

Original Research Article

Congruence of partial sum of binomial coefficients

Abstract: Let $p > 3$ be a prime, using the constant term method and the binomial coefficient expansion, we obtained the congruences of partial sum of binomial coefficients module p^2 ,

$$\sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i-1} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \pmod{p^2}.$$

Keywords: Binomial coefficient, Congruence, Constant term method, Power series expansion

1 Introduction

The congruence property of binomial coefficient sum is an important research subject in mathematics. In 1878, Lucas^[1] proved the congruence theorem of binomial coefficients. Let n_0, k_0 be the remainder of n, k with respect to the module p , then

$$\binom{n}{k} \equiv \binom{\left\lfloor \frac{n}{p} \right\rfloor}{\left\lfloor \frac{k}{p} \right\rfloor} \binom{n_0}{k_0} \pmod{p},$$

where $[x]$ is the greatest integer not greater than x . More generally, for any positive integers n and k , if

$$n = n_0 + n_1p + n_2p^2 + \cdots + n_dp^d, k = k_0 + k_1p + k_2p^2 + \cdots + k_dp^d,$$

are the base p expansions of n and k respectively, where $0 \leq n_i, k_i \leq p-1$, then

$$\binom{n}{k} \equiv \binom{n_0}{k_0} \binom{n_1}{k_1} \cdots \binom{n_d}{k_d} \pmod{p}.$$

In 2006, Sun Zhiwei and Pan Hao^[2] proved that for any prime number $p > 3$, then

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3} \right) \pmod{p},$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre Symbol. In 2011, Sun Zhiwei and Tauraso^[3] extended and

proved the congruence module p^2 ,

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{1}{k+1} \binom{2k}{k} \equiv \frac{3}{2} \left(\frac{p}{3}\right) - \frac{1}{2} \pmod{p^2}$$

where $p \geq 5$. Similar congruence on sums of combinatorial sequences have been studied in [9].

q-Analogues of congruences were also widely discussed by several authors (see, for instance, [10–14]).

Apagodu and Zeiberger^[6] conjectured that for any prime $p > 3$ and any positive integer r ,

$$\sum_{k=0}^{rp-1} \binom{2k}{k} \equiv \begin{cases} \alpha_r \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}, \\ -\alpha_r \pmod{p^2}, & \text{if } p \equiv 2 \pmod{3}, \end{cases}$$

where

$$\alpha_r = \sum_{k=0}^{r-1} \binom{2k}{k}.$$

Liu^[7] confirmed the conjecture and Sun^[8] proved a stronger version,

$$\frac{1}{n} \left(\sum_{k=0}^{pn-1} \binom{2k}{k} - \left(\frac{p}{3}\right) \sum_{k=0}^{n-1} \binom{2k}{k} \right) \equiv 0 \pmod{p^2}.$$

Based on the research of Sun Zhiwei and Tauraso^[3], the following conclusions are obtained:

Theorem 1 Let p be an odd prime number, if $p \equiv 1 \pmod{3}$, we have

$$\sum_{i=1}^{\left[\frac{p}{3}\right]} \binom{2p}{3i} \equiv \sum_{i=1}^{\left[\frac{p}{3}\right]} \binom{2p}{3i-1} \pmod{p^2}.$$

If $p \equiv -1 \pmod{3}$, we have

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\left[\frac{p}{3}\right]} \binom{2p}{3i} \pmod{p^2}.$$

2 Preliminaries

Definition 1^[4] (Constant term method) Given Laurent polynomial $P(x_1, x_2, \dots, x_n)$, define

$$CT[P(x_1, x_2, \dots, x_n)]$$

is the constant term of $P(x_1, x_2, \dots, x_n)$ and define

$$COEFF_{[x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}]} P(x_1, x_2, \dots, x_n)$$

is the coefficient of $x_1^{m_1} x_2^{m_2} \dots x_n^{m_n}$.

