Original Research Article

On Direct Fuzzy Stability of General Quartic Functional Equation

Abstract

The goal of this paper is to study the Generalized Hyers - Ulam - Rassias (HUR) stability of general quartic functional equation (GQFE)

$$f_q(kr_1+(k-1)r_2)+f_q(kr_1-(k-1)r_2)=2k^4f_q(r_1)+2(k-1)^4f_q(r_2)$$
$$+6k^2(k-1)^2[f_q(r_1+r_2)+f_q(r_1-r_2)]-12k^2(k-1)^2[f_q(r_1)+f_q(r_2)]$$

in fuzzy normed spaces (F.N. spaces) using direct method.

Keywords: Generalised Hyers - Ulam - Rassias (HUR) Stability, Fuzzy Normed Space, General Quartic Functional Equation

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1 Introduction

First of all, let us recollect the chronicle in the stability theory for functional equations (FEs). The stability problem for the FEs about the stability of group homomorphisms was started by Ulam[17]. The Ulam′s question was to an extent solved by Hyers[5]. Subsequently, Hyers' result was extended by several mathematicians like Aoki[1], Th.M.Rassias[14], Găvruta[4] and more.

The QFE was first introduced by Rassias [13], who solved its Ulam stability problem. Later, Lee et al. [10] remodified Rassias' QFE and obtained its general solution. Numerous mathematicians have extensively studied the stability problems of various QFE in a variety of spaces, including intuitionistic fuzzy normed spaces, random normed spaces, non-Archimedean fuzzy normed spaces, Banach spaces and many other (see [8, 9, 11, 12, 15]).

Katsaras [7] explicated the postulation of the fuzzy norm over linear space. Since then, various mathematicians [2, 3, 18] gave the meaning of fuzzy norm over vector space from different perspectives.

Definition 1.1. [2]"A function $\mathcal{M} : \mathcal{F}_b \times \mathbb{R} \to [0,1]$ ($\mathcal{F}_b$ being a real vector space over field F) is labeled as a *fuzzy norm* over $\mathcal{F}_b$ if, $\forall\ x, y \in \mathcal{F}_b$ and all $c, s \in \mathbb{R}, k \in F$:

1. $\mathcal{M}(x, c)$ = 0 for $c \leq 0,$
2. x=0 iff $\mathcal{M}(x, c) = 1$ for all $c > 0,$
3. $\mathcal{M}(kx, c) = \mathcal{M}(x, \frac{c}{|k|})$ if k $\neq 0,$
4. $\mathcal{M}(x + y, s + c) \geq min\{\mathcal{M}(x, s), \mathcal{M}(y, c)\},$
5. $\mathcal{M}(x, .)$ is non-decreasing function in $\mathbb{R}$ and $\lim\limits_{t\to\infty} \mathcal{M}(x, c) = 1,$
6. $\mathcal{M}(x, .)$ is continuous on $\mathbb{R}$, $x \neq 0.$

The pair $(\mathcal{F}_b, \mathcal{M})$ is called as a *fuzzy normed vector space*(F.N. space)[2]. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called a *fuzzy Banach space*."

F. Skof [16] has studied H.U. stability of F.E.

$$f_q(x_1 + x_2) + f_q(x_1 - x_2) = 2f_q(x_1) + 2f_q(x_2)$$

for mapping q from real normed space to Banach space. Under the same set-up, K. W. Jun et. al. [6] analyzed the stability problem of following cubic F.E.

$$f_c(2x_1 + x_2) + f_c(2x_1 - x_2) = 2[f_c(x_1 + x_2) + f_c(x_1 - x_2) + 6f_c(x_1)].$$

In this paper, we prove the Generalised HUR stability of the GQFE

$$\begin{aligned} f_q(kr_1 + (k-1)r_2) + f_q(kr_1 - (k-1)r_2) &= 2k^4 f_q(r_1) + 2(k-1)^4 f_q(r_2) \\ &+6k^2(k-1)^2[f_q(r_1 + r_2) + f_q(r_1 - r_2)] - 12k^2(k-1)^2[f_q(r_1) + f_q(r_2)] \end{aligned} \tag{1.1}$$

in F.N. spaces using direct method.

