

On Generalized Bigollo Numbers

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Abstract. It is aimed to describe generalized Bigollo sequences and to examine Bigollo and Bigollo-Lucas sequences in this article. For this purpose, we give Binet's formulas, Simson formulas, generating functions and we introduce a lot of main features of these sequences. Also, we get some formulas and give special matrices for these generalized sequences. Finally, we have determined some close relationships between Bigollo and Bigollo-Lucas numbers and Mersenne, Mersenne-Lucas numbers.

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1. Introduction

The Mersenne number M_n is given by $M_n = 2^n - 1$ and the Mersenne sequence $\{M_n\}_{n \geq 0}$ is defined recurrence relation:

$$M_n = 3M_{n-1} - 2M_{n-2} \quad (1.1)$$

for $M_0 = 0, M_1 = 1$. In addition a Mersenne-Lucas number H_n , is defined by $H_n = 2^n + 1$, extensively also the Mersenne-Lucas sequence $\{H_n\}_{n \geq 0}$ is given recursively by,

$$H_n = 3H_{n-1} - 2H_{n-2} \quad (1.2)$$

for $H_0 = 2, H_1 = 3$. $\{M_n\}_{n \geq 0}$ and $\{H_n\}_{n \geq 0}$ are the sequences with numbers A000225, A000051 in the OEIS respectively [31].

The sequences $\{M_n\}_{n \geq 0}$ and $\{H_n\}_{n \geq 0}$ are given to negative subscripts with defining

$$\begin{aligned} M_{-n} &= \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)}, \\ H_{-n} &= \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)}, \end{aligned}$$

for $n = 1, 2, 3, \dots$. So, (1.1) and (1.2) are true for every integer n .

It should be noted that Mersenne-Lucas numbers are known as Fermat numbers. Actually, there are two definitions of the Fermat numbers. The number in the form $2^n + 1$ whose first few terms are 2, 3, 5, 9, 17, 33, ... is less widely known (OEIS A000051). However, the more common Fermat numbers are a special case, defined by $F_n = 2^{2^n} + 1$. Some of them are as follows: 3, 5, 17, 257, 65537, ... (OEIS A000215).

Many authors have worked on the Mersenne sequence and for more detail on this sequence the following resources can be preferred: [4,6,7,8,11,12,16,17,18,21,24,28,62,29,32,33,36,50,51,59,70].

Now, we define two sequences associated with Mersenne, Mersenne-Lucas numbers. Bigollo and Bigollo-Lucas numbers are defined by

$$B_n = 3B_{n-1} - 2B_{n-2} + 1, \quad \text{for } B_0 = 0, B_1 = 1, \quad n \geq 2,$$

and

$$C_n = 3C_{n-1} - 2C_{n-2}, \quad \text{for } C_0 = 3, C_1 = 4, \quad n \geq 2,$$

respectively. Also, the first few Bigollo and Bigollo-Lucas numbers are

$$0, 1, 4, 11, 26, 57, 120, 247, \dots$$

and

$$3, 4, 6, 10, 18, 34, 66, 130, \dots$$

respectively. A third-order linear recurrence relation for $\{B_n\}$ and $\{C_n\}$ sequences can be given as:

$$\begin{aligned} B_n &= 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, & B_0 = 0, B_1 = 1, B_2 = 4, \\ C_n &= 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, & C_0 = 3, C_1 = 4, C_2 = 6. \end{aligned}$$

The identities giving the important interrelationships between Bigollo, Bigollo-Lucas and Mersenne, Mersenne-Lucas numbers can be written as follows:

$$\begin{aligned} B_n &= 2M_n - n, \\ C_n &= H_n + 1, \end{aligned}$$

and

$$\begin{aligned} B_n &= 4H_{n+1} - 6H_n - n, \\ 2C_n &= 2M_{n+1} - 2M_n + 4. \end{aligned}$$

This article intends to describe the generalization of these sequence of numbers (i.e., Bigollo, Bigollo-Lucas numbers). Now, let's remember the basic structure of generalized Tribonacci numbers.

The generalized Tribonacci sequence

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n \geq 0}$$

(or shortly $\{W_n\}_{n \geq 0}$) is defined by

$$W_0 = a, W_1 = b, W_2 = c \text{ and for } n \geq 3, W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} \quad (1.3)$$

where $r, s, t \in \mathbb{R}$ and a, b, c are real or complex numbers (arbitrary).

