{j+1}+2(r+1)(-s)W_j)H_{m+2}H_m+(((r+1)(s-r)-3(-s))W_{j+2}+2((s-r)^2+(r+1)(-s))W_{j+1}+4(s-r)(-s)W_j)H_{m+1}H_m),$

$z\Theta_5 = z((-2((r+1)^2+3(s-r))W_{j+2}+(2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+1}+((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)W_j)H_{m+2}+((2(r+1)^3+7(r+1)(s-r)+9(-s))W_{j+2}-2((r+1)^4+4(r+1)^2(s-r)+6(r+1)(-s)+(s-r)^2)W_{j+1}-((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))W_j)H_{m+1}+(((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))W_{j+2}-(4(r+1)(s-r)^2+(r+1)^3(s-r)-(r+1)^2(-s)+6(s-r)(-s))W_{j+1}-2(-s)((r+1)^3+4(r+1)(s-r)+9(-s))W_j)H_m),$

$\Theta_6 = (4(r+1)^3(-s)-(r+1)^2(s-r)^2-4(s-r)^3+27(-s)^2+18(r+1)(s-r)(-s))W_j,$

and

$\Gamma_W(z) = \Gamma_1(z)+\Gamma_2(z)+\Gamma_3(z)+\Gamma_4(z),$

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2)),$

$z^2\Gamma_2 = z^2(((r+1)^2+3(s-r))H_{m+2}^2+((r+1)^4+4(r+1)^2(s-r)+(s-r)^2+6(r+1)(-s))H_{m+1}^2+(-s)((r+1)^3+4(r+1)(s-r)+9(-s))H_m^2-(2(r+1)^3+7(r+1)(s-r)+9(-s))H_{m+2}H_{m+1}-((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))H_{m+2}H_m+(r+1)(s-r)((r+1)^2+4(s-r))H_{m+1}H_m-(-s)((r+1)^2-6(s-r))H_{m+1}H_m),$

$z\Gamma_3 = z(-4(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-27(-s)^2-18(r+1)(s-r)(-s))H_m,$

$\Gamma_4 = 4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2.$

(b): *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)(z-b)(z-c) = 0$ *for some* $u, a, b, c \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b \neq c$, *i.e.,* $z = a$ *or* $z = b$ *or* $z = c$ *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(c): *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^2(z-b) = 0$ *for some* $u, a, b \in \mathbb{C}$ *with* $u \neq 0$ *and* $a \neq b$, *i.e.,* $z = a$ *or* $z = b$ *then for* $z = a$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)z^{n+1}\Theta_1 + (n+2)(n+1)z^n\Theta_2 + (n+1)nz^{n-1}\Theta_3 + 2\Theta_4}{6z\Gamma_1 + 2\Gamma_2},$$

and for $z = b$ *we get*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)z^{n+2}\Theta_1 + (n+2)z^{n+1}\Theta_2 + (n+1)z^n\Theta_3 + 2z\Theta_4 + \Theta_5}{3z^2\Gamma_1 + 2z\Gamma_2 + \Gamma_3}.$$

(d): *If* $z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4 = u(z-a)^3 = 0$ *for some* $u, a \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $z = a$, *then*

$$\sum_{k=0}^{n} z^k W_{mk+j} = \frac{(n+3)(n+2)(n+1)z^n\Theta_1 + (n+2)(n+1)nz^{n-1}\Theta_2 + (n+1)n(n-1)z^{n-2}\Theta_3}{6\Gamma_1}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Theorem 65]. □

8. Generating Function of Generalized Horadam-Leonardo Polynomials

In this section, we present generating function of the sequence W_{mn+j} and its special cases.

8.1. Generating Function of Generalized Horadam-Leonardo Polynomials via Generalized Horadam-Leonardo Polynomials.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials).

Lemma 30. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, 1\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$ *is the ordinary generating function of the generalized Horadam-Leonardo (sequence of) polynomials* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 25 (a))