2 Generalized HUR Stability of the GQFE in Fuzzy Normed Spaces

Throughout this section, let $\mathcal{V}_s$ be a real linear space, and $(\mathcal{F}_n, \mathcal{M}^{'})$ and $(\mathcal{F}_b, \mathcal{M})$ be a F.N. space and fuzzy Banach space respectively. Also, define

$$\begin{aligned} \hat{D}f_q(r_1, r_2) = f_q(kr_1 + (k-1)r_2) + f_q(kr_1 - (k-1)r_2) - 2k^4 f_q(r_1) - 2(k-1)^4 f_q(r_2) \\ -6k^2(k-1)^2[f_q(r_1 + r_2) + f_q(r_1 - r_2)] + 12k^2(k-1)^2[f_q(r_1) + f_q(r_2)] \end{aligned}$$

for all $r_1, r_2 \in \mathcal{V}_s$.

2.1 Theorem

Let $\text{ţ} \in \{1, -1\}$ be fixed and $\eta : \mathcal{V}_s^2 \to \mathcal{F}_n$ be a function such that

$$\mathcal{M}^{'}(\eta(k^{\text{ţ}} r_1, 0), c) \geq \mathcal{M}^{'}(p^{\text{ţ}} \eta(r_1, 0), c) \tag{2.1}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$, where $p > 0$ with $\left(\frac{p}{k^4}\right)^{\text{ţ}} < 1$, and

$$\lim_{b\to\infty} \mathcal{M}^{'}(\eta(k^{\text{ţ}b} r_1, k^{\text{ţ}b} r_2), k^{4\text{ţ}b} c) = 1$$

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for all $r_1, r_2 \in \mathcal{V}_s$ and all $c > 0$. Let $f_q : \mathcal{V}_s \to \mathcal{F}_b$ be a mapping which maps zero to zero and satisfies

$$\mathcal{M}(\hat{D}f_q(r_1, r_2), c) \geq \mathcal{M}^{'}(\eta(r_1, r_2), c) \tag{2.2}$$

for all $r_1, r_2 \in \mathcal{V}_s$ and all $c > 0$. Then the limit

$$Q_n(r_1) = \mathcal{M} - \lim_{b\to\infty} \frac{1}{k^{4\text{ţ}b}} f_q(k^{4\text{ţ}b} r_1)$$

exist for all $r_1 \in \mathcal{V}_s$ with a unique quartic mapping $Q_n : \mathcal{V}_s \to \mathcal{F}_b$ such that

$$\mathcal{M}(f_q(r_1) - Q_n(r_1), c) \geq \mathcal{M}^{'}(\eta(r_1, 0), \frac{c|k^4 - p|}{2}) \tag{2.3}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$.

Proof. Let j= 1. Replace (r_1, r_2) by $(r_1, 0)$ in (2.2) to get

$$\mathcal{M}(2f_q(kr_1) - 2k^4 f_q(r_1), c) \geq \mathcal{M}^{'}(\eta(r_1, 0), c),$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Or we can say,

$$\mathcal{M}\Big(f_q(kr_1) - k^4 f_q(r_1), \frac{c}{2}\Big) \geq \mathcal{M}^{'}(\eta(r_1, 0), c). \tag{2.4}$$

After replacing r_1 by $k^b r_1$ in (2.4) we get,

$$\mathcal{M}\Big(\frac{f_q(k^{b+1} r_1)}{k^4} - f_q(k^b r_1), \frac{c}{2k^4}\Big) \geq \mathcal{M}^{'}(\eta(k^b r_1, 0), c), \tag{2.5}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Using (2.1), we obtain

$$\mathcal{M}\Big(\frac{f_q(k^{b+1} r_1)}{k^4} - f_q(k^b r_1), \frac{c}{2k^4}\Big) \geq \mathcal{M}^{'}\Big(\eta(r_1, 0), \frac{c}{p^b}\Big), \tag{2.6}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Replacing c by $p^b c$ in (2.6), we get

$$\mathcal{M}\Big(\frac{f_q(k^{b+1} r_1)}{k^{4(b+1)}} - \frac{f_q(k^b r_1)}{k^{4b}}, \frac{p^b c}{2k^{4(b+1)}}\Big) \geq \mathcal{M}^{'}(\eta(r_1, 0), c), \tag{2.7}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Next, we have

$$\frac{f_q(k^b r_1)}{k^{4b}} - f_q(r_1) = \sum_{j=0}^{b-1} \Big(\frac{f_q(k^{j+1} r_1)}{k^{4(j+1)}} - \frac{f_q(k^j r_1)}{k^{4j}}\Big).$$

