Original Research Article

Generalized Bigollo Numbers

Abstract. In this paper, we investigate the generalized Bigollo sequences and we deal with, in detail, two special cases, namely, Bigollo and Bigollo-Lucas sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences. Furthermore, we show that there are close relations between Bigollo and Bigollo-Lucas numbers and Mersenne, Mersenne-Lucas numbers.

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1. Introduction

Mersenne sequence $\{M_n\}_{n\geq 0}$ (OEIS: $A000225$, [17]) and Mersenne-Lucas sequence $\{H_n\}_{n\geq 0}$ (OEIS: $A000051$, [17]) (is also called as Fermat sequence) are defined by the second-order recurrence relations

$$M_n = 3M_{n-1} - 2M_{n-2}, \qquad M_0 = 0, M_1 = 1, \tag{1.1}$$

$$H_n = 3H_{n-1} - 2H_{n-2}, \qquad H_0 = 2, H_1 = 3, \tag{1.2}$$

The sequences $\{M_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = \frac{3}{2}M_{-(n-1)} - \frac{1}{2}M_{-(n-2)},$$

$$H_{-n} = \frac{3}{2}H_{-(n-1)} - \frac{1}{2}H_{-(n-2)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.1) and (1.2) hold for all integer n.

Mersenne sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to this sequence, see for example, [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,18,19,20,24,25].

Now, we define two sequences related to Mersenne, Mersenne-Lucas numbers. Bigollo and Bigollo-Lucas numbers are defined as

$$B_n = 3B_{n-1} - 2B_{n-2} + 1, \quad \text{with} \quad B_0 = 0, B_1 = 1, \quad n \geq 2,$$

and

$$C_n = 3C_{n-1} - 2C_{n-2}, \quad \text{with} \quad C_0 = 3, C_1 = 4, \quad n \geq 2,$$

respectively. The first few values of Bigollo and Bigollo-Lucas numbers are

$$0, 1, 4, 11, 26, 57, 120, 247, ...$$

and

$$3, 4, 6, 10, 18, 34, 66, 130, \ldots$$

respectively. The sequences $\{B_n\}$ and $\{C_n\}$ satisfy the following third order linear recurences:

$$\begin{aligned} B_n &= 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, \quad B_0 = 0, B_1 = 1, B_2 = 4, \\ C_n &= 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, \quad C_0 = 3, C_1 = 4, C_2 = 6. \end{aligned}$$

There are close relations between Bigollo and Bigollo-Lucas and Mersenne, Mersenne-Lucas numbers. For example, they satisfy the following interrelations:

$$\begin{aligned} B_n &= 2M_n - n, \\ C_n &= H_n + 1, \end{aligned}$$

and

$$\begin{aligned} B_n &= 4H_{n+1} - 6H_n - n, \\ 2C_n &= 2M_{n+1} - 2M_n + 4. \end{aligned}$$

The purpose of this article is to generalize and investigate these interesting sequence of numbers (i.e., Bigollo, Bigollo-Lucas numbers). First, we recall some properties of generalized Tribonacci numbers.

The generalized (r, s, t) sequence (or generalized Tribonacci sequence or generalized 3-step Fibonacci sequence)

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.3}$$

where a, b, c are arbitrary complex (or real) numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [22]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.3) holds for all integer n. As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation is

$$x^3 - rx^2 - sx - t = 0 \tag{1.4}$$

whose roots are

$$\begin{aligned} \alpha &= \frac{r}{3} + A + B, \\ \beta &= \frac{r}{3} + \omega A + \omega^2 B, \\ \gamma &= \frac{r}{3} + \omega^2 A + \omega B, \end{aligned}$$

where

$$\begin{aligned} A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta}\right)^{1/3}, \ B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta}\right)^{1/3}, \\ \Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \quad \omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3). \end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

