Congruence of partial sum of binomial coefficients

Abstract : In this paper, the sum formula containing the central binomial coefficient is studied by the constant term method. Let $p>3$ be a prime, we obtain the following congruence

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\pmod{p^2}.$$

Keywords: Binomial coefficient, Congruence, Constant term method, Power series expansion

1 Introduction

The congruence property of binomial coefficient sum is an important research subject in mathematics. In 1878, Lucas[0] proved the congruence theorem of binomial coefficients. Let n_0, k_0 be the remainder of n, k with respect to the module p, then

$$\binom{n}{k}\equiv\binom{\left[\frac{n}{p}\right]}{\left[\frac{k}{p}\right]}\binom{n_0}{k_0}\pmod{p},$$

where $[x]$ is the greatest integer not greater than x. More generally, for any positive integers n and k, if

$$n=n_0+n_1p+n_2p^2+\cdots+n_dp^d,\ k=k_0+k_1p+k_2p^2+\cdots+k_dp^d,$$

are the base p expansions of n and k respectively, where $0\le n_i,k_i\le p-1$, then

$$\binom{n}{k}\equiv\binom{n_0}{k_0}\binom{n_1}{k_1}\cdots\binom{n_d}{k_d}\pmod{p}.$$

In 2006, Sun Zhiwei and Pan Hao[0] proved that for any prime number $p>3$, then

$$\sum_{k=0}^{p-1}\binom{2k}{k}\equiv\left(\frac{p}{3}\right)\pmod{p},$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre Symbol. In 2011, Sun Zhiwei and Tauraso[0] extended and proved the congruence module p^2,

$$\sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{1}{k+1}\binom{2k}{k} \equiv \frac{3}{2}\left(\frac{p}{3}\right)-\frac{1}{2} \pmod{p^2}$$

where $p \geq 5$.

Apagodu and Zeiberger[6] conjectured that for any prime $p>3$ and any positive integer r,

$$\sum_{k=0}^{rp-1}\binom{2k}{k} \equiv \begin{cases} \alpha_r \pmod{p^2}, \text{ if } p \equiv 1 \pmod 3, \\ -\alpha_r \pmod{p^2}, \text{ if } p \equiv 2 \pmod 3, \end{cases}$$

where

$$\alpha_r = \sum_{k=0}^{r-1}\binom{2k}{k}.$$

Liu[7] confirmed the conjecture and Sun[8] proved a stronger version,

$$\frac{1}{n}\left(\sum_{k=0}^{pn-1}\binom{2k}{k}-\left(\frac{p}{3}\right)\sum_{k=0}^{n-1}\binom{2k}{k}\right) \equiv 0 \pmod{p^2}.$$

Based on the research of Sun Zhiwei and Tauraso0, the following conclusions are obtained:

Theorem 1 Let p be an odd prime number, if $p \equiv 1 \pmod 3$, we have

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i} \equiv \sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1} \pmod{p^2}.$$

If $p \equiv -1 \pmod 3$, we have

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2} \equiv \sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i} \pmod{p^2}.$$

2 Preliminaries

Definition 10 (Constant term method) Given Laurent polynomial $P(x_1, x_2, \ldots, x_n)$, define

$$CT\left[\mathrm{P}(x_1, x_2, \cdots, x_n)\right]$$

is the constant term of $\mathrm{P}(x_1, x_2, \cdots, x_n)$ and define

$$COEEF_{\left[x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}\right]} P(x_1, x_2, \cdots, x_n)$$

is the coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$.