Hence, from above equation and (2.7) we get,

$$\begin{aligned}
\mathcal{M}\Big(\frac{f_q(k^b r_1)}{k^{4b}} \quad &- \quad f_q(r_1), \sum_{j=0}^{b-1} \frac{p^j c}{2k^{4(j+1)}}\Big) \\
&\geq \quad min\Big\{\mathcal{M}\Big(\frac{f_q(k^{j+1} r_1)}{k^{4(j+1)}} - \frac{f_q(k^j r_1)}{k^{4j}}, \frac{p^j c}{2k^{4(j+1)}}\Big) : j = 0, 1, ..., b-1\Big\} \\
&\geq \quad \mathcal{M}^{'}(\eta(r_1, 0), c),
\end{aligned} \tag{2.8}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$. Replacing r_1 by $k^a r_1$ in (2.8), we get

$$\begin{aligned}
\mathcal{M}\Big(\frac{f_q(k^{b+a} r_1)}{k^{4(b+a)}} - \frac{f_q(k^a r_1)}{k^{4a}}, \sum_{j=0}^{b-1} \frac{p^j c}{2k^{4(j+a+1)}}\Big) \quad &\geq \quad \mathcal{M}^{'}(\eta(k^a r_1, 0), c) \\
&\geq \quad \mathcal{M}^{'}\Big(\eta(r_1, 0), \frac{c}{p^a}\Big).
\end{aligned} \tag{2.9}$$

Or simply,

$$\mathcal{M}\Big(\frac{f_q(k^{b+a}r_1)}{k^{4(b+a)}}-\frac{f_q(k^ar_1)}{k^{4a}},\sum_{j=a}^{b+a-1}\frac{p^jc}{2k^{4(j+1)}}\Big)\geq\mathcal{M}^{'}(\eta(r_1,0),c), \tag{2.10}$$

for all $r_1\in\mathcal{V}_s$, $c>0$ and all $a,b\geq 0$. Again replacing c by $\frac{c}{\sum_{j=a}^{b+a-1}\frac{p^j}{2k^{4(j+1)}}}$ in (2.10), we get,

$$\mathcal{M}\Big(\frac{f_q(k^{b+a}r_1)}{k^{4(b+a)}}-\frac{f_q(k^ar_1)}{k^{4a}},c\Big)\geq\mathcal{M}^{'}\Big(\eta(r_1,0),\frac{c}{\sum_{j=a}^{b+a-1}\frac{p^j}{2k^{4(j+1)}}}\Big), \tag{2.11}$$

for all $r_1\in\mathcal{V}_s$, $c>0$ and all $a,b\geq 0$. Since $0<p<k^4$ and $\sum_{j=0}^{\infty}\Big(\frac{p}{k^4}\Big)^j<\infty$, the cauchy criterion for convergence and definition of *fuzzy normed space* implies that $\Big\{\frac{f_q(k^br_1)}{k^{4b}}\Big\}$ is a Cauchy sequence in$(\mathcal{F}_b,\mathcal{M})$ is a *fuzzy Banach space*, hence the sequence converges to a point $Q_n(r_1)\in\mathcal{F}_b$. Define $Q_n(r_1)=\mathcal{V}_s\to\mathcal{F}_b$ by

$$Q_n(r_1)=\mathcal{M}-\lim_{b\to\infty}\frac{f_q(k^br_1)}{k^{4b}}$$

all $r_1\in\mathcal{V}_s$. Letting $a=0$ in (2.11), we get

$$\mathcal{M}\Big(\frac{f_q(k^br_1)}{k^{4b}}-f_q(r_1),c\Big)\geq\mathcal{M}^{'}\Big(\eta(r_1,0),\frac{c}{\sum_{j=0}^{b-1}\frac{p^j}{2k^{4(j+1)}}}\Big), \tag{2.12}$$

for all $r_1\in\mathcal{V}_s$ and all $c>0$. Taking the limit $b\to\infty$ and again using definition of *fuzzy normed space*, we get,

$$\mathcal{M}(f_q(r_1)-Q_n(r_1),c)\geq\mathcal{M}^{'}\Big(\eta(r_1,0),\frac{c}{2}(k^4-p)\Big),$$

for all $r_1\in\mathcal{V}_s$ and all $c>0$. Next we claim that Q_n is quartic. Replacing r_1,r_2 by k^br_1,k^br_2 in (2.2), we have,

$$\mathcal{M}\Big(\frac{1}{k^{4b}}\hat{D}f_q(k^br_1,k^br_2),c\Big)\geq\mathcal{M}^{'}(\eta(k^br_1,k^br_2),k^{4b}c)$$