THEOREM 1. *(Two Distinct Roots Case: $\alpha = \beta \neq \gamma$) Binet's formula of generalized Tribonacci numbers is*

$$W_n = (A_1 + A_2 n) \times \alpha^n + A_3\gamma^n \tag{1.5}$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\alpha W_1 - \gamma(2\alpha - \gamma)W_0}{(\alpha - \gamma)^2}, \\ A_2 &= \frac{W_2 - (\alpha + \gamma)W_1 + \alpha\gamma W_0}{\alpha(\alpha - \gamma)}, \\ A_3 &= \frac{W_2 - 2\alpha W_1 + \alpha^2 W_0}{(\alpha - \gamma)^2}. \end{aligned}$$

2. Generalized Bigollo Sequence

In this paper, we consider the case $r = 4, s = -5, t = 2$. A generalized Bigollo sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 4W_{n-1} - 5W_{n-2} + 2W_{n-3} \tag{2.1}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = \frac{5}{2}W_{-(n-1)} - 2W_{-(n-2)} + \frac{1}{2}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) holds for all integer n.

(1.5) can be used to obtain Binet formula of generalized Bigollo numbers. Binet formula of generalized Bigollo numbers (two distinct roots case: $\alpha \neq \beta = \gamma$) can be given as

$$W_n = (A_1 + A_2 n) \times \beta^n + A_3 \times \alpha^n = (A_1 + A_2 n) + A_3 \times 2^n$$

where

$$\begin{aligned} A_1 &= \frac{-W_2 + 2\beta W_1 - \alpha(2\beta - \alpha)W_0}{(\beta - \alpha)^2} = -W_2 + 2W_1, \\ A_2 &= \frac{W_2 - (\beta + \alpha)W_1 + \beta\alpha W_0}{\beta(\beta - \alpha)} = -W_2 + 3W_1 - 2W_0, \\ A_3 &= \frac{W_2 - 2\beta W_1 + \beta^2 W_0}{(\beta - \alpha)^2} = W_2 - 2W_1 + W_0, \end{aligned}$$

i.e.

$$W_n = ((-W_2 + 2W_1) + (-W_2 + 3W_1 - 2W_0)n) + (W_2 - 2W_1 + W_0) \times 2^n.$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 4x^2 + 5x - 2 = (x^2 - 3x + 2)(x - 1) = (x - 2)(x - 1)(x - 1) = 0.$$

Moreover

$$\begin{aligned} \alpha &= 2, \\ \beta &= 1, \\ \gamma &= 1. \end{aligned}$$

Note that

$$\begin{aligned} \alpha + \beta + \gamma &= 4, \\ \alpha\beta + \alpha\gamma + \beta\gamma &= 5, \\ \alpha\beta\gamma &= 2. \end{aligned}$$

The first few generalized Bigollo numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Bigollo numbers

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n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{2}(5W_0 - 4W_1 + W_2)$
2	W_2	$\frac{1}{4}(17W_0 - 18W_1 + 5W_2)$
3	$2W_0 - 5W_1 + 4W_2$	$\frac{1}{8}(49W_0 - 58W_1 + 17W_2)$
4	$8W_0 - 18W_1 + 11W_2$	$\frac{1}{16}(129W_0 - 162W_1 + 49W_2)$
5	$22W_0 - 47W_1 + 26W_2$	$\frac{1}{32}(321W_0 - 418W_1 + 129W_2)$
6	$52W_0 - 108W_1 + 57W_2$	$\frac{1}{64}(769W_0 - 1026W_1 + 321W_2)$
7	$114W_0 - 233W_1 + 120W_2$	$\frac{1}{128}(1793W_0 - 2434W_1 + 769W_2)$
8	$240W_0 - 486W_1 + 247W_2$	$\frac{1}{256}(4097W_0 - 5634W_1 + 1793W_2)$
9	$494W_0 - 995W_1 + 502W_2$	$\frac{1}{512}(9217W_0 - 12802W_1 + 4097W_2)$
10	$1004W_0 - 2016W_1 + 1013W_2$	$\frac{1}{1024}(20481W_0 - 28674W_1 + 9217W_2)$
11	$2026W_0 - 4061W_1 + 2036W_2$	$\frac{1}{2048}(45057W_0 - 63490W_1 + 20481W_2)$
12	$4072W_0 - 8154W_1 + 4083W_2$	$\frac{1}{4096}(98305W_0 - 139266W_1 + 45057W_2)$
13	$8166W_0 - 16343W_1 + 8178W_2$	$\frac{1}{8192}(212993W_0 - 303106W_1 + 98305W_2)$

