__Original Research Article__

Inverse Hamza Distribution: Properties and Applications to Lifetime Data

Abstract

In this paper, a new distribution, named 'the Inverted Hamza distribution', was introduced. It is an extension of the Hamza distribution and its capable of modelling real life data with upside down bathtub shape and heavy tails was introduced. Mathematical and statistical characteristics such as the quantile function, moments, entropy measure, stochastic ordering and distribution of order statistics have been derived. Furthermore, reliability measures like survival function, hazard function have been derived. The method of maximum likelihood was used for estimating the parameters of the distribution. To demonstrate the applicability of the distribution, a numerical example was given. Based on the results, the proposed distribution performed better than the competing distributions.

Keyword: Hamza Distribution, Inverse Hamza, Inverted Ishita distribution, Inverted Lomax Distribution, Inverted Lindley Distribution, quantile function, lifetime data, order statistics,

1. Introduction

Most statistical investigations have been focused on modelling life time data, and this has led to the proposition of diverse statistical distribution [1]. In modelling life time data, the consideration of the character of the hazard rate is a strong determiner. For instance, in real life we have some life time data with monotone (increasing and non-increasing) hazard rate and some with non-monotone (bathtub and upside down bathtub or unimodal) hazard rates, and several statistical distributions has been proposed to fit each of this categories of data.

[1] proposed a two parameter lifetime distribution with the following probability density function (pdf) and cumulative distribution function respectively:

$$f_H(y) = \frac{\theta^6}{\alpha\theta^5 + 120}\left(\alpha + \frac{\theta}{6} y^6\right) e^{-\theta y} \tag{1}$$

and

$$F_H(y) = 1 - \left[1 + \frac{\theta x\left(\theta^5 y^5 + 6\theta^4 y^4 + 30\theta^3 y^3 + 120\theta^2 y^2 + 360\theta y + 720\right)}{\left(\alpha\theta^5 + 120\right)}\right] e^{-\theta y} \tag{2}$$

For $y > 0$, $\theta > 0$ and $\alpha > 0$.

This distribution is known as the Hamza distribution. The mathematical and statistical properties including the parameter estimation can be shown in [1]. An application from biological and engineering data, have been described in their paper to show its importance, and a discussion of its superiority over other one parameter lifetime distributions such as Lindley distribution due to [2], Ishita distribution by [3], and Pranav distribution by [4], respectively.

The aim of this work is to introduce a new distribution called the inverse Hamza distribution, which is an extension of the Hamza distribution. There has been other proposed distributions were different kinds of transformation techniques were utilized, we have distributions such as; the inverse Ishita distribution, the inverse Rama distribution, inverse Lomax distribution, the inverse Gamma distribution, the two parameter inverse exponential distribution, the modified inverse Rayleigh distribution, the extended inverse lindley distribution, the inverse power Akash distribution, the Weibull inverse Lomax distribution, the inverse power Ishita distribution, transmuted exponentiated exponential distribution, transmuted inverse lindley distribution, the transmuted fretchet distribution, the power inverse Lindley distribution proposed by [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], and [18] respectively . From the literature reviewed in this paper, the inverse and power inverse transformation technique were used to transform distributions that cannot model lifetime data with upside-down bathtub shape and they have shown to produce distribution that are more flexible, more useful for analyzing complex data structure in various field of life, than their corresponding baseline distributions.

The organization of the rest of the paper is divided into eleven sections, introduction of proposed study is discussed in the first section. Inverse Hamza distribution has been defined in the second section. Survival and hazards rate function are discussed in third section. Moment has been derived in fourth section. In the fifth section, the derivation of the quantile function, in the sixth section stochastic ordering has been discussed. Renyi Entropy measure has been discussed in seventh section. Order statistics of proposed distribution has been discussed in eighth section. Maximum likelihood estimation method has been derived for estimation of the parameter of proposed distribution in ninth section. Asymptotic Confidence Interval estimations of the parameter for the proposed distribution was derived in the tenth section. In the eleventh section, application of proposed distribution on real lifetime data has been presented. Conclusions have been given in the last section.