for all $r_1,r_2\in\mathcal{V}_s$ and all $c>0$. Since,

$$\lim_{b\to\infty}\mathcal{M}^{'}(\eta(k^br_1,k^br_2),k^{4b}c)=1,$$

Q_n satisfies (1.1). Hence, $Q_n(r_1)=\mathcal{V}_s\to\mathcal{F}_b$ is quartic.
For uniqueness, let $Q_n^{'}(r_1)=\mathcal{V}_s\to\mathcal{F}_b$ be another quartic mapping satisfying (2.3). For $r_1\in\mathcal{V}_s$, we have $Q_n(k^br_1)=k^{4b}Q_n(r_1)$ and $Q_n^{'}(k^br_1)=k^{4b}Q_n^{'}(r_1)$ for all $b\in\mathbb{N}$. It follows from (2.3) that

$$\begin{aligned}\mathcal{M}(Q_n(r_1)-Q_n^{'}(r_1),c)&=\mathcal{M}\Big(\frac{Q_n(k^br_1)}{k^{4b}}-\frac{Q_n^{'}(k^br_1)}{k^{4b}},c\Big)\\&\geq min\Big\{\mathcal{M}\Big(\frac{Q_n(k^br_1)}{k^{4b}}-\frac{Q_n^{'}(k^br_1)}{k^{4b}},\frac{c}{2}\Big),\mathcal{M}\Big(\frac{Q_n(k^br_1)}{k^{4b}}-\frac{Q_n^{'}(k^br_1)}{k^{4b}},\frac{c}{2}\Big)\Big\}\\&\geq\mathcal{M}^{'}\Big(\eta(k^br_1,0),\frac{k^{4b}c(k^4-p)}{4}\Big)\\&\geq\mathcal{M}^{'}\Big(\eta(r_1,0),\frac{k^{4b}c(k^4-p)}{4p^b}\Big)\end{aligned}$$

for all $r_1\in\mathcal{V}_s$ and all $c>0$. Since, $\lim\limits_{b\to\infty}\frac{k^{4b}c(k^4-p)}{4p^b}=\infty$, we obtain

$$\lim_{b\to\infty}\mathcal{M}^{'}\Big(\eta(r_1,0),\frac{k^{4b}c(k^4-p)}{4p^b}\Big)=1.$$

So, we conclude that $Q_n(r_1) = Q_n^{'}(r_1)$ for all $r_1 \in \mathcal{V}_s$. Thus $Q_n : \mathcal{V}_s \to \mathcal{F}_b$ is unique quartic mapping as desired.
We can demonstrate the result in the same manner for $j = -1$. This completes the proof. ☐

2.2 Corollary

Let $f_q : \mathcal{V}_s \to \mathcal{F}_b$ be a mapping which maps zero to zero and satisfies

$$\mathcal{M}(\hat{D}f_q(r_1, r_2), c) \geq \begin{cases} \mathcal{M}^{'}(\Lambda, c), & \\ \mathcal{M}^{'}(\Lambda(||r_1||^\beta + ||r_2||^\beta), c), & \beta \neq 4 \\ \mathcal{M}^{'}(\Lambda(||r_1||^\beta ||r_2||^\beta + ||r_1||^{2\beta} + ||r_2||^{2\beta}), c), & \beta \neq 2 \end{cases} \tag{2.13}$$

for all $r_1, r_2 \in \mathcal{V}_s$ and all $c > 0$, where Λ and β are real constants with $\Lambda > 0$. Then there exist a unique quartic mapping $Q_n : \mathcal{V}_s \to \mathcal{F}_b$ such that

$$\mathcal{M}(f_q(r_1) - Q_n(r_1), c) \geq \begin{cases} \mathcal{M}^{'}(\Lambda, \frac{c|k^4-1|}{2}), & \\ \mathcal{M}^{'}(\Lambda||r_1||^\beta, \frac{c|k^4-k^\beta|}{2}), & \beta \neq 4 \\ \mathcal{M}^{'}(\Lambda||r_1||^{2\beta}, \frac{c|k^4-k^{2\beta}|}{2}), & \beta \neq 2 \end{cases} \tag{2.14}$$

for all $r_1 \in \mathcal{V}_s$ and all $c > 0$.

Proof. Taking

$$\eta(r_1, r_2) = \begin{cases} \Lambda, & \\ \Lambda(||r_1||^\beta + ||r_2||^\beta), & \beta \neq 4 \\ \Lambda(||r_1||^\beta ||r_2||^\beta + ||r_1||^{2\beta} + ||r_2||^{2\beta}), & \beta \neq 2 \end{cases}$$

for all $r_1, r_2 \in V$ in Theorem [2.1], we get the desired result. ☐

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