Now we define two special cases of the sequence $\{W_n\}$. Bigollo sequence $\{B_n\}_{n\geq 0}$ and Bigollo-Lucas sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$B_n = 4B_{n-1} - 5B_{n-2} + 2B_{n-3}, \qquad B_0 = 0, B_1 = 1, B_2 = 4, \tag{2.2}$$

$$C_n = 4C_{n-1} - 5C_{n-2} + 2C_{n-3}, \qquad C_0 = 3, C_1 = 4, C_2 = 6. \tag{2.3}$$

The sequences $\{B_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$B_{-n} = \frac{5}{2}B_{-(n-1)} - 2B_{-(n-2)} + \frac{1}{2}B_{-(n-3)},$$
$$C_{-n} = \frac{5}{2}C_{-(n-1)} - 2C_{-(n-2)} + \frac{1}{2}C_{-(n-3)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (2.2)-(2.3) hold for all integer n.

B_n and C_n are the sequences A000295 (Eulerian numbers), A052548 in [17], respectively.

Next, we present the first few values of the Bigollo and Bigollo-Lucas numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
B_n	0	1	4	11	26	57	120	247	502	1013	2036	4083	8178	16369
B_{-n}		0	$\frac{1}{2}$	$\frac{5}{4}$	$\frac{17}{8}$	$\frac{49}{16}$	$\frac{129}{32}$	$\frac{321}{64}$	$\frac{769}{128}$	$\frac{1793}{256}$	$\frac{4097}{512}$	$\frac{9217}{1024}$	$\frac{20481}{2048}$	$\frac{45057}{4096}$
C_n	3	4	6	10	18	34	66	130	258	514	1026	2050	4098	8194
C_{-n}		$\frac{5}{2}$	$\frac{9}{4}$	$\frac{17}{8}$	$\frac{33}{16}$	$\frac{65}{32}$	$\frac{129}{64}$	$\frac{257}{128}$	$\frac{513}{256}$	$\frac{1025}{512}$	$\frac{2049}{1024}$	$\frac{4097}{2048}$	$\frac{8193}{4096}$	$\frac{16385}{8192}$

For all integers n, Bigollo and Bigollo-Lucas numbers can be expressed using Binet's formulas as

$$B_n = 2^{n+1} - n - 2,$$

$$C_n = 2^n + 2,$$

respectively.

Note that Binet's formulas of Mersenne and Mersenne-Lucas numbers, respectively, are

$$M_n = 2^n - 1,$$

$$H_n = 2^n + 1,$$

and so

$$B_n = 2M_n - n, \quad (2.4)$$

$$C_n = H_n + 1. \quad (2.5)$$

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence W_n.

LEMMA 2. *Suppose that* $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ *is the ordinary generating function of the generalized Bigollo sequence* $\{W_n\}_{n\geq 0}$. *Then,* $\sum_{n=0}^{\infty} W_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 4W_0)x + (W_2 - 4W_1 + 5W_0)x^2}{1 - 4x + 5x^2 - 2x^3}.$$

Proof. Take $r = 4, s = -5, t = 2$ in Soykan [22, Lemma 1.1]. □

The previous lemma gives the following results as particular examples.

COROLLARY 3. *Generated functions of Bigollo and Bigollo-Lucas numbers are*

$$\sum_{n=0}^{\infty} B_n x^n = \frac{x}{1 - 4x + 5x^2 - 2x^3},$$

$$\sum_{n=0}^{\infty} C_n x^n = \frac{3 - 8x + 5x^2}{1 - 4x + 5x^2 - 2x^3},$$

respectively.

3. Simson Formulas

There is a well-known Simson Identity (formula) for Mersenne sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized Bigollo sequence $\{W_n\}_{n\geq 0}$.