Lemma 1^0 Let p be an odd prime number, then

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv \begin{cases} 0 \pmod{p^2}, \text{ if } p = 3, \\ 1 \pmod{p^2}, \text{ if } p \equiv 1 \pmod 3, \\ -1 \pmod{p^2}, \text{ if } p \equiv -1 \pmod 3. \end{cases}$$

Lemma 2$^{\mathbf{0}}$ For any real number a, we have

$$\sum_{i=1}^{n} aq^{i-1} = \begin{cases} \dfrac{a(1-q^n)}{1-q}, \text{ if } q \neq 1, \\ na, \qquad \text{ if } q = 1, \end{cases}$$

and

$$\sum_{n=1}^{\infty} aq^{n-1} = \begin{cases} \infty, \text{ if } |q| \geq 1, \\ \dfrac{a}{1-q}, \text{ if } |q| < 1. \end{cases}$$

3 Proof of Theorem 1

By Definition 1, we have

$$\sum_{n=0}^{p-1} \binom{2n}{n} = \sum_{n=0}^{p-1} CT\left(\frac{(1+x)^{2n}}{x^n}\right) = CT \sum_{n=0}^{p-1} \left[\frac{(1+x)^2}{x}\right]^n. \tag{1}$$

By Lemma 2, we obtain

$$\sum_{n=0}^{p-1} \left[\frac{(1+x)^2}{x}\right]^n = \frac{1-\left(\dfrac{(1+x)^2}{x}\right)^p}{1-\dfrac{(1+x)^2}{x}}. \tag{2}$$

By (1) and (2) , we get

$$\sum_{n=0}^{p-1}\binom{2n}{n} = CT\frac{1-\left(\frac{(1+x)^2}{x}\right)^p}{1-\frac{(1+x)^2}{x}}. \tag{3}$$

Multiply x^p on the numerator and denominator of the right-hand side of equation (3) , then

$$\sum_{n=0}^{p-1}\binom{2n}{n} = CT\frac{x^p-(1+x)^{2p}}{x^p-(1+x)^2x^{p-1}} = CT\frac{x^p-(1+x)^{2p}}{x^{p-1}\left[x-(1+x)^2\right]} = CT\frac{(1+x)^{2p}-x^p}{\left(1+x+x^2\right)x^{p-1}}. \tag{4}$$

Since

$$1+x+x^2=\frac{1-x^3}{1-x}. \tag{5}$$

Substitute (5) into (4) to get

$$\sum_{n=0}^{p-1}\binom{2n}{n} = CT\frac{\left[(1+\mathrm{x})^{2p}-x^p\right](1-x)}{\left(1-x^3\right)x^{p-1}}, \tag{6}$$

When $|x|<1$, we have

$$\frac{1}{1-x^3}=\sum_{i=0}^{\infty}x^{3i}. \tag{7}$$

By (6) and (7) , we get

$$\begin{aligned}\sum_{n=0}^{p-1}\binom{2n}{n} &= CT\frac{1}{x^{p-1}}\left[(1+x)^{2p}-x^p\right](1-x)\sum_{i=0}^{\infty}x^{3i}\\ &= CT\frac{1}{x^{p-1}}\left[(1+x)^{2p}-x^p\right]\left(\sum_{i=0}^{\infty}x^{3i}-\sum_{i=0}^{\infty}x^{3i+1}\right)\\ &= COEEF_{\left[x^{p-1}\right]}\left[(1+x)^{2p}-x^p\right]\left(\sum_{i=0}^{\infty}x^{3i}-\sum_{i=0}^{\infty}x^{3i+1}\right).\end{aligned} \tag{8}$$

When $p\equiv 1 \pmod 3$, we have

$$\begin{aligned}&COEEF_{\left[x^{p-1}\right]}\left[(1+x)^{2p}-x^p\right]\left(\sum_{i=0}^{\infty}x^{3i}-\sum_{i=0}^{\infty}x^{3i+1}\right)\\ &=1-\binom{2p}{2}+\binom{2p}{3}-\binom{2p}{5}+\binom{2p}{6}-\ldots\binom{2p}{p-2}+\binom{2p}{p-1}.\end{aligned} \tag{9}$$