Many authors have examined different features of this sequence, see for example [52]. $\{W_n\}_{n \geq 0}$ is adapted to negative subscripts by defining them as follows:

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $t \neq 0$ and $n \in \{1, 2, 3, \dots\}$. So, (1.3) gets for each integer n . The characteristic equation of $\{W_n\}$, which has a third-order recurrence sequence, is as follows:

$$x^3 - rx^2 - sx - t = 0 \quad (1.4)$$

where the roots α, β and γ ;

$$\begin{aligned} \alpha &= \frac{r}{3} + A + B, \\ \beta &= \frac{r}{3} + \omega A + \omega^2 B, \\ \gamma &= \frac{r}{3} + \omega^2 A + \omega B, \end{aligned}$$

and

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta} \right)^{1/3}, \quad B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta} \right)^{1/3}, \\ \Delta &= \Delta(r, s, t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3). \end{aligned}$$

Binet's formula is given using the recurrence relation and roots in the next theorem for generalized Tribonacci numbers:

THEOREM 1. (*Two Distinct Roots Case: $\alpha = \beta \neq \gamma$*)

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3 \gamma^n \quad (1.5)$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2}, \\ A_2 &= \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)}, \\ A_3 &= \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}. \end{aligned}$$

2. Generalized Bigollo Sequence

Throughout this article, we deal with the case of $r = 4, s = -5, t = 2$. A generalized Bigollo sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1, W_2)\}_{n \geq 0}$ is given by the third-order recurrence relations

$$W_n = 4W_{n-1} - 5W_{n-2} + 2W_{n-3} \quad (2.1)$$

for $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. Using the negative subscripts we write next formula:

$$W_{-n} = \frac{5}{2}W_{-(n-1)} - 2W_{-(n-2)} + \frac{1}{2}W_{-(n-3)}$$

for $n \in \{1, 2, 3, \dots\}$. Hence, (2.1) is valid for each integer n .

It is obtained using the Binet formula (1.5) for generalized Bigollo numbers (two distinct roots case: $\alpha \neq \beta = \gamma$) by

$$W_n = (A_1 + A_2n) \times \beta^n + A_3 \times \alpha^n = (A_1 + A_2n) + A_3 \times 2^n$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\beta W_1 - \alpha(2\beta - \alpha)W_0}{(\beta - \alpha)^2} = -W_2 + 2W_1, \\ A_2 &= \frac{W_2 - (\beta + \alpha)W_1 + \beta\alpha W_0}{\beta(\beta - \alpha)} = -W_2 + 3W_1 - 2W_0, \\ A_3 &= \frac{W_2 - 2\beta W_1 + \beta^2 W_0}{(\beta - \alpha)^2} = W_2 - 2W_1 + W_0, \end{aligned}$$

i.e.

$$W_n = ((-W_2 + 2W_1) + (-W_2 + 3W_1 - 2W_0)n) + (W_2 - 2W_1 + W_0) \times 2^n.$$

Here, α, β and γ are the roots of the characteristic equation

$$x^3 - 4x^2 + 5x - 2 = (x - 2)(x - 1)(x - 1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= 2, \\ \beta &= 1, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned}\alpha + \beta + \gamma &= 4, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 5, \\ \alpha\beta\gamma &= 2.\end{aligned}$$

Using the positive and negative subscript, we present the first few generalized Bigollo numbers in next table.

Table 1. A few generalized Bigollo numbers

| n | W_n | W_{-n} |
|-----|--------------------------------|--|
| 0 | W_0 | W_0 |
| 1 | W_1 | $\frac{1}{2}(5W_0 - 4W_1 + W_2)$ |
| 2 | W_2 | $\frac{1}{4}(17W_0 - 18W_1 + 5W_2)$ |
| 3 | $2W_0 - 5W_1 + 4W_2$ | $\frac{1}{8}(49W_0 - 58W_1 + 17W_2)$ |
| 4 | $8W_0 - 18W_1 + 11W_2$ | $\frac{1}{16}(129W_0 - 162W_1 + 49W_2)$ |
| 5 | $22W_0 - 47W_1 + 26W_2$ | $\frac{1}{32}(321W_0 - 418W_1 + 129W_2)$ |
| 6 | $52W_0 - 108W_1 + 57W_2$ | $\frac{1}{64}(769W_0 - 1026W_1 + 321W_2)$ |
| 7 | $114W_0 - 233W_1 + 120W_2$ | $\frac{1}{128}(1793W_0 - 2434W_1 + 769W_2)$ |
| 8 | $240W_0 - 486W_1 + 247W_2$ | $\frac{1}{256}(4097W_0 - 5634W_1 + 1793W_2)$ |
| 9 | $494W_0 - 995W_1 + 502W_2$ | $\frac{1}{512}(9217W_0 - 12802W_1 + 4097W_2)$ |
| 10 | $1004W_0 - 2016W_1 + 1013W_2$ | $\frac{1}{1024}(20481W_0 - 28674W_1 + 9217W_2)$ |
| 11 | $2026W_0 - 4061W_1 + 2036W_2$ | $\frac{1}{2048}(45057W_0 - 63490W_1 + 20481W_2)$ |
| 12 | $4072W_0 - 8154W_1 + 4083W_2$ | $\frac{1}{4096}(98305W_0 - 139266W_1 + 45057W_2)$ |
| 13 | $8166W_0 - 16343W_1 + 8178W_2$ | $\frac{1}{8192}(212993W_0 - 303106W_1 + 98305W_2)$ |