THEOREM 4 (Simson Formula of Generalized Bigollo Numbers). *For all integers n, we have*

$$\begin{vmatrix} W_{n+2} & W_{n+1} & W_n \\ W_{n+1} & W_n & W_{n-1} \\ W_n & W_{n-1} & W_{n-2} \end{vmatrix} = -2^{n-2}(W_2 - 2W_1 + W_0)(W_2 - 3W_1 + 2W_0)^2.$$

Proof. Take $r = 4, s = -5, t = 2$ in Soykan [21, Theorem 2.2]. □

The previous theorem gives the following results as particular examples.

COROLLARY 5. *For all integers n, Simson formula of Bigollo and Bigollo-Lucas numbers are given as*

$$\begin{vmatrix} B_{n+2} & B_{n+1} & B_n \\ B_{n+1} & B_n & B_{n-1} \\ B_n & B_{n-1} & B_{n-2} \end{vmatrix} = -2^{n-1},$$

$$\begin{vmatrix} C_{n+2} & C_{n+1} & C_n \\ C_{n+1} & C_n & C_{n-1} \\ C_n & C_{n-1} & C_{n-2} \end{vmatrix} = 0,$$

respectively.

4. Some Identities

In this section, we obtain some identities of Bigollo and Bigollo-Lucas numbers. First, we can give a few basic relations between $\{W_n\}$ and $\{B_n\}$.

LEMMA 6. *The following equalities are true:*

- **(a):** $8W_n = (49W_0 - 58W_1 + 17W_2)B_{n+4} + 2(98W_1 - 81W_0 - 29W_2)B_{n+3} + (129W_0 - 162W_1 + 49W_2)B_{n+2}$.
- **(b):** $4W_n = (17W_0 - 18W_1 + 5W_2)B_{n+3} - 2(29W_0 - 32W_1 + 9W_2)B_{n+2} + (49W_0 - 58W_1 + 17W_2)B_{n+1}$.
- **(c):** $2W_n = (5W_0 - 4W_1 + W_2)B_{n+2} - 2(9W_0 - 8W_1 + 2W_2)B_{n+1} + (17W_0 - 18W_1 + 5W_2)B_n$.
- **(d):** $W_n = W_0B_{n+1} + (-4W_0 + W_1)B_n + (5W_0 - 4W_1 + W_2)B_{n-1}$.
- **(e):** $W_n = W_1B_n + (W_2 - 4W_1)B_{n-1} + 2W_0B_{n-2}$.
- **(f):** $2(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = -(-5W_1^2 - W_2^2 + 2W_0W_1 + 4W_1W_2)W_{n+4} + 2(-9W_1^2 - 2W_2^2 + 4W_0W_1 - W_0W_2 + 8W_1W_2)W_{n+3} + (4W_0^2 + 25W_1^2 + 5W_2^2 - 20W_0W_1 + 8W_0W_2 - 22W_1W_2)W_{n+2}$.
- **(g):** $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = (W_1^2 - W_0W_2)W_{n+3} + (2W_0^2 - 5W_0W_1 + 4W_0W_2 - W_1W_2)W_{n+2} + (5W_1^2 + W_2^2 - 2W_0W_1 - 4W_1W_2)W_{n+1}$.
- **(h):** $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = (2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{n+2} + (W_2^2 - 2W_0W_1 + 5W_0W_2 - 4W_1W_2)W_{n+1} - 2(-W_1^2 + W_0W_2)W_n$.
- **(i):** $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2B_n = (8W_0^2 + 16W_1^2 + W_2^2 - 22W_0W_1 + 5W_0W_2 - 8W_1W_2)W_{n+1} - (10W_0^2 + 18W_1^2 - 25W_0W_1 + 2W_0W_2 - 5W_1W_2)W_n + 2(2W_0^2 + 4W_1^2 - 5W_0W_1 - W_1W_2)W_{n-1}$.