2. The Inverse Hamza (IH) Distribution

If a random variable Y has a Hamza distribution, the variable $X = \frac{1}{Y}$ will have an inverse Hamza distribution (IHD) of equation 1. A random variable X is said to have an inverse Hamza distribution with scale parameter θ, α and its probability density function (pdf) and cumulative density function (cdf) are defined respectively by;

$$f_{IHD}(x) = \frac{\theta^6}{(\alpha\theta^5 + 120)} \left(\frac{\alpha}{x^2} + \frac{\theta}{6x^8} \right) e^{-\frac{\theta}{x}} \; ; x > 0, \theta, \alpha > 0$$

(3)

$$F_{IHD}(x) = \frac{\theta^6}{(\alpha\theta^5 + 120)} \left[\left(\frac{\alpha\theta^5 + 120}{\theta^6} + \frac{1}{6x^6} + \frac{1}{\theta x^5} + \frac{5}{\theta^2 x^4} + \frac{20}{\theta^3 x^3} + \frac{60}{\theta^4 x^2} + \frac{120}{\theta^5 x} \right) e^{-\frac{\theta}{x}} \right]$$

$; x > 0, \theta, \alpha > 0$

(4)

The behavior of the proposed distribution for varying value of θ, α has been presented in figure 1 and figure 2. It is observed from figure 1, the pdf of IHD is

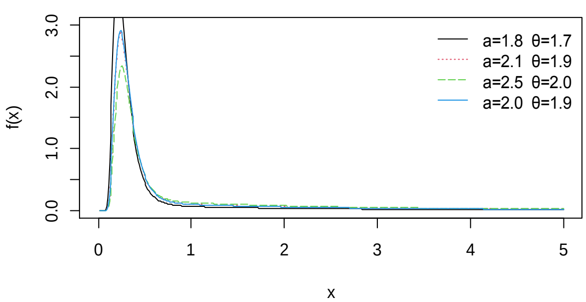

Fig 1a:pdf plot of IHD

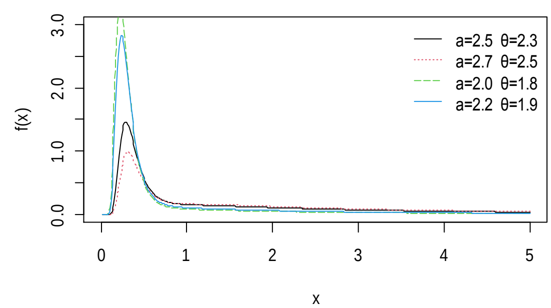

Fig 1b:pdf plot of IHD

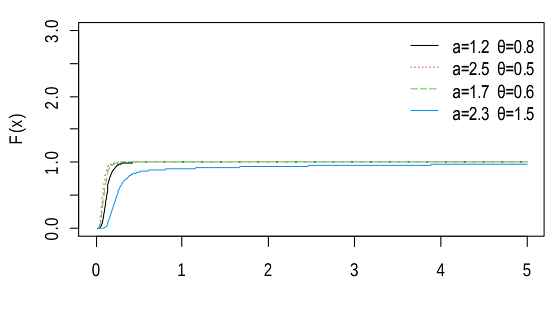

Fig 2a:cdf plot of IHD

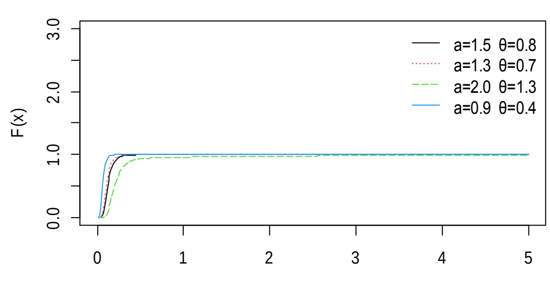

Fig 2b:cdf plot of IHD

3. Survival and Hazard function

Survival function $S(x;\theta,\alpha)$ of IHD can be defined as

$$S(x;\theta,\alpha)=1-F_{IHD}(x)$$

$$S(x;\theta,\alpha)=1-\frac{\theta^6}{(\alpha\theta^5+120)}\left[\left(\frac{\alpha\theta^5+120}{\theta^6}+\frac{1}{6x^6}+\frac{1}{\theta x^5}+\frac{5}{\theta^2x^4}+\frac{20}{\theta^3x^3}+\frac{60}{\theta^4x^2}+\frac{120}{\theta^5x}\right)e^{-\frac{\theta}{x}}\right] \qquad ;x>0,\theta,\alpha>0$$

(5)

And hazard function $h(x;\theta,\alpha)$ of IHD can be defined as

$$h(x;\theta,\alpha)=\frac{f_{IHD}(x;\theta,\alpha)}{S(x;\theta,\alpha)} \quad h(x)=\frac{\frac{\theta^6}{\alpha\theta^5+120}\left(\frac{\alpha}{x^2}+\frac{\theta}{6x^8}\right)e^{-\theta/x}}{1-\frac{\theta^6}{(\alpha\theta^5+120)}\left[\left(\frac{\alpha\theta^5+120}{\theta^6}+\frac{1}{6x^6}+\frac{1}{\theta x^5}+\frac{5}{\theta^2x^4}+\frac{20}{\theta^3x^3}+\frac{60}{\theta^4x^2}+\frac{120}{\theta^5x}\right)e^{-\frac{\theta}{x}}\right]} \quad (6)$$

Figure 3 and 4 below is a presentation of the behavior of the survival and hazard function of IHD respectively, for varying values of the parameter θ,α .

