Original Research Article

FIXED POINTS IN S-METRIC SPACES VIA SIMULATION FUNCTION

Abstract. We introduce the concept of generalized β-γ-Z-contraction mapping with respect to a simulation function ξ in S-metric spaces and study the existence of fixed points for such mappings in complete S-metric spaces. Further, we extend it to partially ordered complete S-metric spaces.

Mathematics Subject Classification. 47H10; 54H25

Keywords. β-γ-continuous mapping; triangular β-orbital admissible mapping with respect to γ; generalized β-γ-Z-contraction mapping.

1. INTRODUCTION

The famous Banach contraction principle, established by Banach[5], guarantees the presence and uniqueness of fixed points in full metric spaces for a contraction mapping. By introducing various contractions in various ambient spaces, several researchers generalized and extended this theory. (See [1], [2], [4], [6], [8], [10], [12], [13]).

Definition 1.1.[17]

"Let X be a non-empty set, an S-metric on X is a function $: X^3 \to [0,+\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

1. $\mathcal{S}(x, y, z) \geq 0$,
2. $\mathcal{S}(x, y, z) = 0$ if and only if $x = y = z$,
3. $\mathcal{S}(x, y, z) \leq \mathcal{S}(x, x, a) + \mathcal{S}(y, y, a) + \mathcal{S}(z, z, a)$,

for all $x, y, z, a \in X$". The pair $(X, \mathcal{S})$ is called an $\mathcal{S}$-metric space.

Definition 1.2. [16]

Let $(X, \mathcal{S})$ be an S-metric space.

(i) A sequence $\{x_n\} \subset X$ converges to $x \in X$ if $\mathcal{S}(x_n, x_n, x) \to 0$ as $n \to +\infty$. That is, for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $n \geq n_0$ we have $\mathcal{S}(x_n, x_n, x) < \epsilon$.

(ii) A sequence $\{x_n\} \subset X$ is a Cauchy sequence if $\mathcal{S}(x_n, x_n, x_m) \to 0$ as $n, m \to +\infty$. That is for each $\epsilon > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$ we have $\mathcal{S}(x_n, x_n, x_m) < \epsilon$.

(iii) S-metric space $(X, \mathcal{S})$ is complete if every Cauchy sequence is a convergent sequence.

Theorem 1.3. Let $(X, \mathcal{S})$ be a S-metric space with coefficient s = 2. Let a mapping $T : X \to X$ satisfy

$\mathcal{S}(T_x, T_x, T_y) \leq \varphi(\mathcal{S}(x, x, y)$ for all $x, y \in X$

Where $\varphi: [0, \infty) \to [0, \infty)$ is an increasing function such that $\lim_{n\to\infty} \varphi^n(t) = 0$ for all $t > 0$. Then T has exactly one fixed point b in X and $\lim_{n\to\infty} \mathcal{S}(T^n(x), T^n(x), b) = 0$ for all $x \in X$.

Lemma 1.4. Suppose $(X, \mathcal{S})$ is a metric space. Let us consider a sequence $\{x_n\}$ in X such that $\mathcal{S}(x_n, x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{r_k\}$ and $\{s_k\}$ with $s_k > r_k > k$ such that $\mathcal{S}(x_{r_k}, x_{r_k}, x_{s_k}) \geq \epsilon$. For each $k > 0$, corresponding to r_k, we can choose s_k to be the smallest positive integer such that $\mathcal{S}(x_{r_k}, x_{r_k}, x_{s_k}) \geq \epsilon$, $\mathcal{S}(x_{r_k}, x_{r_k}, x_{s_{k-1}}) < \epsilon$ and

i. $\lim_{k\to\infty} \mathcal{S}(x_{s_{k-1}}, x_{s_{k-1}}, x_{r_{k+1}}) = \epsilon$

ii. $\lim_{k\to\infty} \mathcal{S}(x_{s_k}, x_{s_k}, x_{r_k}) = \epsilon$

iii. $\lim_{k\to\infty} \mathcal{S}(x_{r_{k-1}}, x_{r_{k-1}}, x_{s_k}) = \epsilon$

iv. $\lim_{k\to\infty} \mathcal{S}(x_{s_k}, x_{s_k}, x_{r_{k+1}}) = \epsilon$.