(j): $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)^2 B_n = (22W_0^2 + 46W_1^2 + 4W_2^2 - 63W_0W_1 + 18W_0W_2 - 27W_1W_2)W_n - (36W_0^2 + 72W_1^2 + 5W_2^2 - 100W_0W_1 + 25W_0W_2 - 38W_1W_2)W_{n-1} + 2(8W_0^2 + 16W_1^2 + W_2^2 - 22W_0W_1 + 5W_0W_2 - 8W_1W_2)W_{n-2}$.

Proof. Note that all the identities hold for all integers n. We prove (a). To show (a), writing

$$W_n = a \times B_{n+4} + b \times B_{n+3} + c \times B_{n+2}$$

and solving the system of equations

$$\begin{aligned} W_0 &= a \times B_4 + b \times B_3 + c \times B_2 \\ W_1 &= a \times B_5 + b \times B_4 + c \times B_3 \\ W_2 &= a \times B_6 + b \times B_5 + c \times B_4 \end{aligned}$$

we find that $a = \frac{1}{8}(49W_0 - 58W_1 + 17W_2), b = \frac{1}{4}(98W_1 - 81W_0 - 29W_2), c = \frac{1}{8}(129W_0 - 162W_1 + 49W_2)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Next, we present a few basic relations between $\{W_n\}$ and $\{C_n\}$.

Lemma 7. *The following equalities are true:*

(a): $4(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = (10W_0 - 19W_1 + 9W_2)W_{n+4} - 2(14W_0 - 27W_1 + 13W_2)W_{n+3} + (18W_0 - 35W_1 + 17W_2)W_{n+2}$.

(b): $2(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = (6W_0 - 11W_1 + 5W_2)W_{n+3} - 2(8W_0 - 15W_1 + 7W_2)W_{n+2} + (10W_0 - 19W_1 + 9W_2)W_{n+1}$.

(c): $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = (4W_0 - 7W_1 + 3W_2)W_{n+2} - 2(5W_0 - 9W_1 + 4W_2)W_{n+1} + (6W_0 - 11W_1 + 5W_2)W_n$.

(d): $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = 2(3W_0 - 5W_1 + 2W_2)W_{n+1} - 2(7W_0 - 12W_1 + 5W_2)W_n + 2(4W_0 - 7W_1 + 3W_2)W_{n-1}$.

(e): $(W_0 - 2W_1 + W_2)(2W_0 - 3W_1 + W_2)C_n = 2(5W_0 - 8W_1 + 3W_2)W_n - 2(11W_0 - 18W_1 + 7W_2)W_{n-1} + 4(3W_0 - 5W_1 + 2W_2)W_{n-2}$.

Now, we give a few basic relations between $\{B_n\}$ and $\{C_n\}$.

Lemma 8. *The following equalities are true*

$$\begin{aligned} 8C_n &= 17B_{n+4} - 50B_{n+3} + 33B_{n+2}, \\ 4C_n &= 9B_{n+3} - 26B_{n+2} + 17B_{n+1}, \\ 2C_n &= 5B_{n+2} - 14B_{n+1} + 9B_n, \\ C_n &= 3B_{n+1} - 8B_n + 5B_{n-1}, \\ C_n &= 4B_n - 10B_{n-1} + 6B_{n-2}. \end{aligned}$$

5. Relations Betwen Special Numbers

In this section, we present identities on Bigollo and Bigollo-Lucas numbers and Mersenne and Mersenne-Lucas numbers. We know that

$$\begin{aligned} B_n &= 2M_n - n, \\ C_n &= H_n + 1. \end{aligned}$$

We also note that from Lemma 8, we have

$$2C_n = 5B_{n+2} - 14B_{n+1} + 9B_n$$

and from Soykan [19, Lemma 11], we get

$$M_n = 2H_{n+1} - 3H_n.$$

Using the above identities, we see that

$$\begin{aligned} B_n &= 4H_{n+1} - 6H_n - n, \\ 2C_n &= 2M_{n+1} - 2M_n + 4. \end{aligned}$$

