Original Research Article

Automorphisms of Zero Divisor Graphs of Power Four Radical Zero Completely Primary Finite Rings

Abstract

Let R be a commutative unital finite rings and $Z(R)$ be its set of zero divisors. The study of automorphisms of algebraic structures via zero divisor graphs is still an active area of research. Perhaps, because of the fact that automorphisms have got real life application in capturing the symmetries of algebraic structures. In this study, the automorphisms zero divisor graphs of such rings in which the product of any four zero divisor is zero has been determined.

Keywords: Automorphisms; Zero Divisor Graphs; Completely Primary Finite Rings

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1 Introduction

The classification of automorphisms of graphs would have been exhausted if it was possible to find necessary and sufficient conditions to determine the full automorphism group. The classification is still open even though it has been done for some families of graphs. Graphs and graph automorphisms are two important structures studied in mathematics. Interestingly the theory of graphs and graph automorphisms are deeply connected. For instance, Evariste Galois characterized the general quintic univariate polynomial f over rationals by showing that the root of such polynomial cannot be expressed interms of radicals via automorphisms of structures of the splitting fields of f. Some contributions on automorphisms can be mentioned. Ojiema et'al [15] did considerable work on automorphisms unit groups of square radical completely primary finite rings. The research on automorphisms of direct product of finite groups was extensively done by Bidwell [1] while Chikunji [7] computed automorphisms of square radical and cube radical zero completely primary finite rings while Ojiema et'al [16] characterized automorphisms of unit groups of power four radical zero finite commutative completely primary rings. The classification of the 4-nilpotent radical of Jacobson finite rings was advanced in [15] and later by Evgeniy [9] where in both instances they determined the structures of the unit groups of primary and local rings. Moreover, an exposition of the automorphisms of such unit groups can be attributed to the findings of [16]. In all these studies, specific cases of these classes of rings have been considered. For recent survey on automorphisms of zero divisor graphs reference can be made to [10, 11, 12]. We therefore advance further, the classification by considering the graphs and the automorphisms of the zero divisor graphs of such classes of rings for the characteristics $p,\ p^2,\ p^3$, and p^4.

2 Power Four Radical Zero Finite Rings of Characteristic p

Let $R_0 = GR(p^r, p)$ be a Galois ring, U, V and W are finitely generated R_0 modules. Suppose U, V and W are the generators of U, V and W respectively, so that $R = R_0 \oplus R_0u \oplus R_0v \oplus R_0w$ is an additive abelian group. On R define multiplication as follows: $(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0, r_0s_1 + r_1s_0, r_0s_2 + r_2s_0 + r_1s_1, r_0s_3 + r_3s_0 + r_1s_2 + r_2s_1)$. It is well known that the multiplication turns R into a commutative ring with identity $(1, 0, 0, 0)$. Further, it has been proved in Ojiema and Owino [16] that

$$Z(R) = R_0u \oplus R_0v \oplus R_0w,$$

$$(Z(R))^2 = R_0v \oplus R_0w,$$

$$(Z(R))^3 = R_0w,$$

$$(Z(R))^4 = (0).$$

Consequently, the next result in the sequel holds:

Theorem 2.1. *Let* $R_0 = GR(p^r, p)$ *and* $R = R_0 \oplus R_0u \oplus R_0v \oplus R_0w$ *is a ring with respect to multiplication in this section. Then,*

$$Aut((\Gamma(R))) \cong S_{p^r-1} \times S_{p^{2r}-1} \times S_{p^{3r}-p^{2r}}.$$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0$ form a basis for R_0 over its prime subfield R_0/pR_0. With respect to the multiplication in this section, $Ann(Z(R)) = (Z(R))^3 = p^3R_0$. Let $V_1 = Ann(Z(R)) \setminus \{0\}$. Then $|V_1| = p^r - 1$. Since each vertex $x \in V_1$ is adjacent to every other vertex in the graph, $deg(x) = p^{3r} - 2$. Consider $V_2 = \{\xi_iv + a\xi_iw \mid a \in R_0\}$. Then $|V_2| = p^{2r} - p^r$ and each vertex in V_2 is adjacent to a vertex in the form $\xi_iv + \xi_jw$. So the degree of $y \in V_2$ is $p^{2r} - 1$. Finally, $V_3 = \{\xi_iu + b\xi_iv + c\xi_iw,\ b, c \in R_0\}$. So $|V_3| = p^{3r} - p^{2r}$ and the degree of each $z \in V_3$ is $p^r - 1$ since each vertex in V_3 is adjacent to a vertex in V_1 but not V_2. ☐