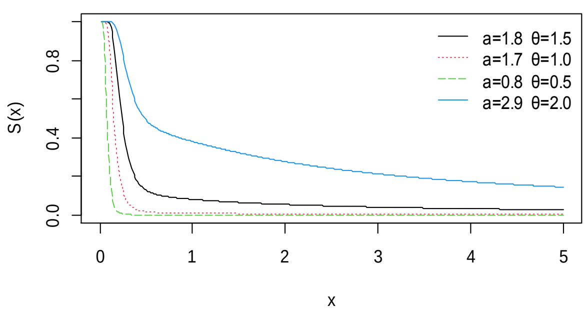

Fig 3A:Survival rate function plot of IHD distribution

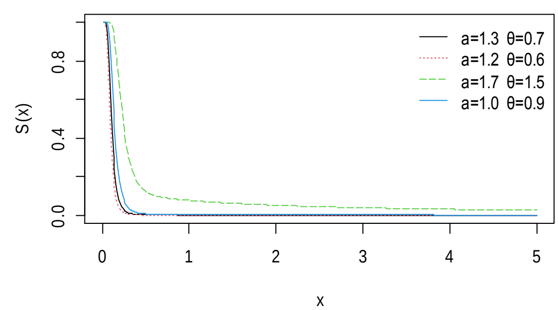

Fig 3B:Survival rate function plot of IHD distribution

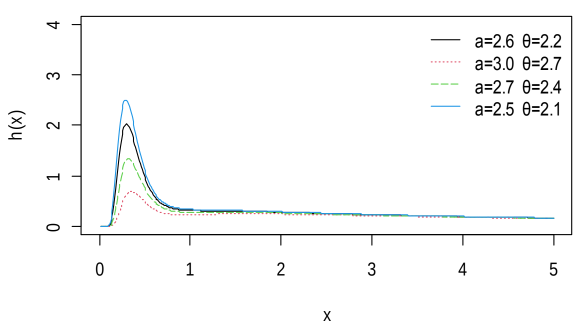

Fig 4a:Hazard rate plot of IHD

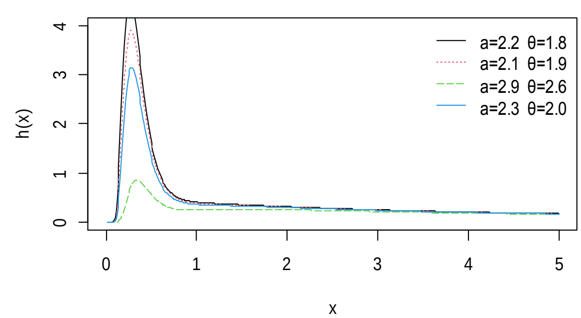

Fig 4b:Hazard rate plot of IHD

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4. Moments

The moments of the distribution is an most important aspect when studying the characteristics of a distribution, and it includes the mean, variance, skewness, kurtosis, etc. [1]. The rth moment about the origin, $\mu_r^{'}$ of IHD can be expressed explicitly in terms of complete gamma functions [1].

Theorem1: Suppose X follows IHD (θ, α).Then the rth moment about the origin, $\mu_r^{'}$, of IHD is

$$\mu_r^{'} = \frac{\theta^r \Gamma(7-r)}{6(\alpha\theta^5 + 120)}$$

Proof: The rth moment X~IHD (θ, α) is obtained as follows

$$\mu_r^{'} = E\left(X^r\right) = \int_0^\infty x^r f_{IHD}(x)dx = \int_0^\infty x^r \frac{\theta^6}{\left(\alpha\theta^5 + 120\right)}\left(\frac{\alpha}{x^2} + \frac{\theta}{6x^8}\right)e^{-\theta x^{-1}}$$

$$= \frac{\theta^6}{\left(\alpha\theta^5 + 120\right)}\left[\alpha\int_0^\infty x^{r-2}e^{-\theta x^{-1}}dx + \frac{\theta}{6}\int_0^\infty x^{r-8}e^{-\theta x^{-1}}dx\right]$$

$$= \frac{\theta^6}{\left(\alpha\theta^5 + 120\right)}\left[\alpha\int_0^\infty x^{r-1-1}e^{-\theta x^{-1}}dx + \frac{\theta}{6}\int_0^\infty x^{r-7-1}e^{-\theta x^{-1}}dx\right]$$

Using inverse gamma function $\int_0^\infty y^{n-\alpha-1}e^{-\theta/y}dy = \frac{\Gamma(\alpha - n)}{\theta^{\alpha-n}}$,

$$= \frac{\theta^6}{\left(\alpha\theta^5 + 120\right)}\left[\frac{\theta}{6}\left(\frac{\Gamma 7 - r}{\theta^{7-r}}\right)\right] = \frac{\theta^r \Gamma 7 - r}{6(\alpha\theta^5 + 120)}$$

(7)

Eq. (7) completes the computation of the rth crude moment of the Inverse Hamza distribution. And it will exist if $r \leq 6$, therefore only the 1st, 2nd, 3rd, 4th, 5th, and 6th non-central moment and the central moment can be obtained.