Now we define two special cases of $\{W_n\}$. Using the third-order recurrence relations, Bigollo sequence $\{B_n\}_{n \geq 0}$ and Bigollo-Lucas sequence $\{C_n\}_{n \geq 0}$ are defined,

$$B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, \quad B_0 = 0, B_1 = 1, B_2 = 4, \quad (2.2)$$

$$C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, \quad C_0 = 3, C_1 = 4, C_2 = 6. \quad (2.3)$$

The sequences $\{B_n\}_{n \geq 0}$ and $\{C_n\}_{n \geq 0}$ is extended to negative subscripts by defining

$$\begin{aligned}B_{-n} &= \frac{5}{2}B_{-(n-1)} - 2B_{-(n-2)} + \frac{1}{2}B_{-(n-3)}, \\ C_{-n} &= \frac{5}{2}C_{-(n-1)} - 2C_{-(n-2)} + \frac{1}{2}C_{-(n-3)},\end{aligned}$$

for $n \in \{1, 2, 3, \dots\}$. That is, (2.2)-(2.3) are valid for each integer n .

B_n and C_n are the sequences A000295 (Eulerian numbers), A052548 in [31].

Now, for positive and negative subscripts we give some values of the Bigollo and Bigollo-Lucas numbers.

Table 2. The first few values of the special third-order numbers.

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|----------|---|---------------|---------------|----------------|-----------------|-----------------|------------------|-------------------|-------------------|--------------------|---------------------|---------------------|----------------------|----------------------|
| B_n | 0 | 1 | 4 | 11 | 26 | 57 | 120 | 247 | 502 | 1013 | 2036 | 4083 | 8178 | 16369 |
| B_{-n} | | 0 | $\frac{1}{2}$ | $\frac{5}{4}$ | $\frac{17}{8}$ | $\frac{49}{16}$ | $\frac{129}{32}$ | $\frac{321}{64}$ | $\frac{769}{128}$ | $\frac{1793}{256}$ | $\frac{4097}{512}$ | $\frac{9217}{1024}$ | $\frac{20481}{2048}$ | $\frac{45057}{4096}$ |
| C_n | 3 | 4 | 6 | 10 | 18 | 34 | 66 | 130 | 258 | 514 | 1026 | 2050 | 4098 | 8194 |
| C_{-n} | | $\frac{5}{2}$ | $\frac{9}{4}$ | $\frac{17}{8}$ | $\frac{33}{16}$ | $\frac{65}{32}$ | $\frac{129}{64}$ | $\frac{257}{128}$ | $\frac{513}{256}$ | $\frac{1025}{512}$ | $\frac{2049}{1024}$ | $\frac{4097}{2048}$ | $\frac{8193}{4096}$ | $\frac{16385}{8192}$ |

For every $n \in \mathbb{Z}$, using Binet's formulas Bigollo and Bigollo-Lucas numbers can be written as

$$B_n = 2^{n+1} - n - 2,$$

$$C_n = 2^n + 2.$$

Also, Binet's formulas of Mersenne and Mersenne-Lucas numbers, are

$$M_n = 2^n - 1,$$

$$H_n = 2^n + 1,$$

and so

$$B_n = 2M_n - n, \quad (2.4)$$

$$C_n = H_n + 1. \quad (2.5)$$

Now, we present the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of W_n .

LEMMA 2. Let $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Bigollo sequence $\{W_n\}_{n \geq 0}$. In this case, $\sum_{n=0}^{\infty} W_n x^n$ is obtained with

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 4W_0)x + (W_2 - 4W_1 + 5W_0)x^2}{1 - 4x + 5x^2 - 2x^3}.$$

Proof. If $r = 4, s = -5, t = 2$ are chosen in [52, Lemma 1.1], we get this equality. \square

The following results can be obtained from the previous lemma.