$z^2\Theta_4 = z^2((W_0W_{j+2} + (W_1 - (r+1)W_0)W_{j+1} + (W_2 - (r+1)W_1 - (s-r)W_0)W_j)W_{m+2}^2 + ((W_2 - (s-r)W_0)W_{j+2} + ((-s)W_0 + (r+1)(s-r)W_0)W_{j+1} + ((s-r)^2W_0 + ((r+1)(s-r) + (-s))W_1 - (s-r)W_2)W_j)W_{m+1}^2 + (-s)(W_1W_{j+2} + (W_2 - (r+1)W_1)W_{j+1} + (-s)W_0W_j)W_m^2 + (-(W_1 + (r+1)W_0)W_{j+2} + ((r+1)^2W_0 - W_2)W_{j+1} + (-(r+1)W_2 + (r+1)^2W_1 + ((r+1)(s-r) - (-s))W_0)W_j)W_{m+1}W_{m+2} + (((r+1)W_1 - W_2)W_{j+2} + ((r+1)W_2 - ((s-r) + (r+1)^2)W_1 - (-s)W_0)W_{j+1} + (-s)((r+1)W_0 - W_1)W_j)W_{m+2}W_m + (((s-r)W_1 - (-s)W_0)W_{j+2} + ((s-r)W_2 - ((r+1)(s-r) + (-s))W_1 + (r+1)(-s)W_0)W_{j+1} + (-s)(-W_2 + (r+1)W_1 + 2(s-r)W_0)W_j) W_{m+1}W_m)$,

$z\Theta_5 = z(((W_1^2 - W_0W_2)W_{j+2} + ((-s)W_0^2 - W_1W_2 + (r+1)W_0W_2 + (s-r)W_0W_1)W_{j+1} + (-2W_2^2 - (r+1)^2W_1^2 - (-s)(r+1)W_0^2 + 3(r+1)W_1W_2 + 2(s-r)W_0W_2 + (2(-s) - (s-r)(r+1))W_0W_1)W_j)W_{m+2} + (((-s)W_0^2 - W_1W_2 + (r+1)W_0W_2 + (s-r)W_0W_1)W_{j+2} + (W_2^2 - ((r+1)^2 + (s-r))W_0W_2 - ((r+1)(s-r) + (-s))W_0W_1) W_{j+1} + ((r+1)W_2^2 - 2((-s) + (r+1)(s-r))W_1^2 - 2(s-r)(-s)W_0^2 + (2(s-r) - (r+1)^2)W_1W_2 + (2(-s) - (r+1)(s-r))W_0W_2 - (2(s-r)^2 + (r+1)(-s))W_0W_1)W_j)W_{m+1} + ((W_2^2 - (s-r)W_1^2 - (r+1)W_1W_2 - (-s)W_0W_1)W_{j+2} + (-(r+1)W_2^2 + ((r+1)(s-r) + (-s))W_1^2 + (r+1)^2W_1W_2 - (-s)W_0W_2 + (r+1)(-s)W_0W_1) W_{j+1} + (-s)(-(r+1)W_1^2 - 2(-s)W_0^2 + 2W_1W_2 - (r+1)W_0W_2 - 2(s-r)W_0W_1)W_j)W_m - (r+1)(-s)W_0^2W_{j+1} W_{m+1}),$

$\Theta_6 = (W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 + ((r+1)^2 - (s-r))W_1^2W_2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + 2(s-r)(-s)W_0^2W_1 + (r+1)(-s)W_0^2W_2 + ((r+1)(s-r) - 3(-s)) W_0W_1W_2)W_j,$

and

$z^3\Gamma_1 = z^3(-(-s)^m(W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 + ((r+1)^2 - (s-r))W_1^2W_2 - 2(r+1)W_1W_2^2 - (s-r)W_0W_2^2 + (r+1)(-s)W_0^2W_2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2)),$

$z^2\Gamma_2 = z^2((3W_2 - 2(r+1)W_1 - (s-r)W_0)W_{m+2}^2 + (((r+1)^2 - (s-r))W_2 + (3(r+1)(s-r) + 3(-s))W_1 + ((s-r)^2 + (r+1)(-s))W_0)W_{m+1}^2 + (-s)((r+1)W_2 + 2(s-r)W_1 + 3(-s)W_0)W_m^2 + (-4(r+1)W_2 + 2((r+1)^2 - (s-r))W_1 + ((r+1)(s-r) - 3(-s))W_0)W_{m+2}W_{m+1} + (-2(s-r)W_2 + ((r+1)(s-r) - 3(-s))W_1 + 2(r+1)(-s)W_0) W_{m+2}W_m + (((r+1)(s-r) - 3(-s))W_2 + 2((s-r)^2 + (r+1)(-s))W_1 + 4(s-r)(-s)W_0)W_{m+1}W_m),$

$z\Gamma_3 = z((-3W_2^2 + ((s-r) - (r+1)^2)W_1^2 - (-s)(r+1)W_0^2 + 4(r+1)W_1W_2 + 2(s-r)W_0W_2 + (3(-s) - (s-r)(r+1))W_0W_1)W_{m+2} + (2(r+1)W_2^2 - (3(r+1)(s-r) + 3(-s))W_1^2 - 2(s-r)(-s)W_0^2 + (2(s-r) - 2(r+1)^2)W_1W_2 + (3(-s) - (r+1)(s-r))W_0W_2 - 2((s-r)^2 + (r+1)(-s))W_0W_1)W_{m+1} + ((s-r)W_2^2 - ((s-r)^2 + (r+1)(-s))W_1^2 - 3(-s)^2W_0^2 + (3(-s) - (r+1)(s-r))W_1W_2 - 2(r+1)(-s)W_0W_2 - 4(s-r)(-s)W_0W_1) W_m),$