The mean of the IH distribution is obtained by setting k=1 in (7). Thus,

$$\mu_1^{'} = \frac{20\theta}{(\alpha\theta^5 + 120)}$$

(8)

The mathematical function for obtaining the 2nd, 3rd, and 4th non central moment are given below;

$$\mu_2^{'} = \frac{4\theta^2}{(\alpha\theta^5 + 120)}$$

(9)

$$\mu_3^{'} = \frac{\theta^3}{(\alpha\theta^5 + 120)}$$

(10)

$$\mu_4^{'} = \frac{\theta^4}{3(\alpha\theta^5 + 120)}$$

(11)

The variance of the IH distribution is obtained as follows;

$$\mathrm{var}(x) = E(x^2) - (E(x))^2$$

(12)

$$\mathrm{var}(x) = \frac{4\theta^2}{\alpha\theta^5 + 120} - \frac{400\theta^2}{\left(\alpha\theta^5 + 120\right)^2}$$

$$\operatorname{var}(x) = \mu_2 = \frac{4\alpha\theta^7 + 80\theta^2}{(\alpha\theta^5 + 120)^2}$$

(13)

5. Quantile function

The quantile function is used for the generation of random numbers. It can also be used to derive percentile of a distribution [1]. The quantile function is defined by;

$$Q(u) = F_{IHD}(x; \theta, \alpha, \beta)$$

Where u is distributed as random distribution, $Q(u) \sim [0,1]$, and $F_{IHD}(x; \theta, \alpha, \beta)$ is the cdf of inverse Hamza distribution.

Thus, given that X is a random variable having the pdf IHD, then the quantile $Q(p)$ function is obtained as follows;

$$Q(p) = \frac{\theta^6}{(\alpha\theta^5 + 120)}\left[\left(\frac{\alpha\theta^5 + 120}{\theta^6} + \frac{1}{6x^6} + \frac{1}{\theta x^5} + \frac{5}{\theta^2 x^4} + \frac{20}{\theta^3 x^3} + \frac{60}{\theta^4 x^2} + \frac{120}{\theta^5 x}\right) e^{-\frac{\theta}{x}}\right]$$

$$e^{\theta x^{-1}} = \frac{\theta^6}{(\alpha\theta^5 + 120)Q(p)}\left[\left(\frac{\alpha\theta^5 + 120}{\theta^6} + \frac{1}{6x^6} + \frac{1}{\theta x^5} + \frac{5}{\theta^2 x^4} + \frac{20}{\theta^3 x^3} + \frac{60}{\theta^4 x^2} + \frac{120}{\theta^5 x}\right)\right]$$

$$x = \left(\frac{1}{\theta} 1n\left(\frac{\beta\theta^6}{(\alpha\theta^5 + 120)Q(p)}\left[\left(\frac{\alpha\theta^5 + 120}{\beta\theta^6} + \frac{1}{6\beta x^{6\beta}} + \frac{1}{\theta\beta x^{5\beta}} + \frac{5}{\theta^2\beta x^{4\beta}} + \frac{20}{\theta^3\beta x^{3\beta}} + \frac{60}{\theta^4\beta x^{2\beta}} + \frac{120}{\theta^5\beta x^{\beta}}\right)\right]\right)\right)^{-\frac{1}{\beta}}$$

(14)

6. Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than another random variable Y, where X and Y follows IH distribution, in the

(i) Stochastic order $\left(X \leq_{st} Y\right)$ if $F_X\left(x\right) \geq F_Y\left(x\right)$ for all x.

(ii) Hazard rate order $\left(X \leq_{hr} Y\right)$ if $h_X\left(x\right) \geq h_Y\left(x\right)$ for all x.

(iii) Mean residual life order $\left(X \leq_{mrl} Y\right)$ if $m_X\left(x\right) \geq m_Y\left(x\right)$ for all x.