COROLLARY 3. Generated functions of Bigollo and Bigollo-Lucas numbers are

$$\sum_{n=0}^{\infty} B_n x^n = \frac{x}{1 - 4x + 5x^2 - 2x^3},$$

$$\sum_{n=0}^{\infty} C_n x^n = \frac{3 - 8x + 5x^2}{1 - 4x + 5x^2 - 2x^3},$$

respectively.

3. Simson Formulas

The Simson formula of the Fibonacci sequence $\{F_n\}$ is:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

and in 1753 it was established by R. Simson then this formula is called as Cassini formula. It can be given by

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

Now we give such formulas for the generalized Bigollo sequence $\{W_n\}_{n \geq 0}$.

THEOREM 4 (Simson Formula for Generalized Bigollo Numbers). *For every integers n ,*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -2^{n-2}(W_2 - 2W_1 + W_0)(W_2 - 3W_1 + 2W_0)^2.$$

is written.

Proof. If we take $r = 4, s = -5, t = 2$ in [37, Theorem 2.2], we have this equality. \square

From the Theorem 4, we write the following special cases.

COROLLARY 5. *For every $n \in \mathbb{N}$, Simson formula for Bigollo and Bigollo-Lucas numbers are obtained following*

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = -2^{n-1},$$

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = 0.$$

4. Some Identities

Now, we give some identities of Bigollo and Bigollo-Lucas numbers. Firstly, a few important relationships between $\{W_n\}$ and $\{B_n\}$ will be expressed.

LEMMA 6. *For every integers n , the following equalities are valid :*

- (a): $8W_n = (49W_0 - 58W_1 + 17W_2)B_{n+4} + 2(98W_1 - 81W_0 - 29W_2)B_{n+3} + (129W_0 - 162W_1 + 49W_2)B_{n+2}$.
- (b): $4W_n = (17W_0 - 18W_1 + 5W_2)B_{n+3} - 2(29W_0 - 32W_1 + 9W_2)B_{n+2} + (49W_0 - 58W_1 + 17W_2)B_{n+1}$.
- (c): $2W_n = (5W_0 - 4W_1 + W_2)B_{n+2} - 2(9W_0 - 8W_1 + 2W_2)B_{n+1} + (17W_0 - 18W_1 + 5W_2)B_n$.
- (d): $W_n = W_0B_{n+1} + (-4W_0 + W_1)B_n + (5W_0 - 4W_1 + W_2)B_{n-1}$.

$$(e): W_n = W_1 B_n + (W_2 - 4W_1)B_{n-1} + 2W_0 B_{n-2}.$$

$$(f): 2(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_n = -(-5W_1^2 - W_2^2 + 2W_0 W_1 + 4W_1 W_2)W_{n+4} + 2(-9W_1^2 - 2W_2^2 + 4W_0 W_1 - W_0 W_2 + 8W_1 W_2)W_{n+3} + (4W_0^2 + 25W_1^2 + 5W_2^2 - 20W_0 W_1 + 8W_0 W_2 - 22W_1 W_2)W_{n+2}.$$

$$(g): (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_n = (W_1^2 - W_0 W_2)W_{n+3} + (2W_0^2 - 5W_0 W_1 + 4W_0 W_2 - W_1 W_2)W_{n+2} + (5W_1^2 + W_2^2 - 2W_0 W_1 - 4W_1 W_2)W_{n+1}.$$

$$(h): (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_n = (2W_0^2 + 4W_1^2 - 5W_0 W_1 - W_1 W_2)W_{n+2} + (W_2^2 - 2W_0 W_1 + 5W_0 W_2 - 4W_1 W_2)W_{n+1} - 2(-W_1^2 + W_0 W_2)W_n.$$

$$(i): (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_n = (8W_0^2 + 16W_1^2 + W_2^2 - 22W_0 W_1 + 5W_0 W_2 - 8W_1 W_2)W_{n+1} - (10W_0^2 + 18W_1^2 - 25W_0 W_1 + 2W_0 W_2 - 5W_1 W_2)W_n + 2(2W_0^2 + 4W_1^2 - 5W_0 W_1 - W_1 W_2)W_{n-1}.$$

$$(j): (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_n = (22W_0^2 + 46W_1^2 + 4W_2^2 - 63W_0 W_1 + 18W_0 W_2 - 27W_1 W_2)W_n - (36W_0^2 + 72W_1^2 + 5W_2^2 - 100W_0 W_1 + 25W_0 W_2 - 38W_1 W_2)W_{n-1} + 2(8W_0^2 + 16W_1^2 + W_2^2 - 22W_0 W_1 + 5W_0 W_2 - 8W_1 W_2)W_{n-2}.$$