Using the above identities (and Lemma 6) we obtain the following Binet's formula of generalized Bigollo numbers in the following forms:

$$\begin{aligned} 2W_n &= (5W_0 - 4W_1 + W_2)B_{n+2} - 2(9W_0 - 8W_1 + 2W_2)B_{n+1} + (17W_0 - 18W_1 + 5W_2)B_n \\ &= 2(3W_2 - 10W_1 + 7W_0)M_n - 2(W_2 - 4W_1 + 3W_0)M_{n+1} - 2(W_2 - 3W_1 + 2W_0)n + 2W_2 - 8W_1 + 8W_0 \\ &= 2(3W_2 - 8W_1 + 5W_0)H_{n+1} - 2(5W_2 - 14W_1 + 9W_0)H_n - 2(W_2 - 3W_1 + 2W_0)n + 2W_2 - 8W_1 + 8W_0. \end{aligned}$$

6. On the Recurrence Properties of Generalized Bigollo Sequence

Taking $r = 4, s = -5, t = 2$ in Soykan [23, Theorem 2], we obtain the following Proposition.

PROPOSITION 9. *For $n \in \mathbb{Z}$, generalized Bigollo numbers (the case $r = 4, s = -5, t = 2$) have the following identity:*

$$W_{-n} = 2^{-n}(W_{2n} - C_n W_n + \frac{1}{2}(C_n^2 - C_{2n})W_0).$$

From the above Proposition and Corollary 6 in [23], we have the following corollary which gives the connection between the special cases of generalized Bigollo sequence at the positive index and the negative index: for modified Bigollo, Bigollo-Lucas and Bigollo numbers: take $W_n = B_n$ with $B_0 = 0, B_1 = 1, B_2 = 4$ and take $W_n = C_n$ with $C_0 = 3, C_1 = 4, C_2 = 6$, respectively. Note that in this case $C_n = H_n$.

COROLLARY 10. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a): *Bigollo sequence:*

$$B_{-n} = 2^{-n}(B_{2n} - B_n C_n).$$

(b): *Bigollo-Lucas sequence:*

$$C_{-n} = 2^{-n-1}\left(C_n^2 - C_{2n}\right).$$

By using the identity $2C_n = 5B_{n+2} - 14B_{n+1} + 9B_n$ (and Proposition 9 or Corollary 10), we get

$$B_{-n} = \frac{1}{2^{n+1}}(14B_nB_{n+1} - 5B_nB_{n+2} - 9B_n^2 + 2B_{2n}).$$

Note also that since $B_n = 2M_n - n$ and $M_{-n} = -\frac{1}{2^n}M_n = \frac{-2^n+1}{2^n}$, we get

$$B_{-n} = -2^{-n+1}M_n + n$$

and since $C_n = H_n + 1$ and $H_{-n} = \frac{1}{2^n}H_n = \frac{2^n+1}{2^n}$ we obtain

$$C_{-n} = 2^{-n}H_n + 1.$$

7. Sums

The following Corollary gives sum formulas of Mersenne and Mersenne-Lucas numbers.

Corollary 11. *For $n \geq 0$, Mersenne and Mersenne-Lucas numbers have the following properties:*

(1)

- **(a):** $\sum_{k=0}^{n} M_k = -(n-1)M_n + 2(n+1)M_{n-1} + 1$.
- **(b):** $\sum_{k=0}^{n} M_{2k} = \frac{1}{3}(-(n-3)M_{2n} + 4(n+1)M_{2n-2} + 3)$.
- **(c):** $\sum_{k=0}^{n} M_{2k+1} = \frac{1}{3}(-(n-3)M_{2n+1} + 4(n+1)M_{2n-1} + 2)$.

(2)

- **(a):** $\sum_{k=0}^{n} H_k = -(n-1)H_n + 2(n+1)H_{n-1} - 3$.
- **(b):** $\sum_{k=0}^{n} H_{2k} = \frac{1}{3}(-(n-3)H_{2n} + 4(n+1)H_{2n-2} - 5)$.
- **(c):** $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{3}(-(n-3)H_{2n+1} + 4(n+1)H_{2n-1} - 6)$.

