Original Research Article

Some Generalized Formula For Sums of Cube

Abstract

The study of integer representations as sums of n cube is still an open area of research. Let $a_1, a_2, a_3, \cdots, a_n$ and d be any positive integers such that $a_n - a_{n-1} = a_{n-1} - a_{n-2} = \cdots = a_2 - a_1 = d$. This study formulates some general results for sums of n cube. In particular, this research introduces and develops the diophantine equation $I = (a_1 + a_2 + a_3 + \cdots + a_n)L = a_1^3 + a_2^3 + a_3^3 + \cdots + a_n^3$ for some integer L. The method involves decomposing integer I into sums of n cube and determination of general representation of integer L using case by case basis.

Keywords: Diophantine Equation; Sums of Cube

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1 Introduction

The process of integer factorization of composite number in the field of cryptography has contributed immensely in E-commerce and banking industry. The security of most of the popular cryptosystems lies purely on the difficulty of the integer decomposition problem. Indeed a number of scholars have obtained results demonstrating the problem of integer factorization via integer decomposition. Due to this direct application of integer decomposition in the field of computer science, researchers have devoted their attention in the subject of factoring large integers since advent of cryptography. For detailed studies regarding integer factorization see [1, 2]. Recent survey on problems of integer decomposition has demonstrated the importance of integer factorization in the study of the diophantine equations. This research is therefore meant to contribute to the study of integer factorization. Current literature's have shown that many researchers have continued to rely on existing formulas on integer decomposition with no generation of new formulas on integer factorization. Perhaps , may be because of the fact that integer decomposition and representation is not an easy task and has been a very long standing problems. This is therefore set to contribute to this knowledge gap by formulating some general results on sums of cube. The study of integer I for which $I = (a_1 + a_2 + a_3 + \cdots + a_n)L = a_1^3 + a_2^3 + a_3^3 + \cdots + a_n^3$ is still hardily available. Most of the attempts have provided solution to diophantine equations problems related to Fermat Last theorem, Ramanujan Nagell equation and with some few exception on the study of polynomial of degree less than 5. For recent work on polynomials of degree less than 5 reference can be made to [1, 2,3,10,11,12] and for detailed recap on studies on Fermat Last Theorem and Ramanujan Nagell equations the reader may see [4,5,6,7,8,9,13,14,15]. In most of this studies the literature on the studies of integer I for which $I = (a_1 + a_2 + a_3 + \cdots + a_n)L = a_1^3 + a_2^3 + a_3^3 + \cdots + a_n^3$ is not known . This study is therefore, set to contribute to this knowledge gap by introducing and developing the formula for integer representation $I = (a_1+a_2+a_3+\cdots+a_n)L = a_1^3+a_2^3+a_3^3+\cdots+a_n^3$ by determining the general integer representation of L.

2 Main Results

In the sequel we present our results and solve some specific cases of our formula $I = (a_1 + a_2 + a_3 + \cdots + a_n)L = a_1^3 + a_2^3 + a_3^3 + \cdots + a_n^3$. Throughout this study the following will be standard $a_n > a_{n-1} > \cdots > a_1$.

Case $i : n = 3$

Proposition 2.1. *$I = (a_1 + a_2 + a_3)(a_2^2 + 2d^2) = a_1^3 + a_2^3 + a_3^3$ has solution in integers if $a_3 - a_2 = a_2 - a_1 = d \geqslant 1$.*

Proof. Suppose $a_2 = a_1 + d$ and $a_3 = a_1 + 2d$. To prove $I = a_1^3 + a_2^3 + a_3^3 = (a_1 + a_2 + a_3)(a_2^2 + 2d^2) \cdots (*)$. It is adequate to establish the equality of equation $(*)$. Thus, $I = a_1^3 + a_2^3 + a_3^3 = I = a_1^3 + (a_1 + d)^3 + (a_1 + 2d)^3 = a_1^3 + a_1^3 + 3a_1^2 d + +3a_1 d^2 + d^3 + a_1^3 + 6a_1^2 d + 12a_1 d^2 + 8d^3 = 3a_1^3 + 9a_1^2 d + 15a_1 d^2 + 9d^3 \cdots (**)$.
On the other hand, $(a_1 + a_2 + a_3)(a_2^2 + 2d^2) = a_1(a_2^2 + 2d^2) + a_2(a_2^2 + 2d^2) + a_3(a_2^2 + 2d^2) = a_1 a_2^2 + 2a_1 d^2 + a_2^3 + 2a_2^2 d^2 + a_2^2 a_3 + 2a_3 d^2. \cdots (**)$. But $a_2 = a_1 + d$ and $a_3 = a_1 + 2d$. So, equation $(**)$ becomes $a_1(a_1 + d)^2 + 2a_1 d^2 + (a_1 + d)^3 + 2(a_1 + d)d^2 + (a_1 + d)^2(a_2 + 2d) + 2(a_1 + 2d)d^2 = a_1(a_1^2+2a_1 d+d^2)+2a_1 d^2+a_1^3+3a_1^2 d+3a_1 d^2+d^3+2a_1 d^2+2d^3+(a_1^2+2a_1 d+d^2)(a_1+2d)+2a_1 d^2+4d^3 = a_1^3 + 2a_1^2 d + a_1 d^2 + a_1 d^2 + 2a_1^2 d + 4a_1 d^2 + 2d^3 + 2a_1 d^2 + 4d^3 = 3a_1^3 + 9a_1^2 d + 15a_1 d^2 + 9d^3 \cdots (***)$.
Since equation $(*)$ is equal to $(***)$ the results easily follows. $\square$