$\Gamma_4 = W_2^3 + ((-s) + (r+1)(s-r))W_1^3 + (-s)^2W_0^3 - 2(r+1)W_1W_2^2 + ((r+1)^2 - (s-r))W_1^2W_2 - (s-r)W_0W_2^2 + ((s-r)^2 + (r+1)(-s))W_0W_1^2 + (r+1)(-s)W_0^2W_2 + 2(s-r)(-s)W_0^2W_1 + ((r+1)(s-r) - 3(-s))W_0W_1W_2.$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Lemma 66]. □

Now, we consider special cases of the last Lemma.

Corollary 31. *The ordinary generating functions of the sequences W_n, W_{2n}, W_{2n+1}, W_{-n}, W_{-2n}, W_{-2n+1} are given as follows:*

(a): $(m = 1, j = 0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, 1\})$.

$$\sum_{n=0}^{\infty} W_n z^n = \frac{z^2(W_2 - (r+1)W_1 - (s-r)W_0) + z(W_1 - (r+1)W_0) + W_0}{z^3 s + z^2(-1)(s-r) + z(-1)(r+1) + 1}.$$

(b): $(m=2, j=0, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\sum_{n=0}^{\infty} W_{2n}z^n = \frac{z^2(-(s-r)W_2 - r(r-s+1)W_1 + (r^2-rs+s^2+s)W_0) + z(W_2-(r^2+2s+1)W_0) + W_0}{z^3(-s^2)+z^2(r^2+s^2+2s)+z(-1)(r^2+2s+1)+1}.$$

(c): $(m=2, j=1, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\sum_{n=0}^{\infty} W_{2n+1}z^n = \frac{z^2(-s)(W_2-(r+1)W_1-(s-r)W_0) + z((r+1)W_2-(r^2+r+s+1)W_1+(-s)W_0) + W_1}{z^3(-s^2)+z^2(r^2+s^2+2s)+z(-1)(r^2+2s+1)+1}.$$

(d): $(m=-1, j=0, |z|<\min\{|\alpha|, |\beta|, 1\})$.

$$\sum_{n=0}^{\infty} W_{-n}z^n = \frac{z^2W_1 + z(W_2-(r+1)W_1) + (-s)W_0}{z^3(-1)+z^2(r+1)+z(s-r)+(-s)}.$$

(e): $(m=-2, j=0, |z|<\min\{|\alpha|^{2}, |\beta|^{2}, 1\})$.

$$\sum_{n=0}^{\infty} W_{-2n}z^n = \frac{z^2W_2 + z(-(s-r)W_2 - r(r-s+1)W_1 + (r+1)(-s)W_0) + s^2W_0}{z^3(-1)+z^2(r^2+2s+1)+z(-1)(r^2+s^2+2s)+s^2}.$$

(f): $(m=-2, j=1, |z|<\min\{|\alpha|^{2}, |\beta|^{2}, 1\})$.

$$\sum_{n=0}^{\infty} W_{-2n+1}z^n = \frac{z^2((r+1)W_2+(s-r)W_1-sW_0) + z(-sW_2-(r^2+s^2+s-rs)W_1 - s(r-s)W_0) + s^2W_1}{z^3(-1)+z^2(r^2+2s+1)+z(-1)(r^2+s^2+2s)+s^2}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 67.]. □

The last Lemma gives the following results as particular examples (generating functions of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials).

Corollary 32. *Assume that $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, 1\}$. Generating functions of (r,s)-Horadam-Leonardo and (r,s)-Horadam-Leonardo-Lucas polynomials are given, respectively, as follows:*

(a):

$$\sum_{n=0}^{\infty} G_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}$$

where (as Theorem 25 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2G_{j+1} + (r+1)G_{m+1}^2G_{j+2} + (-s)G_{m+1}^2G_j + (-s)G_m^2G_{j+2} - G_{m+1}G_{m+2}G_{j+2} + (s-r)G_mG_{m+1}G_{j+2} - (r+1)G_{m+1}G_{m+2}G_{j+1} - (s-r)G_mG_{m+2}G_{j+1} - (-s)G_mG_{m+1}G_{j+1} - (-s)G_mG_{m+2}G_j)$,