Proof. It is given in Soykan [19, Corollary 25]. □

The following Corollary presents sum formulas of Bigollo and Bigollo-Lucas numbers.

Corollary 12. *For $n \geq 0$, Bigollo and Bigollo-Lucas numbers have the following properties:*

(1)

- **(a):** $\sum_{k=0}^{n} B_k = \frac{1}{2}(-4(n-1)M_n + 8(n+1)M_{n-1} - n^2 - n + 4)$.
- **(b):** $\sum_{k=0}^{n} B_{2k} = \frac{1}{3}(-2(n-3)M_{2n} + 8(n+1)M_{2n-2} - 3(n-1)(n+2))$
- **(c):** $\sum_{k=0}^{n} B_{2k+1} = \frac{1}{3}(-2(n-3)M_{2n+1} + 8(n+1)M_{2n-1} + 4 - 3(n+1)^2)$.

(2)

- **(a):** $\sum_{k=0}^{n} C_k = -(n-1)H_n + 2(n+1)H_{n-1} + n - 2$.
- **(b):** $\sum_{k=0}^{n} C_{2k} = \frac{1}{3}(-(n-3)H_{2n} + 4(n+1)H_{2n-2} + 3n - 2)$.
- **(c):** $\sum_{k=0}^{n} C_{2k+1} = \frac{1}{3}(-(n-3)H_{2n+1} + 4(n+1)H_{2n-1} + 3n - 3)$.

Proof. The proof follows from Corollary 11 and the identities (2.4) and (2.5), i.e.,

$$\begin{aligned} B_n &= 2M_n - n, \\ C_n &= H_n + 1. \quad \square \end{aligned}$$

8. Matrices Related With Generalized Bigollo Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{8.1}$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 2$. From (2.1) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} W_{n+1} \\ W_n \\ W_{n-1} \end{pmatrix} \tag{8.2}$$

and from (8.1) (or using (8.2) and induction) we have

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}.$$

If we take $W = B$ in (8.2) we have

$$\begin{pmatrix} B_{n+2} \\ B_{n+1} \\ B_n \end{pmatrix} = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} B_{n+1} \\ B_n \\ B_{n-1} \end{pmatrix}.$$

We also define

$$N_n = \begin{pmatrix} B_{n+1} & -5B_n + 2B_{n-1} & 2B_n \\ B_n & -5B_{n-1} + 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & -5B_{n-2} + 2B_{n-3} & 2B_{n-2} \end{pmatrix}$$

and

$$U_n = \begin{pmatrix} W_{n+1} & -5W_n + 2W_{n-1} & 2W_n \\ W_n & -5W_{n-1} + 2W_{n-2} & 2W_{n-1} \\ W_{n-1} & -5W_{n-2} + 2W_{n-3} & 2W_{n-2} \end{pmatrix}$$

Theorem 13. *For all integer m, n, we have*

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(a): $N_n = A^n$, *i.e.,*

$$A^n = \begin{pmatrix} 4 & -5 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} B_{n+1} & -5B_n + 2B_{n-1} & 2B_n \\ B_n & -5B_{n-1} + 2B_{n-2} & 2B_{n-1} \\ B_{n-1} & -5B_{n-2} + 2B_{n-3} & 2B_{n-2} \end{pmatrix}.$$

(b): $U_1 A^n = A^n U_1$

(c): $U_{n+m} = U_n N_m = N_m U_n$.

Proof. Take $r = 4, s = -5, t = 2$ in Soykan [22, Theorem 5.1.]. □

Some properties of matrix A^n can be given as

$$\begin{aligned} A^n &= 4A^{n-1} - 5A^{n-2} + 2A^{n-3}, \\ A^{n+m} &= A^n A^m = A^m A^n, \\ \det(A^n) &= 2^n, \end{aligned}$$

for all integer m and n.

Using the above last Theorem and the identities

$$\begin{aligned} B_n &= 2M_n - n, \\ B_n &= 4H_{n+1} - 6H_n - n, \end{aligned}$$

we obtain the following identities for Mersenne and Mersenne-Lucas numbers.