Case $ii : n = 4$

Proposition 2.2. $I = (a_1 + a_2 + a_3 + a_4)(a_2^2 + d(a_3 + a_4 - a_1)) = a_1^3 + a_2^3 + a_3^3 + a_4^3$ *has solution in integers if* $a_4 - a_3 = a_3 - a_2 = a_2 - a_1 = d \geqslant 1.$

Proof. Let $a_4 - a_3 = a_3 - a_2 = a_2 - a_1 = d$. Then, $a_2 = a_1 + d, a_3 = a_1 + 2d, a_4 = a_1 + 3d$. To prove $a_1^3+a_2^3+a_3^3+a_4^3 = (a_1+a_2+a_3+a_4)(a_2^2+d(a_3+a_4-a_1)) \cdots (*)$. It is sufficient to satisfy the equality of equation $(*)$. Thus, $a_1^3+a_2^3+a_3^3+a_4^3 = a_1^3+(a_1+d)^3+(a_1+2d)^3(a_1+3d)^3 = a_1^3+a_1^3+3a_1^2d+3a_1d^2+d^3+a_1^3+6a_1^2d+12a_1d^2+8d^3+a_1^3+9a_1^2d+27a_1d^2+27d^3 = 4a_1^3+18a_1^2d+42a_1d^2+36d^3 \cdots (**)$. On the other hand, $(a_1+a_2+a_3+a_4)(a_2^2+d(a_3+a_4-a_1)) = a_1(a_2^2+d(a_3+a_4-a_1))+a_2(a_2^2+d(a_3+a_4-a_1))+a_3(a_2^2+d(a_3+a_4-a_1))+a_4(a_2^2+d(a_3+a_4-a_1)) \cdots (***)$. But $a_2 = a_1+d, a_3 = a_1+2d, a_4 = a_1+3d$. So, $(a_1+a_2+a_3+a_4)(a_2^2+d(a_3+a_4-a_1)) = a_1^3+3a_1^2d+6a_1d^2+a_1^3+4a_1^2d+9a_1d^2+6d^3+a_1^3+5a_1^2d+12a_1d^2+12d^3+a_1^3+6a_1^2d+15a_1d^2+18d^3 = 4a_1^3+18a_1d^2+4a_1d^2+36d^3 \cdots (****)$. Since, equation $(**)$ and $(****)$ are equal, consequently $a_1^3+a_2^3+a_3^3+a_4^3 = (a_1+a_2+a_3+a_4)(a_2^2+d(a_3+a_4-a_1))$ concluding the proof. ☐

Case $iii : n = 5$

Proposition 2.3. $I = (a_1 + a_2 + a_3 + a_4 + a_5)(a_3^2 + 6d^2) = a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3$ *has solution in integers if* $a_5 - a_4 = a_4 - a_3 = a_3 - a_2 = a_2 - a_1 = d \geqslant 1.$