Lemma 1^[3] Let p be an odd prime number, then

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv \begin{cases} 0 \pmod{p^2}, & \text{if } p = 3, \\ 1 \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}, \\ -1 \pmod{p^2}, & \text{if } p \equiv -1 \pmod{3}. \end{cases}$$

Lemma 2^[5] For any real number a , we have

$$\sum_{i=1}^n aq^{i-1} = \begin{cases} \frac{a(1-q^n)}{1-q}, & \text{if } q \neq 1, \\ na, & \text{if } q = 1, \end{cases}$$

and

$$\sum_{n=1}^{\infty} aq^{n-1} = \begin{cases} \infty, & \text{if } |q| \geq 1, \\ \frac{a}{1-q}, & \text{if } |q| < 1. \end{cases}$$

3 Proof of Theorem 1

By Definition 1, we have

$$\sum_{n=0}^{p-1} \binom{2n}{n} = \sum_{n=0}^{p-1} CT \left(\frac{(1+x)^{2n}}{x^n} \right) = CT \sum_{n=0}^{p-1} \left[\frac{(1+x)^2}{x} \right]^n. \quad (1)$$

By Lemma 2, we obtain

$$\sum_{n=0}^{p-1} \left[\frac{(1+x)^2}{x} \right]^n = \frac{1 - \left(\frac{(1+x)^2}{x} \right)^p}{1 - \frac{(1+x)^2}{x}}. \quad (2)$$

By (1) and (2), we get

$$\sum_{n=0}^{p-1} \binom{2n}{n} = CT \frac{1 - \left(\frac{(1+x)^2}{x} \right)^p}{1 - \frac{(1+x)^2}{x}}. \quad (3)$$

Multiply x^p on the numerator and denominator of the right-hand side of equation (3), then

$$\sum_{n=0}^{p-1} \binom{2n}{n} = CT \frac{x^p - (1+x)^{2p}}{x^p - (1+x)^2 x^{p-1}} = CT \frac{x^p - (1+x)^{2p}}{x^{p-1} [x - (1+x)^2]} = CT \frac{(1+x)^{2p} - x^p}{(1+x+x^2)x^{p-1}}. \quad (4)$$

Since

$$1+x+x^2 = \frac{1-x^3}{1-x}. \quad (5)$$

Substitute (5) into (4) to get

$$\sum_{n=0}^{p-1} \binom{2n}{n} = CT \frac{[(1+x)^{2p} - x^p](1-x)}{(1-x^3)x^{p-1}}, \quad (6)$$

When $|x| < 1$, we have

$$\frac{1}{1-x^3} = \sum_{i=0}^{\infty} x^{3i}. \quad (7)$$

By (6) and (7), we get

$$\begin{aligned} \sum_{n=0}^{p-1} \binom{2n}{n} &= CT \frac{1}{x^{p-1}} [(1+x)^{2p} - x^p] (1-x) \sum_{i=0}^{\infty} x^{3i} \\ &= CT \frac{1}{x^{p-1}} [(1+x)^{2p} - x^p] \left(\sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right) \\ &= COEFF_{[x^{p-1}]} [(1+x)^{2p} - x^p] \left(\sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right). \end{aligned} \quad (8)$$

When $p \equiv 1 \pmod{3}$, we have

$$\begin{aligned} &COEFF_{[x^{p-1}]} [(1+x)^{2p} - x^p] \left(\sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right) \\ &= 1 - \binom{2p}{2} + \binom{2p}{3} - \binom{2p}{5} + \binom{2p}{6} - \dots - \binom{2p}{p-2} + \binom{2p}{p-1}. \end{aligned} \quad (9)$$

By Lemma 1, when $p \equiv 1 \pmod{3}$, we have

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv 1 \pmod{p^2}. \quad (10)$$

Combined (8), (9) and (10)

$$1 - \binom{2p}{2} + \binom{2p}{3} - \binom{2p}{5} + \binom{2p}{6} - \dots - \binom{2p}{p-2} + \binom{2p}{p-1} \equiv 1 \pmod{p^2}.$$