3 Power Four Radical Zero Finite Rings of Characteristic p^2

Let $R_0 = GR(p^{2r}, p^2)$ be Galois ring of order p^{2r} and characteristic p^2. Suppose U and V are R_0- modules generated by u_1, u_2 and v respectively, so that $R = R_0 \oplus U \oplus V$. If the multiplication on R is defined by $(r_0, r_1, r_2, r_3)(s_0, s_1, s_2, s_3) = (r_0s_0 + pr_1s_1 + pr_2s_2, r_0s_1 + r_1s_0, r_0s_2 + r_2s_0, r_0s_3 + r_3s_0 +$

$r_1s_1 + r_1s_2 + r_2s_1 + r_2s_2)$ so that $pu_i \neq 0, p^2u_i = 0, 1 \leq i \leq 2$ and $pv = 0$, it has been verified in [20] that the multiplication turns R into a ring with identity $(1, 0, 0, 0)$ and $Z(R)$ satisfies the following properties:

$$Z(R) = pR_0 \oplus R_0u_1 \oplus R_0u_2 \oplus R_0v,$$
$$(Z(R))^2 = pR_0 \oplus pR_0u_1 \oplus pR_0u_2 \oplus R_0v,$$
$$(Z(R))^3 = pR_0u_1 \oplus pR_0u_2,$$
$$(Z(R))^4 = (0).$$

The next result gives the structure of the automorphisms of the zero divisor graphs of R. constructed above.