Proof. We only prove (a), since other equalities is shown similarly. First using

$$W_n = a \times B_{n+4} + b \times B_{n+3} + c \times B_{n+2}$$

we solve the system of equations

$$W_0 = a \times B_4 + b \times B_3 + c \times B_2$$

$$W_1 = a \times B_5 + b \times B_4 + c \times B_3$$

$$W_2 = a \times B_6 + b \times B_5 + c \times B_4$$

then we have $a = \frac{1}{8}(49W_0 - 58W_1 + 17W_2)$, $b = \frac{1}{4}(98W_1 - 81W_0 - 29W_2)$, $c = \frac{1}{8}(129W_0 - 162W_1 + 49W_2)$. \square

Obviously, all identities in Lemma 6 are also shown using induction.

We present some relationships between $\{W_n\}$ and $\{C_n\}$.

LEMMA 7. *The following identities are correct.*

$$(a): 4(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = (10W_0 - 19W_1 + 9W_2)W_{n+4} - 2(14W_0 - 27W_1 + 13W_2)W_{n+3} + (18W_0 - 35W_1 + 17W_2)W_{n+2}.$$

$$(b): 2(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = (6W_0 - 11W_1 + 5W_2)W_{n+3} - 2(8W_0 - 15W_1 + 7W_2)W_{n+2} + (10W_0 - 19W_1 + 9W_2)W_{n+1}.$$

$$(c): (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = (4W_0 - 7W_1 + 3W_2)W_{n+2} - 2(5W_0 - 9W_1 + 4W_2)W_{n+1} + (6W_0 - 11W_1 + 5W_2)W_n.$$

$$(d): (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = 2(3W_0 - 5W_1 + 2W_2)W_{n+1} - 2(7W_0 - 12W_1 + 5W_2)W_n + 2(4W_0 - 7W_1 + 3W_2)W_{n-1}.$$

$$(e): (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = 2(5W_0 - 8W_1 + 3W_2)W_n - 2(11W_0 - 18W_1 + 7W_2)W_{n-1} + 4(3W_0 - 5W_1 + 2W_2)W_{n-2}.$$

Then, we also give some basic relationships between $\{B_n\}$ and $\{C_n\}$.

LEMMA 8. *The following identities are true*

$$8C_n = 17B_{n+4} - 50B_{n+3} + 33B_{n+2},$$

$$4C_n = 9B_{n+3} - 26B_{n+2} + 17B_{n+1},$$

$$2C_n = 5B_{n+2} - 14B_{n+1} + 9B_n,$$

$$C_n = 3B_{n+1} - 8B_n + 5B_{n-1},$$

$$C_n = 4B_n - 10B_{n-1} + 6B_{n-2}.$$

5. Identities Between Special Numbers

Now, we give some identities on Bigollo and Bigollo-Lucas numbers and Mersenne and Mersenne-Lucas numbers. It is known that

$$B_n = 2M_n - n,$$

$$C_n = H_n + 1.$$

We also note that from Lemma 8, we have

$$2C_n = 5B_{n+2} - 14B_{n+1} + 9B_n$$

and from Soykan [50, Lemma 11], we get

$$M_n = 2H_{n+1} - 3H_n.$$

If we use the above identities, we have

$$B_n = 4H_{n+1} - 6H_n - n,$$

$$2C_n = 2M_{n+1} - 2M_n + 4.$$

We use these formulas and Lemma 6, we get Binet's formula of generalized Bigollo numbers as:

$$\begin{aligned} 2W_n &= (5W_0 - 4W_1 + W_2)B_{n+2} - 2(9W_0 - 8W_1 + 2W_2)B_{n+1} + (17W_0 - 18W_1 + 5W_2)B_n \\ &= 2(3W_2 - 10W_1 + 7W_0)M_n - 2(W_2 - 4W_1 + 3W_0)M_{n+1} - 2(W_2 - 3W_1 + 2W_0)n + 2W_2 - 8W_1 + 8W_0 \\ &= 2(3W_2 - 8W_1 + 5W_0)H_{n+1} - 2(5W_2 - 14W_1 + 9W_0)H_n - 2(W_2 - 3W_1 + 2W_0)n + 2W_2 - 8W_1 + 8W_0. \end{aligned}$$

6. The Recurrence Relations of Generalized Bigollo Sequence

If we take $r = 4, s = -5, t = 2$ in [40, Theorem 2], we have the next Proposition.