Lemma 1.5.[3] Suppose $(X, \mathcal{S})$ is a S-metric space with coefficient $s \geq 1$ and let $\{x_n\}$ be a sequence in X such that $\mathcal{S}(x_n, x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{r_k\}$ and $\{s_k\}$ with $s_k > r_k \geq k$ such that $\mathcal{S}(x_{r_k}, x_{r_k}, x_{s_k}) \geq \epsilon$, $\mathcal{S}(x_{r_k}, x_{r_k}, x_{s_{k-1}}) < \epsilon$ and

i. $\epsilon \leq \lim_{k\to\infty} \sup \mathcal{S}\left(x_{r_k}, x_{r_k}, x_{s_k}\right) \leq s\epsilon$

ii. $\frac{\epsilon}{s} \leq \lim_{k\to\infty} \sup \mathcal{S}\left(x_{r_{k+1}}, x_{r_{k+1}}, x_{s_k}\right) \leq s^2\epsilon$

iii. $\frac{\epsilon}{s} \leq \lim_{k\to\infty} \sup \mathcal{S}\left(x_{r_k}, x_{r_k}, x_{s_{k+1}}\right) \leq s^2\epsilon$

iv. $\frac{\epsilon}{s^2} \leq \lim_{k\to\infty} \inf \mathcal{S}\left(x_{r_{k+1}}, x_{r_{k+1}}, x_{s_{k+1}}\right) \leq \lim_{k\to\infty} \sup \mathcal{S}\left(x_{r_{k+1}}, x_{r_{k+1}}, x_{s_{k+1}}\right) \leq s^3\epsilon$.

Definition 1.6.[17] Let $T : X \to X$ be a mapping and $\beta : X * X \to [0,\infty)$ be a function. We say that T be a function. We say that T is an β-admissible mapping if $x, y \in X, \beta(x,y) \geq 1$ $\Rightarrow \beta(Tx, Ty) \geq 1$.

Definition 1.7.[16] Let $T : X \to X$ be a mapping and $\beta : X * X \to [0,\infty)$ is a function. Then we say that T be an β-orbital admissible mapping, if $x, y \in X, \beta(x, Tx) \geq 1 \Rightarrow \beta(Tx, T^2x) \geq 1$.

Definition 1.8.[16] Let $T : X \to X$ be a mapping and $\beta : X * X \to [0,\infty)$ is a function. Then T is a triangular β-orbital admissible mapping, if

i. T is an β-orbital admissible mapping and

ii. $\beta(x,y) \geq 1$ and $\beta(y, Ty) \geq 1 \Rightarrow \beta(x, Ty) \geq 1, \; x, y \in X$.

Remark 1.9. Every triangular β-admissible mapping is a triangular β-orbital admissible mapping. An β-admissible triangular orbital mapping does exist, but it is not an β-admissible triangular mapping.

Definition 1.10.[8] Let $T : X \to X$ and $\beta, \gamma : X * X \to [0,\infty)$. Then T is said to be an β-orbital admissible mapping with respect to γ if $\beta(x, Tx) \geq \gamma(x, Tx)$ implies $\beta(Tx, T^2x) \geq \gamma(x, T^2x)$.

Definition 1.11.[8] Let $T : X \to X$ and $\beta, \gamma : X * X \to [0,\infty)$. Then is said to be an β-orbital admissible mapping with respect to γ if

i. β-orbital admissible mapping with respect to γ

ii. $\beta(x,y) \geq \gamma(x,y)$ and $\beta(y,Ty) \geq \gamma(y,Ty)$ implies $\beta(x,Ty) \geq \gamma(x,Ty)$.

Lemma 1.12.[8] Let T is a triangular β-orbital admissible mapping with respect to γ. Assume that there exists $x_0 \in X$ such that $(x_0, Tx_0) \geq \gamma(x_0, Tx_0)$. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then $\beta(x_m, x_n) \geq \gamma(x_m, x_n)$ for all $m, n \in N$ with $m < n$.

Definition 1.13.[11] Let $(X, \mathcal{S})$ be a metric space and $\beta, \gamma : X * X \to [0,\infty)$. A mapping $T : X \to X$ is said to be β-γ-continuous if every sequence $\{x_n\}$ in X with $\beta(x_n, x_{n+1}) \geq \gamma(x_n, x_{n+1})$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

Definition 1.14. Let $(X, \mathcal{S})$ be a S-metric space and $\beta, \gamma : X * X \to [0,\infty)$. A mapping $T : X \to X$ is said to be β-γ-continuous if every sequence $\{x_n\}$ in X with $\beta(x_n, x_{n+1}) \geq \gamma(x_n, x_{n+1})$ for all $n \in N$ and $x_n \to x$ as $n \to \infty$ implies $Tx_n \to Tx$ as $n \to \infty$.

Definition1.15.[14] A simulation function is a mapping

$$\xi : [0,\infty) * [0,\infty) \to (-\infty,\infty)$$

Satisfying the following conditions:

1) $\xi(0,0)=0$;
2) $\xi(g,h)<g-h$, for all $g,h>0$;
3) If $\{g_n\}$, $\{h_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n = m \in (0,\infty)$, then $\lim_{n\to\infty} \sup \xi(g_n,h_n)<0$.

Remark 1.16. Let ξ be a simulation function, if $\{g_n\}$, $\{h_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} g_n = \lim_{n\to\infty} h_n = m \in (0,\infty)$, then $\lim_{n\to\infty} \sup \xi(kg_n,h_n)<0$ for any $k>1$.