Theorem 3.1. *Let R be a ring defined in this section. Then,*
$Aut(\Gamma(R)) \cong S_{p^{3r}-1} \times S_{p^{6r}-2p^{4r}+p^{3r}} \times S_{p^{5r}-p^{4r}-p^{3r}} \times S_{p^{4r}-p^{3r}} \times S_{p^{6r}-3p^{5r}+3p^{4r}-p^{3r}-2}$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0$ form a basis for R_0 over its prime subfield R_0/pR_0. Using the given multiplication, $Ann(Z(R)) = \{pa_1\xi_iu_1 + pa_2\xi_iu_2 + b\xi_iv \mid a_1, a_2, b \in R_0\}$. Let $V_1 = Ann(Z(R)) \setminus \{0\}$. Then $|V_1| = p^{3r} - 1$. Since each vertex in V_1 is adjacent to other vertex in the graph, the degree of $x \in V_1$ is $p^{6r} - 2$.
Let $X = \{a_1\xi_iu_1 + a_2\xi_iu_2 + b\xi_iv \mid a_1, a_2, b \in R_0,\ a_1 + a_2 \not\equiv 0(\text{mod } p)\}$. Then $|X| = p^{2r}(p^{2r} - p).p^r = p^{5r} - p^{4r}$. Each vertex in X is adjacent to a vertex in X or a vertex of the form $a_1^{'}\xi_i + a_2^{'}\xi_i + b^{'}v$ where $a_1^{'} + a_2^{'} \equiv 0(\text{mod } p)$. So the degree of each vertex in V_2 is $p^{3r} - 1 + p^{4r} - p^{3r} = p^{4r} - 1$. If $Y = \{pr_0 + a_1\xi_iu_1 + a_2\xi_iu_2 + b\xi_iv \mid pr_0 \neq 0, a_1, a_2, b \in R_0, a_1 + a_2 \not\equiv 0(\text{mod } p)\}$. Then $|Y| = (p^{4r} - p^{2r})(p^r - 1)p^r = p^{6r} - p^{5r} - p^{4r} + p^{3r}$. Each vertex in Y is adjacent to a vertex in either V_1 or a vertex of the form $pr_0 + pa_1\xi_i + pa_2\xi_i + bv$ where $pr_0 \neq 0,\ a_1 + a_2 \equiv 0(\text{mod } p)$. So the degree of a vertex in Y is $p^{3r} - 1 + p^{4r} - p^{3r} = p^{4r} - 1$. So we consider $V_2 = X \cup Y$. Next, consider $V_3 = \{a_1^{'}\xi_iu_1 + a_2^{'}\xi_iu_2 + b^{'}\xi_iv \mid a_1^{'} + a_2^{'} \equiv 0(\text{mod } p)\}\} \setminus V_1$. So $|V_3|$ is $p^{5r} - p^{4r} - p^{3r}$. Each vertex in V_3 is adjacent to a vertex in V_1, V_2 or V_3. So the degree of a vertex in V_3 is $p^{5r} - p^{4r} + p^{3r} - 1 + p^{5r} - p^{4r} - p^{3r} = 2p^{5r} - 2p^{4r} - 1$. Next, Consider $V_4 = \{pr_0 + pa_1\xi_i + pa_2\xi_i + bv \mid pr_0 \neq 0, a_1, a_2, b \in R_0,\ a_1 + a_2 \equiv 0(\text{mod } p)\}$. Then $|V_4| = (p^r - 1)(p^{2r})p^r = p^{4r} - p^{3r}$. Each vertex in V_4 is adjacent to a vertex in either V_1 or V_4. So the degree of a vertex in V_4 is $p^{4r} - p^{3r} - 1 + p^{3r} - 1 = p^{4r} - 2$. Finally, consider $V_5 = \{pr_0 + a_1\xi_i + a_2\xi_i + bv \mid pr_0 \neq 0,\ a_1, a_2, b \in R_0,\ a_1 + a_2 \equiv 0(\text{mod } p)\} - V_4$. Then $V_5 = (p^r - 1).p^r(p^{2r} - p^r)(p^{2r} - p^r) = (p^{2r} - p^r)(p^{4r} - 2p^{3r} + p^{2r}) = p^{6r} - 3p^{5r} + 3p^{4r} - p^{3r}$. Each vertex in V_5 is adjacent to a vertex in either V_1, V_4 or V_5. Therefore, the degree of each vertex in V_5 is $p^{3r} - 1 + p^{4r} - p^{3r} + p^{6r} - 3p^{5r} + 3p^{4r} - p^{3r} - 1 = p^{6r} - 3p^{5r} + 4p^{4r} - p^{3r} - 2$. The result follows from the fact that $Aut(\Gamma(R))$ permutes V_1, V_2, V_3, V_4 and V_5 independently. □

4 Power Four Radical Zero Finite Rings of Characteristic p^3

Let $R_0 = GR(p^{3r}, p^3)$ be a Galois ring of order p^{3r} and characteristic p^3. Suppose U and V are R_0- module generated by u and v respectively, so that $R = R_0 \oplus R_0u \oplus R_0v$, if the multiplication on R is defined by $(r_0, r_1, r_3)(s_0, s_1, s_3) = (r_0s_0, r_0s_1 + s_0r_1, r_0s_2 + s_0r_2 + r_1s_1)$ so that $p^2u \neq 0,\ p^3u = 0$ and $pv = 0$, it has been verified in [16] that the multiplication turns R into a ring with identity $(1, 0, 0)$ and $Z(R)$ satisfies the following properties:

$$Z(R) = pR_0 \oplus R_0u \oplus R_0v,$$
$$(Z(R))^2 = p^2R_0 \oplus pR_0u \oplus R_0v,$$
$$(Z(R))^3 = pR_0v,$$
$$(Z(R))^4 = (0).$$

The following result summarizes the structure of the automorphisms of the zero divisor graphs of R.