PROPOSITION 9. *For $n \in \mathbb{Z}$, generalized Bigollo numbers get the following equation:*

$$W_{-n} = 2^{-n}(W_{2n} - C_n W_n + \frac{1}{2}(C_n^2 - C_{2n})W_0).$$

Using the Corollary 6 and Proposition 9 in [40], we get the following corollary which presents the relation between the special cases of generalized Bigollo sequence for the negative and the positive index: for modified Bigollo, Bigollo-Lucas and Bigollo numbers: take $W_n = B_n$, $B_0 = 0, B_1 = 1, B_2 = 4$ and taking $W_n = C_n$ and also $C_0 = 3, C_1 = 4, C_2 = 6$.

COROLLARY 10. For $n \in \mathbb{Z}$, it is written the next recurrence relations:

(a): Bigollo sequence:

$$B_{-n} = 2^{-n}(B_{2n} - B_n C_n).$$

(b): Bigollo-Lucas sequence:

$$C_{-n} = 2^{-n-1}(C_n^2 - C_{2n}).$$

By using the equation $2C_n = 5B_{n+2} - 14B_{n+1} + 9B_n$ (and Corollary 10 or Proposition 9),

$$B_{-n} = \frac{1}{2^{n+1}}(14B_n B_{n+1} - 5B_n B_{n+2} - 9B_n^2 + 2B_{2n})$$

is written. Note also that since $B_n = 2M_n - n$ and $M_{-n} = -\frac{1}{2^n}M_n = \frac{-2^n + 1}{2^n}$, we get

$$B_{-n} = -2^{-n+1}M_n + n$$

and using $C_n = H_n + 1$ and $H_{-n} = \frac{1}{2^n}H_n = \frac{2^n + 1}{2^n}$, then we obtain

$$C_{-n} = 2^{-n}H_n + 1.$$

7. Sum Formulas

In next Corollary we give sum formulas of Mersenne and Mersenne-Lucas numbers.

COROLLARY 11. Let $n \geq 0$. For Mersenne and Mersenne-Lucas numbers, the following properties are true :

(1)

(a): $\sum_{k=0}^n M_k = -(n-1)M_n + 2(n+1)M_{n-1} + 1.$

(b): $\sum_{k=0}^n M_{2k} = \frac{1}{3}(-(n-3)M_{2n} + 4(n+1)M_{2n-2} + 3).$

(c): $\sum_{k=0}^n M_{2k+1} = \frac{1}{3}(-(n-3)M_{2n+1} + 4(n+1)M_{2n-1} + 2).$

(2)

(a): $\sum_{k=0}^n H_k = -(n-1)H_n + 2(n+1)H_{n-1} - 3.$

(b): $\sum_{k=0}^n H_{2k} = \frac{1}{3}(-(n-3)H_{2n} + 4(n+1)H_{2n-2} - 5).$

(c): $\sum_{k=0}^n H_{2k+1} = \frac{1}{3}(-(n-3)H_{2n+1} + 4(n+1)H_{2n-1} - 6).$

Proof. It is given in [50, Corollary 25]. \square

In the following Corollary we present sum formulas of Bigollo and Bigollo-Lucas numbers.

COROLLARY 12. For $n \geq 0$, Bigollo and Bigollo-Lucas numbers get the following equalities:

(1)

$$(a): \sum_{k=0}^n B_k = \frac{1}{2}(-4(n-1)M_n + 8(n+1)M_{n-1} - n^2 - n + 4).$$

$$(b): \sum_{k=0}^n B_{2k} = \frac{1}{3}(-2(n-3)M_{2n} + 8(n+1)M_{2n-2} - 3(n-1)(n+2))$$

$$(c): \sum_{k=0}^n B_{2k+1} = \frac{1}{3}(-2(n-3)M_{2n+1} + 8(n+1)M_{2n-1} + 4 - 3(n+1)^2).$$

(2)

$$(a): \sum_{k=0}^n C_k = -(n-1)H_n + 2(n+1)H_{n-1} + n - 2.$$

$$(b): \sum_{k=0}^n C_{2k} = \frac{1}{3}(-(n-3)H_{2n} + 4(n+1)H_{2n-2} + 3n - 2).$$

$$(c): \sum_{k=0}^n C_{2k+1} = \frac{1}{3}(-(n-3)H_{2n+1} + 4(n+1)H_{2n-1} + 3n - 3).$$

Proof. The proof is valid from the identities (2.4) and (2.5) and Corollary 11, i.e.,

$$B_n = 2M_n - n,$$

$$C_n = H_n + 1. \quad \square$$

8. Matrices Formulation of Generalized Bigollo Numbers

Matrix forms of W_n is shown by

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \quad (8.1)$$

A square matrix is defined as:

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. From (2.1) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \quad (8.2)$$

and using (8.1) or (8.2) and induction we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W = B$ in (8.2) we have

$$\begin{pmatrix} B_{n+2} \\ B_{n+1} \\ B_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} B_{n+1} \\ B_n \\ B_{n-1} \end{pmatrix}.$$

We also define

$$N_n = \begin{pmatrix} B_{n+1} & -5B_n + 2B_{n-1} & 2B_n \\ B_n & -5B_{n-1} + 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & -5B_{n-2} + 2B_{n-3} & 2B_{n-2} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & -5W_n + 2W_{n-1} & 2W_n \\ W_n & -5W_{n-1} + 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & -5W_{n-2} + 2W_{n-3} & 2W_{n-2} \end{pmatrix}$$

THEOREM 13. For every $m, n \in \mathbb{Z}$,

(a): $N_n = A^n$, i.e.,

$$A^n = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & -5B_n + 2B_{n-1} & 2B_n \\ B_n & -5B_{n-1} + 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & -5B_{n-2} + 2B_{n-3} & 2B_{n-2} \end{pmatrix}.$$

(b): $U_1 A^n = A^n U_1$

(c): $U_{n+m} = U_n N_m = N_m U_n$.

Proof. If we take $r = 4, s = -5, t = 2$ in [52, Theorem 5.1.], we get these properties. \square

For every integer m and n , some characterizations of A^n is written as

$$\begin{aligned} A^n &= 4A^{n-1} - 5A^{n-2} + 2A^{n-3}, \\ A^{n+m} &= A^n A^m = A^m A^n, \\ \det(A^n) &= 2^n, \end{aligned}$$

Using the above last Theorem and the identities

$$\begin{aligned} B_n &= 2M_n - n, \\ B_n &= 4H_{n+1} - 6H_n - n, \end{aligned}$$

we get next identities of Mersenne and Mersenne-Lucas numbers.

COROLLARY 14. For every $n \in \mathbb{Z}$, we write the following identities for Mersenne and Mersenne-Lucas numbers.

(a): *Mersenne Numbers.*

$$A^n = \begin{pmatrix} 2M_{n+1} - n - 1 & -2M_{n+1} - 4M_n + 3n + 2 & 4M_n - 2n \\ 2M_n - n & 2M_{n+1} - 8M_n + 3n - 1 & -2M_{n+1} + 6M_n - 2n + 2 \\ -M_{n+1} + 3M_n - n + 1 & 4M_{n+1} - 10M_n + 3n - 4 & -3M_{n+1} + 7M_n - 2n + 4 \end{pmatrix}.$$

(b): *Mersenne-Lucas Numbers.*

$$A^n = \begin{pmatrix} 6H_{n+1} - 8H_n - n - 1 & -14H_{n+1} + 20H_n + 3n + 2 & 8H_{n+1} - 12H_n - 2n \\ 4H_{n+1} - 6H_n - n & -10H_{n+1} + 16H_n + 3n - 1 & 6H_{n+1} - 10H_n - 2n + 2 \\ 3H_{n+1} - 5H_n - n + 1 & -8H_{n+1} + 14H_n + 3n - 4 & 5H_{n+1} - 9H_n - 2n + 4 \end{pmatrix}.$$

THEOREM 15. *For every $m, n \in \mathbb{Z}$, we get*

$$W_{n+m} = W_n B_{m+1} + (-5W_{n-1} + 2W_{n-2}) B_m + 2W_{n-1} B_{m-1} \quad (8.3)$$

Proof. Take $r = 4, s = -5, t = 2$ in [52, Theorem 5.2]. \square

Using Lemma 6, we have next equalities.

$$\begin{aligned} & (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_m \\ = & (2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+2} \\ & + (W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{m+1} - 2(-W_1^2 + W_0W_2)W_m \end{aligned}$$

so (8.3) the next can be written as

$$\begin{aligned} & (W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 W_{n+m} \\ = & W_n((2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+3} \\ & + (W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{m+2} - 2(-W_1^2 + W_0W_2)W_{m+1}) \\ & + (-5W_{n-1} + 2W_{n-2})((2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+2} \\ & + (W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{m+1} - 2(-W_1^2 + W_0W_2)W_m) \\ & + 2W_{n-1}((2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{m+1} \\ & + (W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_m - 2(-W_1^2 + W_0W_2)W_{m-1}). \end{aligned}$$