An example of simulation function is as follows:

Example 1.17. Let $\xi : [0,\infty) * [0,\infty) \to (-\infty,\infty)$, be defined by

1) $\xi(g,h) = \delta h - g$ for all $g,h \in [0,\infty)$, where $\delta \in [0,1)$.
2) $\xi(h,g) = \frac{h}{1+h} - g$ for all $g,h \in [0,\infty)$.
3) $\xi(g,h) = h - kg$ otherwise, where $k>1$.
4) $\xi(h,g) = \frac{h}{1+h} - ge^{g}$ for all $g,h \in [0,\infty)$.

Definition 1.18.[14] Let $(X,\mathcal{S})$ be a metric space and T be a selfmap of X. We say that T is a Z-contraction with respect to ξ, if there exists simulation function ξ such that

$$\xi(\mathcal{S}(Tx,Tx,Ty),\mathcal{S}(x,x,y)) \geq 0 \text{ for all } x,y \in X \tag{1.1}$$

Theorem 1.19.[14] Let $(X,\mathcal{S})$ be a S-metric space and $T : X \to X$ be a Z-contraction with respect to a certain simulation function ξ, then T has a unique fixed point in X.

Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Theorem 1.20.[15] Let $(X,\mathcal{S})$ be a S-metric space and T has a unique fixed point in X. If there exists simulation function ξ such that

$$\xi(\mathcal{S}(Tx,Tx,Ty), M_T(x,y)) \geq 0 \text{ for all } x,y \in X, \tag{1.2}$$

Where $M_T(x,y) = \max\{\mathcal{S}(x,x,y),\ \mathcal{S}(x,x,Tx),\ \mathcal{S}(y,y,Ty),\ \frac{\mathcal{S}(x,x,Ty)+\mathcal{S}(y,y,Tx)}{2}\}$, then T has a unique fixed point in X. Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

In Section 2, we prove our main results in which we study the existence of fixed points of generalized β-γ-Z-contraction mapping with respect to a certain simulation function ξ in S-metric spaces. In Section 3, we extend the main results of Section 2 to partially ordered S-metric spaces. In Section 4, we provide corollaries and examples in support of our results.

1. Main results

Theorem 2.1. Let $(X,\mathcal{S})$ be a S-metric space with coefficient $s \geq 1$. Let $T: X \to X$ and $\beta,\gamma: X * X \to [0,\infty)$ be mappings.

Suppose that the following conditions are satisfied:

I. T is a generalized β-γ- Z-contraction with respect to ξ,

II. T is a triangular β-orbital admissible mapping with respect to γ,

III. there exists $x_1 \in X$ such that $\beta(x_1,Tx_1) \geq \gamma(x_1,Tx_1)$ and

IV. T is an β-γ-continuous mapping.

Then T has a fixed point $x^1 \in X$ and $\{T^n x_1\}$ converges to x^1.

Proof. Let $x_1 \in X$ be as in ш, so that $\beta(x_1,Tx_1) \geq \gamma(x_1,Tx_1)$. Now we define a sequence $\{x_h\}$ in X by $x_{h+1} = T^h x_1 = Tx_h$ for all $h \in N$. Suppose that $x_{h_0} = x_{h_{0+1}}$ for some $h_0 \in N$, we have $Tx_{h_0} = x_{h_0}$, so that x_{h_0} is a fixed point of T.

Hence, without loss of generality, we assume that $x_{n+1} \neq x_n$ for all $n \in N$. From (1.3), we have

$$\xi(s^4\, \mathcal{S}(x_{h+1},x_{h+1},x_{h+2}), M_T(x_h,x_{h+1})) = \xi(s^4\, \mathcal{S}(Tx_h,Tx_h,Tx_{h+1}), M_T(x_h,x_{h+1})) \geq 0 \qquad (2.1)$$

where

$$M_T(x_h,x_{h+1}) = \max\{ \mathcal{S}(x_h,x_h,x_{h+1}), \mathcal{S}(x_h,x_h,Tx_h), \mathcal{S}(x_{h+1},x_{h+1},Tx_{h+1}), \frac{\mathcal{S}(x_h,x_h,Tx_{h+1})+\mathcal{S}(x_{h+1},x_{h+1},Tx_h)}{2s}\}$$

$$= \max\{ \mathcal{S}(x_h,x_h,x_{h+1}), \mathcal{S}(x_h,x_h,x_{h+1}), \mathcal{S}(x_{h+1},x_{h+1},x_{h+2}), \frac{\mathcal{S}(x_h,x_h,x_{h+2})+\mathcal{S}(x_{h+1},x_{h+1},x_{h+1})}{2s} \}$$

$$\leq \max\{ \mathcal{S}(x_h,x_h,x_{h+1}),\ \mathcal{S}(x_{h+1},x_{h+1},Tx_{h+2}), \frac{\mathcal{S}(x_h,x_h,x_{h+1})+\mathcal{S}(,x_{h+1},x_{h+1},x_{h+2})}{2s} \}$$

$$= \max\{ \mathcal{S}(x_h,x_h,x_{h+1}),\ \mathcal{S}(x_{h+1},x_{h+1},Tx_{h+2}) \}.$$

Hence $M_T(x_h,x_{h+1}) = \max\{ \mathcal{S}(x_h,x_h,x_{h+1}),\ \mathcal{S}(x_{h+1},x_{h+1},Tx_{h+2}) \}$.