By Lemma 1, when $p\equiv 1 \pmod 3$, we have

$$\sum_{n=0}^{p-1}\binom{2n}{n}\equiv 1 \pmod{p^2}. \tag{10}$$

Combined (8), (9) and (10)

$$1-\binom{2p}{2}+\binom{2p}{3}-\binom{2p}{5}+\binom{2p}{6}-\ldots\binom{2p}{p-2}+\binom{2p}{p-1}\equiv 1 \pmod{p^2}.$$

Then

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1}\pmod{p^2}. \tag{11}$$

When $p\equiv -1 \pmod 3$, we have

$$\begin{aligned}&\mathrm{COEEF}_{\left[x^{p-1}\right]}\left[(1+x)^{2p}-x^p\right]\left(\sum_{i=0}^{\infty}x^{3i}-\sum_{i=0}^{\infty}x^{3i+1}\right)\\&=-1+\binom{2p}{1}-\binom{2p}{3}+\binom{2p}{4}-\binom{2p}{6}+\ldots-\binom{2p}{p-2}+\binom{2p}{p-1}.\end{aligned} \tag{12}$$

By Lemma 1, when $p\equiv -1 \pmod 3$, we have

$$\sum_{n=0}^{p-1}\binom{2n}{n}\equiv -1 \pmod{p^2}. \tag{13}$$

Combined (11)-(13), we have

$$-1+\binom{2p}{1}-\binom{2p}{3}+\binom{2p}{4}-\binom{2p}{6}+\ldots-\binom{2p}{p-2}+\binom{2p}{p-1}\equiv -1 \pmod{p^2}.$$

Then

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\pmod{p^2}. \tag{14}$$

4 Conclusion

By proof of Theorem 1, if prime $p>3$, we have the congruence

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\equiv\begin{cases}\displaystyle\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1}\pmod{p^2}, \text{ if } p\equiv 1 \pmod 3,\\ \displaystyle\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2}\pmod{p^2}, \text{ if } p\equiv -1 \pmod 3.\end{cases}$$

We can't help but ask the congruence

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\equiv\begin{cases}\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1}\left(\bmod p^2\right), \text{if } p\equiv -1 \pmod 3,\\ \sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2}\left(\bmod p^2\right), \text{if } p\equiv 1 \pmod 3.\end{cases}$$

is it established?

It's not hard to prove that it is also true. Since

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2}\equiv\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\frac{2p(2p-1)(2p-2)\cdots(2p-3i+3)}{(3i-2)!}\equiv 2p\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\frac{(-1)^{3i-3}}{3i-2}\left(\bmod p^2\right),$$

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\frac{2p(2p-1)(2p-2)\cdots(2p-3i+1)}{(3i)!}\equiv 2p\sum_{i=1}^{\left[\frac{p}{3}\right]}\frac{(-1)^{3i-1}}{3i}\left(\bmod p^2\right),$$

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\frac{2p(2p-1)(2p-2)\cdots(2p-3i+2)}{(3i-1)!}\equiv 2p\sum_{i=1}^{\left[\frac{p}{3}\right]}\frac{(-1)^{3i-2}}{3i-1}\left(\bmod p^2\right).$$

When $p\equiv 1 \pmod 3$,

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\frac{(-1)^{3i-3}}{3i-2}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\frac{(-1)^{3i-1}}{3i} \pmod p,$$

this is because the i-th item on the left is congruent to the reciprocal i-th item on the right with respect to the module p, thus

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\left(\bmod p^2\right).$$

when $p\equiv -1 \pmod 3$,

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\frac{(-1)^{3i-2}}{3i-1}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\frac{(-1)^{3i-1}}{3i} \pmod p,$$

this is also because the i-th item on the left is congruent with the reciprocal i-th item on the right with respect to the module p, so

$$\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1}\left(\bmod p^2\right).$$

In conclusion, if $p>3$ is a prime, then we obtain the following congruence

$$\sum_{i=1}^{\left[\frac{p+1}{3}\right]}\binom{2p}{3i-2}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i-1}\equiv\sum_{i=1}^{\left[\frac{p}{3}\right]}\binom{2p}{3i}\left(\bmod p^{2}\right).$$

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