Proof. Suppose $a_5 - a_4 = a_4 - a_3 = a_3 - a_2 = a_2 - a_1 = d$. Then,$a_2 = a_1 + d, a_3 = a_1 + 2d, a_4 = a_1 + 3d, a_5 = a_1 + 4d$. To prove $a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = (a_1 + a_2 + a_3 + a_4 + a_5)(a_3^2 + 6d^2) \cdots (*)$ need to guarantee the equality of $(*)$. Now, $a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = a_1^3 + (a_1 + d)^3 + (a_1 + 2d)^3 + (a_1 + 3d)^3 + (a_1 + 4d)^3 = a_1^3 + 3a_1^2d + 3a_1d^2 + d^3 + a_1^3 + 6a_1^2d + 12a_1d^2 + 8d^3 + a_1^3 + 9a_1^2d + 27a_1d^2 + 27d^3 + a_1^3 + 12a_1^2d + 48a_1d^2 + 64d^3 = 5a_1^3 + 30a_1^2d + 90a_1d^2 + 100d^3 \cdots (**)$.
On the other hand $(a_1+a_2+a_3+a_4+a_5)(a_3^2+6d^2) = (a_1+a_1+d+a_1+2d+a_1+3d+a_1+4d)((a_1+2d)^2+6d^2) = (5a_1+10d)(a_1^2+4a_1d+4d^2+10d^2) = 5a_1(a_1^2+4a_1d+4d^2+10d^2)+10d(a_1^2+4a_1d+4d^2+10d^2) = 5a_1^3 + 20a_1^2d + 50a_1d^2 + 10a_1^2d + 40a_1d^2 + 100d^3 = 5a_1^3 + 30a_1^2d + 90a_1d^2 + 100d^3 \cdots (* * *)$. Since, we have equation $(**)$ and $(* * *)$ are equal the proof follows immediately. ☐

In this subsection, we provide some examples to support our results in proposition 2.1. We have the following :

a_1^3	a_2^3	a_3^3	$a_1^3 + a_2^3 + a_3^3 = I = (a_1 + a_2 + a_3)(a_2^2 + 2d^2)$	d
1^3	2^3	3^3	36	1
3^3	5^3	7^3	495	2
2^3	5^3	8^3	645	3
3^3	7^3	11^3	1701	4
5^3	11^3	17^3	6369	6
10^3	20^3	30^3	36000	10
5^3	20^3	35^3	51000	15
30^3	38^3	46^3	179208	8
35^3	44^3	53^3	276936	9

Table 1: Sums of Three Cube

In this subsection, we provide some examples to support our results in proposition 2.2. We have the following :

a_1^3	a_2^3	a_3^3	a_4^3	$a_1^3+a_2^3+a_3^3+a_4^3 = I = (a_1+a_2+a_3+a_4)(a_2^2+d(a_3+a_4-a_1))$	d
1^3	2^3	3^3	4^3	100	1
3^3	5^3	7^3	9^3	1224	2
2^3	5^3	8^3	11^3	1976	3
3^3	7^3	11^3	15^3	5076	4
5^3	11^3	17^3	23^3	18536	6
10^3	20^3	30^3	40^3	100000	10
5^3	20^3	35^3	50^3	51125	15
30^3	38^3	46^3	54^3	336672	8
35^3	44^3	53^3	62^3	515264	9

Table 2: Sums of Four Cube

In this subsection, we provide some examples to support our results in proposition 2.3. We have the following :

a_1^3	a_2^3	a_3^3	a_4^3	a_5^3	$a_1^3+a_2^3+a_3^3+a_4^3+a_5^3 = I = (a_1+a_2+a_3+a_4+a_5)(a_3^2+6d^2)$	d
1^3	2^3	3^3	4^3	5^3	225	1
3^3	5^3	7^3	9^3	11^3	2555	2
2^3	5^3	8^3	11^3	14^3	4720	3
3^3	7^3	11^3	15^3	19^3	11935	4
5^3	11^3	17^3	23^3	29^3	42925	6
10^3	20^3	30^3	40^3	50^3	225000	10
5^3	20^3	35^3	50^3	65^3	450625	15
30^3	38^3	46^3	54^3	62^3	575000	8
35^3	44^3	53^3	62^3	71^3	873175	9

Table 3: Sums of Five Cube

Conjecture 2.1. *$I = (a_1+a_2+a_3+\cdots+a_n)L = a_1^3+a_2^3+a_3^3+\cdots+a_n^3$ has solution in integers if $a_n - a_{n-1} = a_{n-1} - a_{n-2} = \cdots = a_2 - a_1 = d$ for some integer L.*

3 Conclusion

This research, has contributed to problem of integer representation of sum n cubes. In particular, the study has partially determine the value of I for which $I = (a_1+a_2+a_3+\cdots+a_n)L = a_1^3+a_2^3+a_3^3+\cdots+a_n^3$ for $a_n - a_{n-1} = a_{n-1} - a_{n-2} = \cdots = a_2 - a_1 = d$ for some integer L. The study devoted its attention by introducing and developing the formula by way of conjectures and demonstrating the validity of the obtained formula through constructions of tables with no formal proof to the formulas stated. We therefore, encourage other researchers and more especially Number Theorist to devote there attention in this fascinating area of research by providing proof of the conjecture outlined in this research.

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