$z\Theta_5 = z(G_{m+2}G_{j+2} - (r+1)G_{m+1}G_{j+2} - (r+1)G_{m+2}G_{j+1} - (s-r)G_mG_{j+2} + (r+1)^2G_{m+1}G_{j+1} - 2(-s)G_jG_{m+1} + ((-s)+(r+1)(s-r))G_mG_{j+1} + (r+1)(-s)G_mG_j)$,

$\Theta_6 = (-s)G_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^{m+1})$,

$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + (-s)((r+1)^2 + 2(s-r))G_m^2 - 2((r+1)^2 + (s-r))G_{m+2}G_{m+1} - (3(-s)+(r+1)(s-r))G_{m+2}G_m + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_{m+1}G_m)$,

$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r)+3(-s))G_{m+1} - ((s-r)^2 - 2(r+1)(-s))G_m),$

$\Gamma_4 = (-s).$

(b):

$$\sum_{n=0}^{\infty} H_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as Theorem 25 (a))

$z^2\Theta_4 = z^2((3H_{j+2} - 2(r+1)H_{j+1} - (s-r)H_j)H_{m+2}^2 + (((r+1)^2 - (s-r))H_{j+2} + 3((r+1)(s-r) + (-s))H_{j+1} + ((s-r)^2 + (r+1)(-s))H_j)H_{m+1}^2 + (-s)((r+1)H_{j+2} + 2(s-r)H_{j+1} + 3(-s)H_j)H_m^2 + (-4(r+1)H_{j+2} + 2((r+1)^2 - (s-r))H_{j+1} + ((r+1)(s-r) - 3(-s))H_j)H_{m+2}H_{m+1} + (-2(s-r)H_{j+2} + ((r+1)(s-r) - 3(-s))H_{j+1} + 2(r+1)(-s)H_j)H_{m+2}H_m + (((r+1)(s-r) - 3(-s))H_{j+2} + 2((s-r)^2 + (r+1)(-s))H_{j+1} + 4(s-r)(-s)H_j)H_{m+1}H_m),$

$z\Theta_5 = z((-2((r+1)^2 + 3(s-r))H_{j+2} + (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{j+1} + ((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))H_j)H_{m+2} + +((2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{j+2} - (2(r+1)^4 + 8(r+1)^2(s-r) + 3(r+1)(-s) + 2(s-r)^2)H_{j+1} - ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))H_j)H_{m+1} + (((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)H_{j+2} - ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))H_{j+1} - 2(-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))H_j)H_m - 9(r+1)(-s)H_{j+1}H_{m+1}),$

$\Theta_6 = (4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s))H_j,$

and

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2)),$

$z^2\Gamma_2 = z^2(((r+1)^2 + 3(s-r))H_{m+2}^2 + ((r+1)^4 + 4(r+1)^2(s-r) + (s-r)^2 + 6(r+1)(-s))H_{m+1}^2 + (-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))H_m^2 - (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{m+2}H_{m+1} - ((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)H_{m+2}H_m + ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))H_{m+1}H_m),$

$z\Gamma_3 = z(-4(r+1)^3(-s) + (r+1)^2(s-r)^2 + 4(s-r)^3 - 27(-s)^2 - 18(r+1)(s-r)(-s))H_m,$

$\Gamma_4 = 4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s).$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 68.]. □

Now, we consider special cases of the last two Corollaries.

Corollary 33. *The ordinary generating functions of the sequences G_n, G_{2n}, G_{2n+1}, G_{-n}, G_{-2n}, G_{-2n+1} and H_n, H_{2n}, H_{2n+1}, H_{-n}, H_{-2n}, H_{-2n+1} are given as follows:*

(a): $(m = 1, j = 0, |z| < \min\{|\alpha|^{-1}, |\beta|^{-1}, 1\})$.

$$\sum_{n=0}^{\infty} G_n z^n = \frac{z}{1-(r+1)z-(s-r)z^2+sz^3},$$

$$\sum_{n=0}^{\infty} H_n z^n = \frac{3-2(r+1)z-(s-r)z^2}{1-(r+1)z-(s-r)z^2+sz^3}.$$

(b): $(m=2, j=0, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\sum_{n=0}^{\infty} G_{2n}z^n = \frac{(r+1)z+(-s)z^2}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3},$$

$$\sum_{n=0}^{\infty} H_{2n}z^n = \frac{3-2\left(r^2+2s+1\right)z+(r^2+s^2+2s)z^2}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3}.$$

(c): $(m=2, j=1, |z|<\min\{|\alpha|^{-2}, |\beta|^{-2}, 1\})$.