(iv) Likelihood ratio order $\left(X \leq_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The results above due to [34] are well known for establishing stochastic ordering of distributions

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$$

(15)

$$\Downarrow$$
$$X \leq_{st} Y$$

The IHD is ordered with respect to the strongest 'likelihood ratio ordering' as shown in the following theorem:

Theorem 2: Let X and Y~ IHD $\left(\theta_1, \alpha_1\right)$ and $\left(\theta_2, \alpha_2\right)$ respectively. To show the flexibility of the IHD, its likelihood ratio is defined as

$$\frac{f_X\left(x; \alpha_1, \theta_1\right)}{f_Y\left(x; \alpha_2, \theta_2\right)} = \frac{\dfrac{\theta_1^6}{\left(\alpha_1 \theta_1^5 + 120\right)}\left(\dfrac{\alpha_1}{x^2} + \dfrac{\theta_1}{6x^8}\right) e^{-\theta_1 x^{-1}}}{\dfrac{\theta_2^6}{\left(\alpha_2 \theta_2^5 + 120\right)}\left(\dfrac{\alpha_2}{x^2} + \dfrac{\theta_2}{6x^8}\right) e^{-\theta_2 x^{-1}}}$$

(16)

$$= \frac{\theta_1^6}{\left(\alpha_1 \theta_1^5 + 120\right)}\left(\frac{\alpha_1}{x^2} + \frac{\theta_1}{6x^8}\right) e^{-\theta_1 x^{-1}} * \left(\frac{\theta_2^6}{\left(\alpha_2 \theta_2^5 + 120\right)}\left(\frac{\alpha_2}{x^2} + \frac{\theta_2}{6x^8}\right) e^{-\theta_2 x^{-1}}\right)^{-1}$$

$$=\frac{\theta_1^6\left(\alpha_2\theta_2^5+120\right)}{\theta_2^6\left(\alpha_1\theta_1^5+120\right)}\left(\frac{6\alpha_1x^6+\theta_1}{6\alpha_2x^6+\theta_2}\right)e^{-\theta_1x^{-1}+\theta_2x^{-1}}$$

(17)

Taking natural log of (17), we have

$$1n\frac{f_X\left(x;\alpha_1,\theta_1\right)}{f_Y\left(x;\alpha_2,\theta_2\right)}=6\ln\left(\frac{\theta_1}{\theta_2}\right)+\ln\left(\frac{\alpha_2\theta_2^5+120}{\alpha_1\theta_1^5+120}\right)+\ln\left(\frac{6\alpha_1x^6+\theta_1}{6\alpha_2x^6+\theta_2}\right)-\frac{1}{x}\left(\theta_1-\theta_2\right)$$

Taking the derivative of $ln\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)}$ gives

Hence, $\frac{\partial y}{\partial x}=\frac{36\left(\alpha_1\theta_2-\alpha_2\theta_1\right)x^5}{\left(6\alpha_1x^6+\theta_1\right)\left(6\alpha_2x^6+\theta_2\right)}-\frac{1}{x^2}\left(\theta_2-\theta_1\right)=0$

Thus, for $\theta_2 \geq \theta_1$ and $\alpha_1 = \alpha_2(or\ for\ \alpha_2 \geq \alpha_1\ and\ \theta_1 = \theta_2,$ $\frac{d}{dx}ln\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)} \leq 0$, This implies that $X \leq_{lr} Y$and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

7. Order Statistics

Let $X_1, X_2,..., X_n$ be a random sample of size n from IHD in Eq. (3). Let $X_{(1)} < X_{(2)} < ... < X_n$ denote the corresponding order statistics. The pdf and cdf of the kth order statistic, say $Y = X_{(k)}$ are given by

$$f_Y\left(y\right)=\frac{n!}{(k-1)!(n-k)!}F^{k-1}\left(y\right)\{1-F\left(y\right)\}^{n-k}f\left(y\right)$$

$$=\frac{n!}{(k-1)!(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}(-1)^l F^{K+l-1}\left(y\right)f\left(y\right)$$

(18)

and

$$F_Y(y) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)\{1 - F(y)\}^{n-j}$$

$$= \sum_{j=k}^{n} \sum_{i=0}^{n-j} \binom{n}{j}\binom{n-j}{l} (-1)^l F^{j+1}(y),$$

(19)

Respectively, for $k = 1,2,3,...,n.$

Thus the pdf and cdf of the kth order statistics of IHD are obtained as

$$f_Y(y) = \frac{n!}{(\omega-1)!(n-\omega)!} \sum_{j=0}^{n-\omega} \binom{n-\omega}{j} (-1)^j \left(\frac{\theta^6}{\alpha\theta^5+120} \left(\frac{\alpha\theta^5+120}{\theta^6} + \frac{y^{-6}}{6} + \frac{y^{-5}}{\theta} + \frac{5y^{-4}}{\theta^2} + \frac{20y^{-3}}{\theta^3} + \frac{60y^{-2}}{\theta^4} + \frac{120y^{-1}}{\theta^5} \right) e^{-\theta y^{-1}} \right)^{\omega+j-1} \frac{\theta^6}{\alpha\theta^5+120} \left(\frac{\alpha}{y^2} + \frac{\theta}{6y^8} \right) e^{-\frac{\theta}{y}}$$