Theorem 4.1. *Let R be a ring defined in this section. Then*

$$Aut(\Gamma(R)) \cong S_{p^{2r}-1} \times S_{p^{3r}-p^{2r}} \times S_{p^{4r}-p^{2r}} \times S_{2p^{5r}-2p^{4r}} \times S_{p^{6r}-2p^{5r}+p^{4r}}$$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0$ form a basis for R_0 over its prime subfield R_0/pR_0. Using the given multiplication, the annihilator of $Z(R)$, $ann(Z(R)) = \{p^2\xi_i u + a\xi_i v \mid a \in R_0\}$. Consider $V_1 = ann(Z(R)) \setminus \{0\}$. Then $|V_1| = p^{2r} - 1$. Every vertex in V_1 is adjacent to all the other vertices in the graph. So, the degree of $x \in V_1$ is $p^{6r} - 2$. Next, consider $V_2 = \{p^2 r_0 + p^2\xi_i u + a\xi_i v \mid p^2 r_0 \neq 0,\ a \in R_0\}$. Then $|V_2| = p^{3r} - p^{2r}$. Each vertex in V_2 is adjacent to all the other vertices in the graph except vertices of the form $pr_0 + \xi_i u + a\xi_i v,\ a \in R_0$ where r_0 is not a multiple of p. So, the degree of a vertex in V_2 is $p^{5r} - 2$. Now, let $X = \{p^2 r_0 + p\xi_i u + a\xi_i v\} \setminus V_1 \cup V_2$. Then $|X| = p^{4r} - p^{3r}$. Each vertex in X is adjacent to a vertex in V_1 or V_2 or X or Y where $Y = \{p\xi_i u + a\xi_i v \mid a \in R_0\} \setminus V_1$. So $|Y| = p^{3r} - p^{2r}$ which implies that the degree of each vertex in X is $p^{2r} - 1 + p^{3r} - p^{2r} + p^{3r} - p^{2r} + p^{4r} - p^{3r} - 1 = p^{4r} + p^{3r} - 2p^{2r} - 2$, and each vertex in Y is adjacent to a vertex in V_1 or V_2 or X or Y. So the degree of a vertex in Y is $p^{4r} + p^{3r} - 2p^{2r} - 2$. Consequently, we consider $V_3 = X \cup Y$. Next, let $W = \{pr_0 + p\xi_i u + a\xi_i v \mid a \in R_0\} \setminus V_1 \cup V_2 \cup V_3$. Then $|W| = p^{5r} - (p^{2r} + p^{4r} - p^{3r} + p^{3r} - p^{2r}) = p^{5r} - p^{4r}$. Each vertex in W is adjacent to a vertex in V_1 or V_2. So the degree of a vertex in W is $p^{2r} - 1 + p^{3r} - p^{2r} = p^{3r} - 1$. Next, let $Z = \{p^2 r_0 + \xi_i u + a\xi_i v \mid a \in R_0\}$. Then $|Z| = p^r(p^{3r} - p^{2r}).p^r = p^{5r} - p^{4r}$. Each vertex in Z to a vertex in V_1 or Y. So the degree of a vertex in Z is $p^{2r} - 1 + p^{3r} - p^{2r} = p^{3r} - 1$. Consequently, we consider $V_4 = W \cup Z$. Finally, let $V_5 = \{pr_0 + \xi_i u + a\xi_i v \mid a \in R_0\} \setminus Z$. Then $|V_5| = p^{2r}(p^{3r} - p^{2r})p^r - (p^{5r} - p^{4r}) = p^{2r}(p^{4r} - p^{3r}) - (p^{5r} - p^{4r}) = p^{6r} - 2p^{5r} + p^{4r}$. Each vertex in V_5 is adjacent to a vertex in V_1 or X. So, the degree of a vertex in V_5 is $p^{2r} - 1 + (p^{4r} - p^{3r}) = p^{4r} - p^{3r} + p^{2r} - 1$. ☐