$$\sum_{n=0}^{\infty} G_{2n+1}z^n = \frac{1-(s-r)z}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3},$$

$$\sum_{n=0}^{\infty} H_{2n+1}z^n = \frac{(r+1)+(rs-2s-r-r^2)z-s(r-s)z^2}{1-(r^2+2s+1)z+(r^2+s^2+2s)z^2-s^2z^3}.$$

(d): $(m=-1, j=0, |z|<\min\{|\alpha|, |\beta|, 1\})$.

$$\sum_{n=0}^{\infty} G_{-n}z^n = \frac{z^2}{(-s)+(s-r)z+(r+1)z^2-z^3},$$

$$\sum_{n=0}^{\infty} H_{-n}z^n = \frac{3(-s)+2(s-r)z+(r+1)z^2}{(-s)+(s-r)z+(r+1)z^2-z^3}.$$

(e): $(m=-2, j=0, |z|<\min\{|\alpha|^{2}, |\beta|^{2}, 1\})$.

$$\sum_{n=0}^{\infty} G_{-2n}z^n = \frac{(-s)z+(r+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3},$$

$$\sum_{n=0}^{\infty} H_{-2n}z^n = \frac{3s^2-2(r^2+s^2+2s)z+(r^2+2s+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3}.$$

(f): $(m=-2, j=1, |z|<\min\{|\alpha|^{2}, |\beta|^{2}, |\gamma|^{2}\})$.

$$\sum_{n=0}^{\infty} G_{-2n+1}z^n = \frac{s^2-(r^2+s^2+2s)z+(r^2+r+s+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3},$$

$$\sum_{n=0}^{\infty} H_{-2n+1}z^n = \frac{(r+1)(-s)^2-(r^3+r^2+3rs+rs^2+2s)z+(r^3+3rs+1)z^2}{s^2-(r^2+s^2+2s)z+(r^2+2s+1)z^2-z^3}.$$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 69.]. □

8.2. Generating Function of Generalized Horadam-Leonardo Polynomials via Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo Polynomials.

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo polynomials).

Lemma 34. *Assume that $|z|<\min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. Suppose that $f_{W_{mn+j}}(z)=\sum_{n=0}^{\infty} W_{mn+j}z^n$ is the ordinary generating function of the generalized Horadam-Leonardo (sequence of) polynomials $\{W_{mn+j}\}$. Then, $\sum_{n=0}^{\infty} W_{mn+j}z^n$ is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4+z\Theta_5+\Theta_6}{z^3\Gamma_1+z^2\Gamma_2+z\Gamma_3+\Gamma_4}$$

where (as in Theorem 27 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2 W_{j+1} + ((r+1)W_{j+2} + (-s)W_j)G_{m+1}^2 + (-s)G_m^2 W_{j+2} - ((r+1)W_{j+1} + W_{j+2})G_{m+1}G_{m+2} - ((s-r)W_{j+1} + (-s)W_j)G_m G_{m+2} + ((s-r)W_{j+2} - (-s)W_{j+1})G_m G_{m+1})$,

$z\Theta_5 = z((W_{j+2} - (r+1)W_{j+1})G_{m+2} + (-(r+1)W_{j+2} + (r+1)^2 W_{j+1} - 2(-s)W_j)G_{m+1} + (-(s-r)W_{j+2} + ((-s)W_{j+1} + (r+1)(s-r)W_{j+1}) + (r+1)(-s)W_j)G_m)$,

$\Theta_6 = (-s)W_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^{m+1})$,

$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + ((r+1)^2(-s) + 2(s-r)(-s))G_m^2 - 2((r+1)^2 + (s-r))G_{m+1}G_{m+2} - ((r+1)(s-r) + 3(-s))G_m G_{m+2} + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_m G_{m+1})$,

$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} + (2(r+1)(-s) - (s-r)^2)G_m)$,

$\Gamma_4 = (-s)$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Lemma 70]. □

Lemma 34 gives the following result as particular example (generating functions of (r,s)-Horadam-Leonardo-Lucas polynomials).

Corollary 35. *Assume that $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. Generating function of (r,s)-Horadam-Leonardo-Lucas polynomials is given, as follows:*

$$\sum_{n=0}^{\infty} H_{mn+j} z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 27 (a))

$z^2\Theta_4 = z^2(G_{m+2}^2 H_{j+1} + ((r+1)H_{j+2} + (-s)H_j)G_{m+1}^2 + (-s)G_m^2 H_{j+2} - ((r+1)H_{j+1} + H_{j+2})G_{m+1}G_{m+2} - ((s-r)H_{j+1} + (-s)H_j)G_m G_{m+2} + ((s-r)H_{j+2} - (-s)H_{j+1})G_m G_{m+1})$,