(20)

and

$$F_Y(y) = \sum_{j=0}^{n} \sum_{l=0}^{n-j} \binom{n}{j}\binom{n-j}{l} (-1)^l \left[\frac{\theta^6}{\alpha\theta^5+120} \left(\frac{\alpha\theta^5+120}{\theta^6} + \frac{y^{-6}}{6} + \frac{y^{-5}}{\theta} + \frac{5y^{-4}}{\theta^2} + \frac{20y^{-3}}{\theta^3} + \frac{60y^{-2}}{\theta^4} + \frac{120y^{-1}}{\theta^5} \right) e^{-\theta y^{-1}} \right]^{j+l}$$

(21)

8. Renyi Entropy

The measure of variation of uncertainty is said to be an entropy of a random variable. A popular entropy measure is [37]. If X is a continuous random variable having probability density function $f(.)$, then Renyi entropy is defined as

$$J_R(\gamma) = \frac{1}{1-\gamma} \log\left(\int_R f^\gamma(x;\theta,\alpha,\beta)dx \right)$$

$$= \frac{1}{1-\gamma}\log\int_0^\infty\left(\frac{\theta^6}{\alpha\theta^5+120}\left(\frac{6\alpha x^6+\theta}{6x^8}\right)e^{-\theta/y}\right)^\gamma dy$$

(22)

$$= \frac{1}{1-\gamma}\log\sum_{r=0}^{\gamma}\begin{bmatrix}\gamma\\ r\end{bmatrix}\frac{\alpha^{\gamma-r}\theta^{6\gamma+r}}{(\alpha\theta^5+120)^\gamma 6^r}\left\{\int_0^\infty x^{-(8\gamma-6\gamma+6r-1)-1}e^{-\theta\gamma/y}dy\right\}$$

Recall that, $\int_0^\infty x^{-\alpha-1}e^{-\theta/x}dx = \frac{\Gamma\alpha}{\theta^\alpha}$ hence we have

$$= \frac{1}{1-\gamma}\log\sum_{r=0}^{\gamma}\begin{bmatrix}\gamma\\ r\end{bmatrix}\frac{\alpha^{\gamma-r}\theta^{6\gamma+r}}{(\alpha\theta^5+120)^\gamma 6^r}\left\{\frac{\Gamma 8\gamma-6\gamma+6r-1}{\theta^{8\gamma-6\gamma+6r-1}\gamma^{8\gamma-6\gamma+6r-1}}\right\}$$

$$= \frac{1}{1-\gamma}\log\sum_{r=0}^{\gamma}\begin{bmatrix}\gamma\\ r\end{bmatrix}\frac{\alpha^{\gamma-r}\theta^{4\gamma-5r-1}}{(\alpha\theta^5+120)^\gamma 6^r}\left\{\frac{\Gamma 8\gamma-6\gamma+6r-1}{\gamma^{8\gamma-6\gamma+6r-1}}\right\}$$

(23)

9. Maximum Likelihood Estimation Method

Let ($x_1, x_2, x_3, ..., x_n$) be a random sample of size n from Eq.3. The likelihood function, L of IHD is given by

$$L(\theta,\alpha) = \prod_{i=1}^{n} f(x_i,\theta,\alpha)$$

(24)

$$= \prod_{i=1}^{n}\frac{\theta^6(6\alpha x_i^{\,6}+\theta)e^{-\theta/x_i}}{(\alpha\theta^5+120)6x_i^{\,8}}$$

(25)

Taking natural log of (25), we obtain $1nL(\theta,\alpha)$ as

$$= 1n\prod_{i=1}^{n}\frac{\theta^6\left(6\alpha x_i^{\ 6}+\theta\right)e^{-\theta/x_i}}{\left(\alpha\theta^5+120\right)6x_i^{\ 8}}$$

$$= 1n\left[\left(\frac{\theta^6}{\alpha\theta^5+120}\right)^n * \sum_{i=1}^{n}\left(\frac{\left(6\alpha x_i^6+\theta\right)}{6x_i^8}\right) * e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x}\right)}\right]$$

$$= n1n\left(\theta^6\right) - n1n\left(\alpha\theta^5+120\right) + \sum_{i=1}^{n}1n\left(6\alpha x_i^6+\theta\right) - \sum_{i=1}^{n}1n\left(6x_i^8\right) - \theta\sum_{i=1}^{n}x_i^{-1}$$

(26)