5 Power Four Radical Zero Finite Rings of Characteristic p^4

Let $R_0 = GR(p^{4r}, p^4)$ be a Galois ring of order p^{4r} and characteristic p^4. Suppose U and V are R_0-module generated by u and v respectively, so that $R = R_0 \oplus R_0 u \oplus R_0 v$. If the multiplication on R is defined by $(r_0, r_1, r_3)(s_0, s_1, s_3) = (r_0 s_0, r_0 s_1 + s_0 r_1, r_0 s_2 + s_0 r_2 + r_1 s_1)$ with $p^4 u = 0,\ p^3 u \neq 0$ and $pv = 0$, it has been verified that R is a ring with identity $(1, 0, 0)$ and satisfies the following properties:

$$Z(R) = pR_0 \oplus R_0 u \oplus R_0 v,$$

$$(Z(R))^2 = p^2 R_0 \oplus R_0 v,$$

$$(Z(R))^3 = p^3 R_0,$$

$$(Z(R))^4 = (0).$$

The following result summarizes the structure of the automorphisms of the zero divisor graphs of R.

Theorem 5.1. *Let R be a ring defined in this section. Then,*

$$Aut(\Gamma(R)) \cong S_{p^{2r}-1} \times S_{p^{3r}-2p^{2r}+p^r} \times S_{p^{3r}-p^{2r}} \times S_{p^{3r}-p^{2r}+2} \times S_{p^{4r}-p^{3r}}.$$

Proof. Let $\xi_1, \cdots, \xi_r \in R_0$ with $\xi_1 = 1$ such that $\bar{\xi_1}, \cdots, \bar{\xi_r} \in R_0$ form a basis for R_0 over its prime subfield R_0/pR_0. Using the given multiplication, the annihilator of $Z(R)$, $ann(Z(R)) = \{p^3 r_0 + a\xi_i v \mid a \in R_0\}$. Let $V_1 = ann(Z(R)) \setminus \{0\}$. Then $|V_1| = p^{2r} - 1$. Each vertex in V_1 is adjacent to every other vertex in the graph. So the degree of $x \in V_1$ is $p^{5r} - 2$. Let $V_2 = \{p^3 r_0 + \xi_i u + a\xi_i v \mid a \in R_0\}$. Then $|V_2| = (p^r - 1)(p^r - 1)p^r = p^{3r} - 2p^{2r} + p^r$. Every vertex in V_2 is adjacent to a vertex of the form $pr_0 + a\xi_i v$. So, the degree of a vertex in V_2 is $p^{6r} - 2$. Next, let $V_3 = \{p^2 r_0 + b\xi_i v \mid b \in R_0\}$. Then $|V_3| = (p^{2r} - p^r)p^r = p^{3r} - p^{2r}$. Every vertex in V_3 is adjacent to a vertex of the form $p^2 s_0 + a\xi_i u + b\xi_i v$. Then the degree of a vertex in V_3 is $p^{4r} - 2$. Next, let $V_4 = \{p^2 r_0 + \xi_i u + \xi_i v\} \setminus V_2$.

Then $|V_4| = p^{3r} - (p^{2r} - 2) = p^{3r} - p^{2r} + 2$. Each vertex in V_4 is adjacent to a vertex in V_1 or V_3. So the degree of a vertex in V_4 is $p^{2r} - 1 + p^{3r} - p^{2r} = p^{3r} - 1$. Finally, let $V_5 = \{pr_0 + \xi_i v\} \setminus V_1 \cup V_3$. Then $|V_5| = p^{4r} - (p^{2r} + p^{3r} - p^{2r}) = p^{4r} - p^{3r}$. Each vertex in V_5 is adjacent to a vertex in V_1 or V_2. So the degree of a vertex in V_5 is $p^{2r} - 1 + p^{3r} - 2p^{2r} + p^r = p^{3r} - p^{2r} + p^r - 1$. □

Conclusion

In this Study we have determined the automorphisms of such rings in which the product of any four zero divisor is zero and revealed the structures and order formulae for automorphisms. This was achieved by partitioning the ring under consideration into mutually disjoint subset of invertible elements and zero divisors, isolation of zero divisors and determination of there graphs using case to case basis discovery of there maps. To this end, we recommend other researchers to carry out more studies regarding automorphisms of zero divisor graphs of power five radicals in future.

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