Then

$$\sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i-1} \pmod{p^2}. \quad (11)$$

When $p \equiv -1 \pmod{3}$, we have

$$\begin{aligned} & \text{COEFF}_{[x^{p-1}]} \left[(1+x)^{2p} - x^p \right] \left(\sum_{i=0}^{\infty} x^{3i} - \sum_{i=0}^{\infty} x^{3i+1} \right) \\ &= -1 + \binom{2p}{1} - \binom{2p}{3} + \binom{2p}{4} - \binom{2p}{6} + \dots - \binom{2p}{p-2} + \binom{2p}{p-1}. \end{aligned} \quad (12)$$

By Lemma 1, when $p \equiv -1 \pmod{3}$, we have

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv -1 \pmod{p^2}. \quad (13)$$

Combined (11)-(13), we have

$$-1 + \binom{2p}{1} - \binom{2p}{3} + \binom{2p}{4} - \binom{2p}{6} + \dots - \binom{2p}{p-2} + \binom{2p}{p-1} \equiv -1 \pmod{p^2}.$$

Then

$$\sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \pmod{p^2}. \quad (14)$$

4 Conclusion

By proof of Theorem 1, if prime $p > 3$, we have the congruence

$$\sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \equiv \begin{cases} \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i-1} \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}, \\ \sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \binom{2p}{3i-2} \pmod{p^2}, & \text{if } p \equiv -1 \pmod{3}. \end{cases}$$

We wonder whether the congruence

$$\sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \equiv \begin{cases} \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i-1} \pmod{p^2}, & \text{if } p \equiv -1 \pmod{3}, \\ \sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \binom{2p}{3i-2} \pmod{p^2}, & \text{if } p \equiv 1 \pmod{3}. \end{cases}$$

is established?

It's not hard to prove that it is also true. Since

$$\begin{aligned} \sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \binom{2p}{3i-2} &\equiv \sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \frac{2p(2p-1)(2p-2)\cdots(2p-3i+3)}{(3i-2)!} \equiv 2p \sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \frac{(-1)^{3i-3}}{3i-2} \pmod{p^2}, \\ \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} &\equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \frac{2p(2p-1)(2p-2)\cdots(2p-3i+1)}{(3i)!} \equiv 2p \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \frac{(-1)^{3i-1}}{3i} \pmod{p^2}, \\ \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i-1} &\equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \frac{2p(2p-1)(2p-2)\cdots(2p-3i+2)}{(3i-1)!} \equiv 2p \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \frac{(-1)^{3i-2}}{3i-1} \pmod{p^2}. \end{aligned}$$

When $p \equiv 1 \pmod{3}$,

$$\sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \frac{(-1)^{3i-3}}{3i-2} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \frac{(-1)^{3i-1}}{3i} \pmod{p},$$

this is because the i -th item on the left is congruent to the reciprocal i -th item on the right with respect to the module p , thus

$$\sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \pmod{p^2}.$$

When $p \equiv -1 \pmod{3}$,

$$\sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \frac{(-1)^{3i-2}}{3i-1} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \frac{(-1)^{3i-1}}{3i} \pmod{p},$$

this is also because the i -th item on the left is congruent with the reciprocal i -th item on the right with respect to the module p , so

$$\sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i-1} \pmod{p^2}.$$

In conclusion, if $p > 3$ is a prime, then we obtain the congruences of partial sum of binomial coefficients classified by 3 module p^2 ,

$$\sum_{i=1}^{\lfloor \frac{p+1}{3} \rfloor} \binom{2p}{3i-2} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i-1} \equiv \sum_{i=1}^{\lfloor \frac{p}{3} \rfloor} \binom{2p}{3i} \pmod{p^2}.$$

In the future, we want to get more congruences of partial sum of binomial coefficients classified by k module p^α ($\alpha \geq 2$).

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