The partial derivatives in terms of the parameter (α,θ), are given as follows

$$\frac{dLL}{d\theta} = \frac{6n}{\theta} - \frac{5n\alpha\theta^4}{\alpha\theta^5+120} + \sum_{i=1}^{n}\frac{1}{\left(6\alpha x_i^6+\theta\right)} - \sum_{i=1}^{n}x_i^{-1} = 0$$

(27)

$$\frac{dLL}{d\alpha} = \frac{-n\theta^5}{\alpha\theta^5+120} + \sum_{i=1}^{n}\left(\frac{6x_i^6}{\left(6\alpha x_i^6+\theta\right)}\right) = 0$$

(28)

10. Asymptotic Confidence Interval of the inverse Hamza distribution

In this section, we present the asymptotic confidence intervals for the parameters of the IHD distribution. Let $\hat{\psi} = \left(\hat{\theta},\hat{\alpha}\right)^T$ be the maximum likelihood estimate of $\psi = (\theta,\alpha)^T$. Under the condition that the parameters are in the interior of the parameter space, but not on the boundary, the asymptotic distribution of $\sqrt{n}\left(\hat{\Psi}-\Psi\right)$is $N_3\left(0, \mathrm{I}^{-1}(\Psi)\right)$, where $\mathrm{I}(\Psi)$ is the expected fisher information matrix. The asymptotic behavior of the expected information matrix can be approximate by the observed information matrix, denoted by $\mathrm{I}_n\left(\hat{\Psi}\right)$. The observed information matrix of the inverse power Hamza is given by

$$\mathrm{I}_n(\Psi)=\begin{bmatrix}\frac{\partial^2 L(\theta,\alpha)}{\partial\theta^2} & \frac{\partial^2 L(\theta,\alpha)}{\partial\theta\partial\alpha}\\ \frac{\partial^2 L(\theta,\alpha)}{\partial\alpha\partial\theta} & \frac{\partial^2 L(\theta,\alpha)}{\partial\alpha^2}\end{bmatrix}$$

(29)

Thus,

$$\mathrm{I}_n^{-1}(\hat{\Psi})=\left(n\mathrm{I}(\hat{\Psi})\right)^{-1}=\begin{bmatrix}\mathrm{var}\left(\hat{\theta}\right) & \mathrm{cov}\left(\hat{\theta},\hat{\alpha}\right)\\ \mathrm{cov}\left(\hat{\alpha},\hat{\theta}\right) & \mathrm{var}\left(\hat{\alpha}\right)\end{bmatrix}$$

(30)

Taking the second order derivatives of (27) and (28) each with respect to θ and α respectively, we obtain the entries of (30) as follows

$$\frac{\partial^2 LL}{\partial\alpha^2}=\frac{n\theta^{10}}{\left(\alpha\theta^5+120\right)^2}-\sum_{i=1}^{n}\left[\frac{36x_i^{12}}{\left(6\alpha x_i^6+\theta\right)^2}\right]$$

(31)

$$\frac{\partial^2 LL}{\partial\theta^2}=-\frac{\left(2400n\alpha\theta^3-5n\alpha^2\theta^8\right)}{\left(\alpha\theta^5+120\right)^2}-\frac{6n}{\theta^2}-\sum_{i=1}^{n}\frac{1}{\left(6\alpha x_i^6+\theta\right)^2}$$

(32)

$$\frac{\partial^2 LL}{\partial\alpha\partial\theta}=-\frac{600n\theta^4}{\left(\alpha\theta^5+120\right)^2}-6\sum_{i=1}^{n}\left[\frac{x_i^6}{\left(6\alpha x_i^6+\theta\right)^2}\right]$$

(33)

$$\frac{\partial^2 LL}{\partial\theta\partial\alpha}=-\frac{600n\theta^4}{\left(\alpha\theta^5+120\right)^2}-6\sum_{i=1}^{n}\left[\frac{x_i^6}{\left(6\alpha x_i^6+\theta\right)^2}\right]$$

(34)

The six equations above are Eq. (31), (32), (33), (34) respectively.

The expectations in the Fisher information matrix can be obtained numerically. The multivariate normal distribution with mean vector $(0,0,0)^T$ and covariance matrix $I^{-1}(\Psi)$ can be used to construct confidence intervals for the model parameters. The approximate $100(1-\eta)\%$ two sided confidence intervals for $\theta, \alpha, and \beta$ are determined by

$$\hat{\theta} \pm Z_{\eta/2}\sqrt{\operatorname{var}(\hat{\theta})}, \hat{\alpha} \pm Z_{\eta/2}\sqrt{\operatorname{var}(\hat{\alpha})}, \hat{\beta} \pm Z_{\eta/2}\sqrt{\operatorname{var}(\hat{\beta})}$$

(35)

respectively, where $Z_{\eta/2}$ is the upper $(\eta/2)th$ percentile of a standard normal distribution.