$z\Theta_5 = z((H_{j+2} - (r+1)H_{j+1})G_{m+2} + (-(r+1)H_{j+2} + (r+1)^2 H_{j+1} - 2(-s)H_j)G_{m+1} + (-(s-r)H_{j+2} + ((-s)H_{j+1} + (r+1)(s-r)H_{j+1}) + (r+1)(-s)H_j)G_m)$,

$\Theta_6 = (-s)H_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^{m+1})$,

$z^2\Gamma_2 = z^2((r+1)G_{m+2}^2 + ((r+1)^3 + 2(r+1)(s-r) + 3(-s))G_{m+1}^2 + ((r+1)^2(-s) + 2(s-r)(-s))G_m^2 - 2((r+1)^2 + (s-r))G_{m+1}G_{m+2} - ((r+1)(s-r) + 3(-s))G_m G_{m+2} + ((r+1)^2(s-r) + 2(s-r)^2 - (r+1)(-s))G_m G_{m+1})$,

$z\Gamma_3 = z((s-r)G_{m+2} - ((r+1)(s-r) + 3(-s))G_{m+1} + (2(r+1)(-s) - (s-r)^2)G_m)$,

$\Gamma_4 = (-s)$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 71.]. □

8.3. Generating Function of Generalized Horadam-Leonardo Polynomials via Generalized Horadam-Leonardo Polynomials and (r,s)-Horadam-Leonardo-Lucas Polynomials. Next, we give

UNDER PEER REVIEW

the ordinary generating function $\sum_{n=0}^{\infty} W_{mn+j}z^n$ of the sequence W_{mn+j} (in terms of elements of the sequence of generalized Horadam-Leonardo polynomials and (r,s)-Horadam-Leonardo-Lucas polynomials).

Lemma 36. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Suppose that* $f_{W_{mn+j}}(z) = \sum_{n=0}^{\infty} W_{mn+j}z^n$ *is the ordinary generating function of the generalized Horadam-Leonardo (sequence of) polynomials* $\{W_{mn+j}\}$. *Then,* $\sum_{n=0}^{\infty} W_{mn+j}z^n$ *is given by*

$$\sum_{n=0}^{\infty} W_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 29 (a))

$z^2\Theta_4 = z^2((3W_{j+2} - 2(r+1)W_{j+1} - (s-r)W_j)H_{m+2}^2 + (((r+1)^2 - (s-r))W_{j+2} + 3((r+1)(s-r) + (-s))W_{j+1} + ((s-r)^2 + (r+1)(-s))W_j)H_{m+1}^2 + (-s)((r+1)W_{j+2} + 2(s-r)W_{j+1} + 3(-s)W_j)H_m^2 + (-4(r+1)W_{j+2} + 2((r+1)^2 - (s-r))W_{j+1} + ((r+1)(s-r) - 3(-s))W_j)H_{m+2}H_{m+1} + (-2(s-r)W_{j+2} + ((r+1)(s-r) - 3(-s))W_{j+1} + 2(r+1)(-s)W_j)H_{m+2}H_m + (((r+1)(s-r) - 3(-s))W_{j+2} + 2((s-r)^2 + (r+1)(-s))W_{j+1} + 4(s-r)(-s)W_j)H_{m+1}H_m)$,

$z\Theta_5 = z((-2((r+1)^2 + 3(s-r))W_{j+2} + (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))W_{j+1} + ((r+1)^2(s-r) - 3(r+1)(-s) + 4(s-r)^2)W_j)H_{m+2} + ((2(r+1)^3 + 7(r+1)(s-r) + 9(-s))W_{j+2} - 2((r+1)^4 + 4(r+1)^2(s-r) + 6(r+1)(-s) + (s-r)^2)W_{j+1} - ((r+1)^3(s-r) + 4(r+1)(s-r)^2 - (r+1)^2(-s) + 6(s-r)(-s))W_j) H_{m+1} + (((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))W_{j+2} - (4(r+1)(s-r)^2 + (r+1)^3(s-r) - (r+1)^2(-s) + 6(s-r)(-s))W_{j+1} - 2(-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))W_j)H_m)$,

$\Theta_6 = (4(r+1)^3(-s) - (r+1)^2(s-r)^2 - 4(s-r)^3 + 27(-s)^2 + 18(r+1)(s-r)(-s))W_j$,

and

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2))$,

$z^2\Gamma_2 = z^2(((r+1)^2 + 3(s-r))H_{m+2}^2 + ((r+1)^4 + 4(r+1)^2(s-r) + (s-r)^2 + 6(r+1)(-s))H_{m+1}^2 + (-s)((r+1)^3 + 4(r+1)(s-r) + 9(-s))H_m^2 - (2(r+1)^3 + 7(r+1)(s-r) + 9(-s))H_{m+2}H_{m+1} - ((r+1)^2(s-r) + 4(s-r)^2 - 3(r+1)(-s))H_{m+2}H_m + (r+1)(s-r)((r+1)^2 + 4(s-r))H_{m+1}H_m - (-s)((r+1)^2 - 6(s-r))H_{m+1}H_m)$,