Corollary 14. *For all integers n, we have the following formulas for Mersenne and Mersenne-Lucas numbers.*

(a): *Mersenne Numbers.*

$$A^n = \begin{pmatrix} 2M_{n+1} - n - 1 & -2M_{n+1} - 4M_n + 3n + 2 & 4M_n - 2n \\ 2M_n - n & 2M_{n+1} - 8M_n + 3n - 1 & -2M_{n+1} + 6M_n - 2n + 2 \\ -M_{n+1} + 3M_n - n + 1 & 4M_{n+1} - 10M_n + 3n - 4 & -3M_{n+1} + 7M_n - 2n + 4 \end{pmatrix}.$$

(b): *Mersenne-Lucas Numbers.*

$$A^n = \begin{pmatrix} 6H_{n+1} - 8H_n - n - 1 & -14H_{n+1} + 20H_n + 3n + 2 & 8H_{n+1} - 12H_n - 2n \\ 4H_{n+1} - 6H_n - n & -10H_{n+1} + 16H_n + 3n - 1 & 6H_{n+1} - 10H_n - 2n + 2 \\ 3H_{n+1} - 5H_n - n + 1 & -8H_{n+1} + 14H_n + 3n - 4 & 5H_{n+1} - 9H_n - 2n + 4 \end{pmatrix}.$$

Theorem 15. *For all integers m, n, we have*

$$W_{n+m} = W_n B_{m+1} + (-5W_{n-1} + 2W_{n-2}) B_m + 2W_{n-1} B_{m-1} \tag{8.3}$$

Proof. Take $r = 4, s = -5, t = 2$ in Soykan [22, Theorem 5.2.]. □

By Lemma 6, we know that

$$\begin{aligned} & (W_0-2W_1+W_2)(2W_0-3W_1+W_2)^2B_m \\ = \; & (2W_0^2+4W_1^2-5W_0W_1-W_1W_2)W_{m+2} \\ & +(W_2^2-2W_0W_1+5W_0W_2-4W_1W_2)W_{m+1}-2(-W_1^2+W_0W_2)W_m \end{aligned}$$

so (8.3) can be written in the following form

$$\begin{aligned} & (W_0-2W_1+W_2)(2W_0-3W_1+W_2)^2W_{n+m} \\ = \; & W_n((2W_0^2+4W_1^2-5W_0W_1-W_1W_2)W_{m+3} \\ & +(W_2^2-2W_0W_1+5W_0W_2-4W_1W_2)W_{m+2}-2(-W_1^2+W_0W_2)W_{m+1}) \\ & +\left(-5W_{n-1}+2W_{n-2}\right)((2W_0^2+4W_1^2-5W_0W_1-W_1W_2)W_{m+2} \\ & +(W_2^2-2W_0W_1+5W_0W_2-4W_1W_2)W_{m+1}-2(-W_1^2+W_0W_2)W_m) \\ & +2W_{n-1}((2W_0^2+4W_1^2-5W_0W_1-W_1W_2)W_{m+1} \\ & +(W_2^2-2W_0W_1+5W_0W_2-4W_1W_2)W_m-2(-W_1^2+W_0W_2)W_{m-1}). \end{aligned}$$

Corollary 16. *For all integers m,n, we have*

$$\begin{aligned} B_{n+m} &= B_nB_{m+1}+\left(-5B_{n-1}+2B_{n-2}\right)B_m+2B_{n-1}B_{m-1}, \\ C_{n+m} &= C_nB_{m+1}+\left(-5C_{n-1}+2C_{n-2}\right)B_m+2C_{n-1}B_{m-1}. \end{aligned}$$

Taking $m=n$ in the last corollary, we obtain the following identities:

$$\begin{aligned} B_{2n} &= B_nB_{n+1}+\left(-5B_{n-1}+2B_{n-2}\right)B_n+2B_{n-1}^2, \\ C_{2n} &= C_nB_{n+1}+\left(-5C_{n-1}+2C_{n-2}\right)B_n+2C_{n-1}B_{n-1}. \end{aligned}$$

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