COROLLARY 16. *For every $m, n \in \mathbb{Z}$, we get*

$$\begin{aligned} B_{n+m} &= B_n B_{m+1} + (-5B_{n-1} + 2B_{n-2}) B_m + 2B_{n-1} B_{m-1}, \\ C_{n+m} &= C_n B_{m+1} + (-5C_{n-1} + 2C_{n-2}) B_m + 2C_{n-1} B_{m-1}. \end{aligned}$$

Taking $m = n$ in this corollary, we get the next equalities:

$$\begin{aligned} B_{2n} &= B_n B_{n+1} + (-5B_{n-1} + 2B_{n-2}) B_n + 2B_{n-1}^2, \\ C_{2n} &= C_n B_{n+1} + (-5C_{n-1} + 2C_{n-2}) B_n + 2C_{n-1} B_{n-1}. \end{aligned}$$

9. Conclusions

Recently, quite a lot of work has been done on the sequences of Horadam numbers and generalized third-order Pell numbers for example Fibonacci, Lucas, Jacobsthal and Pell numbers; third order Jacobsthal, third-order Pell, third-order Pell-Lucas, Narayana, Perrin, Padovan, Padovan-Perrin and third order

Jacobsthal-Lucas numbers. The sequences of numbers have been frequently used in important fields, especially in engineering, physics, nature and architecture.

In our study, we introduced the generalized Bigollo sequence, which is a third order sequence, and the special case of this sequence, Bigollo and Bigollo-Lucas sequences. Also we give Binet's formulas, Simson formulas, generating functions, some identities, the sum formulas, matrices and recurrence relations of these sequences. We found significant relationships between Bigollo, Bigollo-Lucas numbers (which are third order linear recurrences) and special second order linear recurrences (numbers), namely Mersenne and Mersenne-Lucas numbers

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, We give some studies on the applications of second order sequences.

- For some implements of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [1].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [5].
- For the adaptation of Jacobsthal numbers to special matrices, see [66].
- For the adaptation of generalized k-order Fibonacci numbers to hybrid quaternions, see [22].
- For some applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [68].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [69].
- For the adaptation of generalized Fibonacci numbers to binomial sums, see [64].
- For the adaptation of generalized Jacobsthal numbers to hyperbolic numbers, see [41].
- For the adaptation of generalized Fibonacci numbers to dual hyperbolic numbers, see [42].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial for Fibonacci numbers, see [14].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [30].
- For the application of higher order Jacobsthal numbers to quaternions, see [63].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [20].
- For the implements of Fibonacci numbers to lacunary statistical convergence, see [3].
- For the implements of Fibonacci numbers to lacunary statistical convergence in IFNLS, see [25].
- For the implements of Fibonacci numbers to ideal convergence in IFNLS, see [26].
- For some identities on k-Mersenne Numbers, see [65].

Now we give some other implements of third order sequences.

- For the implements of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [10] and [9].
- For the adaptation of Tribonacci numbers to special matrices, see [67].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [34] and [2], respectively.
- For the application of Pell-Padovan numbers to groups, see [13].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [19].
- For the adaptation of Gaussian Tribonacci numbers to various graphs, see [60].
- For the implements of third-order Jacobsthal numbers to hyperbolic numbers, see [15].
- For the implements of Narayan numbers to finite groups see [27].
- For the adaptation of generalized third-order Jacobsthal sequence to binomial transform, see [43].
- For the implements of generalized Generalized Padovan numbers to Binomial Transform, see [44].
- For the implements of generalized Tribonacci numbers to Gaussian numbers, see [45].
- For the implement of generalized Tribonacci numbers to Sedenions, see [46].
- For the adaptation of Tribonacci and Tribonacci-Lucas numbers to matrices, see [47].
- For the adaptation of generalized Tribonacci numbers to circulant matrix, see [48].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [61].

Next, we now list some implements of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [53].
- For the adaptation of generalized Tetranacci numbers to Gaussian numbers, see [54].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [55].
- For the application of generalized Tetranacci numbers to binomial transform, see [56].

Also, we give some applications of fifth order sequences.

- For the adaptation of Pentanacci numbers to matrices, see [35].
- For the adaptation of generalized Pentanacci numbers to quaternions, see [57].
- For the application of generalized Pentanacci numbers to binomial transform, see [49].

We now mention some applications of second order sequences of polynomials.

- For the application of generalized Fibonacci Polynomials to the summation formulas, see [38].
- For some applications of generalized Fibonacci Polynomials, see [39].

We now give some implements of third order sequences of polynomials.

- For some applications of generalized Tribonacci Polynomials, see [58].

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