$z\Gamma_3 = z(-4(r+1)^3(-s) + (r+1)^2(s-r)^2 + 4(s-r)^3 - 27(-s)^2 - 18(r+1)(s-r)(-s))H_m$,

$\Gamma_4 = 4(r+1)^3(-s) - (r+1)^2(s-r)^2 + 18(r+1)(s-r)(-s) - 4(s-r)^3 + 27(-s)^2$.

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Lemma 72.]. □

Lemma 36 gives the following result as particular example (generating function of (r,s)-Horadam-Leonardo polynomials).

Corollary 37. *Assume that* $|z| < \min\{|\alpha|^{-m}, |\beta|^{-m}, |\gamma|^{-m}\}$. *Generating function of* (r,s)*-Horadam-Leonardo polynomials is given, as follows:*

$$\sum_{n=0}^{\infty} G_{mn+j}z^n = \frac{z^2\Theta_4 + z\Theta_5 + \Theta_6}{z^3\Gamma_1 + z^2\Gamma_2 + z\Gamma_3 + \Gamma_4}$$

where (as in Theorem 29 (a))

$z^2\Theta_4 = z^2((3G_{j+2}-2(r+1)G_{j+1}-(s-r)G_j)H^2_{m+2}+(((r+1)^2-(s-r))G_{j+2}+3((r+1)(s-r)+(-s))G_{j+1}+((s-r)^2+(r+1)(-s))G_j)H^2_{m+1}+(-s)((r+1)G_{j+2}+2(s-r)G_{j+1}+3(-s)G_j)H^2_m+(-4(r+1)G_{j+2}+2((r+1)^2-(s-r))G_{j+1}+((r+1)(s-r)-3(-s))G_j)H_{m+2}H_{m+1}+(-2(s-r)G_{j+2}+((r+1)(s-r)-3(-s))G_{j+1}+2(r+1)(-s)G_j)H_{m+2}H_m+(((r+1)(s-r)-3(-s))G_{j+2}+2((s-r)^2+(r+1)(-s))G_{j+1}+4(s-r)(-s)G_j)H_{m+1}H_m),$

$z\Theta_5 = z((-2((r+1)^2+3(s-r))G_{j+2}+(2(r+1)^3+7(r+1)(s-r)+9(-s))G_{j+1}+((r+1)^2(s-r)-3(r+1)(-s)+4(s-r)^2)G_j)H_{m+2}+((2(r+1)^3+7(r+1)(s-r)+9(-s))G_{j+2}-2((r+1)^4+4(r+1)^2(s-r)+6(r+1)(-s)+(s-r)^2)G_{j+1}-((r+1)^3(s-r)+4(r+1)(s-r)^2-(r+1)^2(-s)+6(s-r)(-s))G_j)H_{m+1}+(((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))G_{j+2}-(4(r+1)(s-r)^2+(r+1)^3(s-r)-(r+1)^2(-s)+6(s-r)(-s))G_{j+1}-2(-s)((r+1)^3+4(r+1)(s-r)+9(-s))G_j)H_m),$

$\Theta_6 = (4(r+1)^3(-s)-(r+1)^2(s-r)^2-4(s-r)^3+27(-s)^2+18(r+1)(s-r)(-s))G_j,$

and

$z^3\Gamma_1 = z^3(-(-s)^m(4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2)),$

$z^2\Gamma_2 = z^2(((r+1)^2+3(s-r))H^2_{m+2}+((r+1)^4+4(r+1)^2(s-r)+(s-r)^2+6(r+1)(-s))H^2_{m+1}+(-s)((r+1)^3+4(r+1)(s-r)+9(-s))H^2_m-(2(r+1)^3+7(r+1)(s-r)+9(-s))H_{m+2}H_{m+1}-((r+1)^2(s-r)+4(s-r)^2-3(r+1)(-s))H_{m+2}H_m+(r+1)(s-r)((r+1)^2+4(s-r))H_{m+1}H_m-(-s)((r+1)^2-6(s-r))H_{m+1}H_m),$

$z\Gamma_3 = z(-4(r+1)^3(-s)+(r+1)^2(s-r)^2+4(s-r)^3-27(-s)^2-18(r+1)(s-r)(-s))H_m,$

$\Gamma_4 = 4(r+1)^3(-s)-(r+1)^2(s-r)^2+18(r+1)(s-r)(-s)-4(s-r)^3+27(-s)^2.$

Proof. Replace r, s and t with $r+1$, $s-r$, $-s$, respectively, in [16, Corollary 73.]. □

References

[1] Andrica, D., Bagdasar, O., Recurrent Sequences Key Results, Applications, and Problems, Springer, 2020.