11. Application on real data

This section involves the application of the proposed distribution on two real life datasets and the comparison of the proposed distribution with three other probability distributions, they are the inverse Lomax distribution (ILD), inverse Ishita distribution (IID), and the inverse Rama distribution (IRD). Goodness of fit has been decided using the Akaike information criteria, Bayesian information criteria values respectively, which are calculated for each distribution and also compared. As we know that the basis for calculating best goodness of fit during comparisons of distribution is minimum value of AIC and BIC. Comparison of distribution is shown in table 1 as well as their fitted plots are presented below. Table 1 and 2 shows that AIC and BIC of IHD, ILD, IID, and IRD (Four distributions) have been calculated and compared, and it is observed that inverse Hamza distribution (IHD) has minimum value of AIC and BIC in comparison to ILD, IID, and IRD.

Data set 1: This data set represents uncensored breaking stress of carbon fibres in (Gba).

0.92, 0.928, 0.997, 0.9971, 1.061, 1.117, 1.162, 1.183, 1.187, 1.192, 1.196, 1.213, 1.215, 1.2199, 1.22, 1.224, 1.225, 1.228, 1.237, 1.24, 1.244, 1.259, 1.261, 1.263, 1.276, 1.31, 1.321, 1.329, 1.331, 1.337, 1.351, 1.359, 1.388, 1.408, 1.449, 1.4497, 1.45, 1.459, 1.471, 1.475, 1.477, 1.48, 1.489, 1.501, 1.507, 1.515, 1.53, 1.5304, 1.533, 1.544, 1.5443, 1.552, 1.556, 1.562, 1.566, 1.585, 1.586, 1.599, 1.602, 1.614, 1.616, 1.617, 1.628, 1.684, 1.711, 1.718, 1.733, 1.738, 1.743, 1.759, 1.777, 1.794, 1.799, 1.806, 1.814, 1.816, 1.828, 1.83,

1.884, 1.892, 1.944, 1.972, 1.984, 1.987, 2.02, 2.0304, 2.029, 2.035, 2.037, 2.043, 2.046, 2.059, 2.111, 2.165, 2.686, 2.778, 2.972, 3.504, 3.863, 5.306

Source: see [16]

Data set 2 : Relief time of twenty patients receiving an analgesic

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

Source: see [20]

Table 1. MLEs, S.E, LL, AIC, and BIC (Data 1)

Distribution	Parameter Estimation	Standard Error	Log L	AIC	BIC	AICc
IHD	$\alpha = 2.488390\text{e+}05$	1.186328e+04	149.60	303.2086	308.419	303.4586
	$\theta = 1.529781$	1.528462e-01				
IRD	$\theta = 2.6489423$	0.1438201	168.6203	339.2405	341.8457	339.4905
IID	$\theta = 2.08978$	0.130738	155.5259	313.0518	315.657	313.3018
ILD	$b = 1.486350\text{e+}02$	1.077735e+02	149.9139	303.8279	309.0382	304.0779
	$l = 1.032588\text{e-}02$	7.442275e-03				

Table 2. MLEs, S.E, LL, AIC, and BIC (Data 2)

Distribution	Parameter Estimation	Standard Error	Log L	AIC	BIC	AICc
IHD	$\alpha = 3.366091\text{e+}05$	2.372657e+04	32.66913	69.33827	71.32973	70.83827
	$\theta = 1.724941$	3.855735e-01				

IRD	$\theta = 2.818379$	0.355971	36.1725	74.345	75.34073	75.845
IID	$\theta = 2.25893$	0.33081	33.7432	69.4864	71.48213	70.98639
ILD	$b = 160.13398861$	272.07671	32.72626	69.45251	71.44398	70.95251
	$l = 0.01080690$	0.018263				

Based on the results displayed in table 1 and 2 respectively, it is evident that the inverse Hamza distribution has the smallest AIC, BIC, AICc, and log-likelihood values among all competing models, and so it could be chosen as the best model among all distributions which have been fitted to the two data sets.

12. Conclusion

A two-parameter lifetime distribution called the inverse Hamza distribution is introduced as an inverse transformation extension of the Hamza distribution. Its several properties including moments, survival and hazard function, quantile function, stochastic ordering, ordered statistics, Renyi entropy, have been discussed. The parameters of the distribution have been estimated by known method of maximum likelihood estimator. Finally, the performance of the model has been examined, being applied to two data sets and compared with Inverse Ishita distribution, Inverse Lomax distribution, and Inverse Rama distribution. Result shows that the Inverse Hamza distribution gives an adequate fit for the data sets.

Availability of Data and Materials

Source of data used in this work have been properly cited and provided in the work.

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