[2] Cerda-Moralez, G., On Third-Order Jacobsthal Polynomials and Their Properties, Miskolc Mathematical Notes, Vol. 22(1), 123–132, 2021. DOI: 10.18514/MMN.2021.3227

[3] Djordjević, G.B., Milovanović, G.V., Special Classes of Plynomials, University of Niš, Faculty of Technology, Leskovac, 2014. http://www.mi.sanu.ac.rs/~gvm/Teze/Special%20Classes%20of%20Polynomials.pdf

[4] Frei, G., Binary Lucas and Fibonacci Polynomials, I, Math. Nadir. 96, 83-112, 1980.

[5] Flórez, R., McAnally, N., Mukherjee, A., Identities for the Generalized Fibonacci Polynomial, Integers, 18B, 2018.

[6] He,T.X., Peter J.-S. Shiue, P.J.S., On Sequences of Numbers and Polynomials Defined by Linear Recurrence Relations of Order 2, International Journal of Mathematics and Mathematical Sciences, Volume 2009, Article ID 709386, 21 pages, doi:10.1155/2009/709386.

[7] Koshy, T., Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.

[8] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 1 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), Second Edition, John Wiley&Sons, New York, 2018.

[9] Koshy, T., Fibonacci and Lucas Numbers with Applications, Volume 2 (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts), John Wiley&Sons, New York, 2019.

[10] Merzouka, H., Boussayoudb, A., Chelghamc, M., Generating Functions of Generalized Tribonacci and Tricobsthal Polynomials, Montes Taurus Journal of Pure and Applied Mathematics, 2(2), 7–37, 2020.

[11] Özkan, E., Altun, İ., (2019) Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials, Communications in Algebra, 47(10), 2019. DOI: 10.1080/00927872.2019.1576186

[12] Ricci, P.E., A Note on Q-matrices and Higher Order Fibonacci Polynomials, Notes on Number Theory and Discrete Mathematics, 27(1), 2021, 91-100. DOI: 10.7546/nntdm.2021.27.1.91-100

[13] Soykan, Y., Generalized Fibonacci Numbers: Sum Formulas, Minel Yayın, 2022. https://www.minelyayin.com/generalized-fibonacci-numbers-sum-formulas-51

[14] Soykan, Y., On Generalized Fibonacci Polynomials: Horadam Polynomials, Earthline Journal of Mathematical Sciences, 11(1), 23-114, 2023. https://doi.org/10.34198/ejms.11123.23114

[15] Soykan, Y., Sums and Generating Functions of Generalized Fibonacci Polynomials via Matrix Methods, International Journal of Advances in Applied Mathematics and Mechanics, 10(4), 23-71, 2023.

[16] Soykan, Y., Generalized Tribonacci Polynomials, Earthline Journal of Mathematical Sciences, 13(1), 1-120, 2023. https://doi.org/10.34198/ejms.13123.1120

[17] Soykan, Y., Generalized Leonardo Numbers, Journal of Progressive Research in Mathematics, 18(4), 58-84, 2021.

[18] Soykan, Y., Generalized John Numbers, Journal of Progressive Research in Mathematics, 19(1), 17-34, 2022.

[19] Soykan, Y., Generalized Ernst Numbers, Asian Journal of Pure and Applied Mathematics, 4(3), 1-15, 2022.

[20] Soykan, Y., Okumuş, İ, Taşdemir, E., Generalized Pisano Numbers, Notes on Number Theory and Discrete Mathematics, 28(3), 477-490, 2022. DOI: 10.7546/nntdm.2022.28.3.477-490

[21] Soykan, Y., Generalized Edouard Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 9(3), 41-52, 2022.

[22] Soykan, Y., Okumuş, İ., Bilgin, N.G., Generalized Bigollo Numbers, Summitted.

[23] Vajda, S,. Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications, Dover Publications Inc.; 2008.

[24] Wang, J., Some New Results for the (p,q)-Fibonacci and Lucas Polynomials, Advances in Difference Equations, 2014. http://www.advancesindifferenceequations.com/content/2014/1/64

[25] Wang, W., Wang, H., Generalized-Humbert Polynomials via Generalized Fibonacci Polynomials, Applied Mathematics and Computation 307, 204–216, 2017. https://doi.org/10.1016/j.amc.2017.02.050.