Suppose that $(x_h,x_h,x_{h+1}) \leq \mathcal{S}(x_{h+1},x_{h+1},Tx_{h+2})$ for some $h \in N$. Then we have

$M_T(x_h,x_{h+1}) = \max\{ \mathcal{S}(x_h,x_h,x_{h+1}),\ \mathcal{S}(x_{h+1},x_{h+1},Tx_{h+2})\} = \mathcal{S}(x_{h+1},x_{h+1},Tx_{h+2})$.

Hence from (2.1), we have

$$0 \leq \xi(s^4\, \mathcal{S}(Tx_{h+1},Tx_{h+1},Tx_{h+2}), M_T(x_h,x_{h+1}))$$
$$= \xi(s^4\, \mathcal{S}(Tx_{h+1},Tx_{h+1},Tx_{h+2}), \mathcal{S}(Tx_{h+1},Tx_{h+1},Tx_{h+2}))$$
$$< \mathcal{S}(x_{h+1},x_{h+1},x_{h+2}) - s^4\, \mathcal{S}(Tx_{h+1},Tx_{h+1},Tx_{h+2}) \leq 0, \qquad \text{contradiction.}$$

Hence $\mathcal{S}(x_{h+1},x_{h+1},x_{h+2}) < \mathcal{S}(x_h,x_h,x_{h+1})$ for all $h \in N$. Therefore, $\{\mathcal{S}(Tx_h,Tx_h,Tx_{h+1})\}$ is decreasing and bounded below. So there exists $r \geq 0$ such that $\lim_{n\to\infty} \mathcal{S}(Tx_h,Tx_h,Tx_{h+1}) = r$.

Suppose that $r > 0$. Now, using condition (ξ_3), with $t_h = \mathcal{S}(x_{h+1}, x_{h+1}, x_{h+2})$ and

$s_h = \mathcal{S}(x_h, x_h, x_{h+1})$, we have

$0 \leq \lim_{h\to\infty} sup\, \xi(s^4\, \mathcal{S}(x_{h+1}, x_{h+1}, x_{h+2}), \mathcal{S}(x_h, x_h, x_{h+1}) < 0$, a contradiction.

Therefore, $r = 0$ i.e.,

$$\lim_{h\to\infty} \mathcal{S}(x_h, x_h, x_{h+1}) = 0, \tag{2.2}$$

Now, we show that $\{x_h\}$ is a Cauchy sequence. Suppose that $\{x_h\}$ is not a Cauchy sequence. Now, we consider the following two cases

Case 1 : $s = 1$.

In this case $(X, \mathcal{S})$ is a metric space. Then by Lemma 1.4 there exist $\epsilon > 0$ and sequence of positive integers $\{h_k\}$ and $\{g_k\}$ such that $h_k > g_k \geq k$ satisfying

$$\mathcal{S}(x_{g_k}, x_{g_k}, x_{h_k}) \geq \epsilon. \tag{2.3}$$

Let us choose the smallest h_k satisfying (2.3), then we have $h_k > g_k \geq k$ with $\mathcal{S}(x_{g_k}, x_{g_k}, x_{h_k})$ $\geq \epsilon$ and $\mathcal{S}(x_{g_k}, x_{g_k}, x_{h_{k-1}}) < \epsilon$

satisfying (i)-(iv) of Lemma 1.4.

Hence we have

$M_s(x_{g_k}, x_{h_k}) = \max\{\mathcal{S}(x_{g_k}, x_{g_k}, x_{h_k}), \mathcal{S}(x_{g_k}, x_{g_k}, Tx_{g_k}), \mathcal{S}(x_{h_k}, x_{h_k}, Tx_{h_k}),$

$\frac{\mathcal{S}\left(x_{g_k}, x_{g_k}, Tx_{h_k}\right) + \mathcal{S}(x_{h_k}, x_{h_k}, Tx_{g_k})}{2}$

On taking limit as $k \to \infty$, we have $M_s(x_{g_k}, x_{g_k}, x_{h_k}) = \epsilon$

Using condition (ξ_3) with $t_k = \mathcal{S}(x_{g_{k+1}}, x_{g_{k+1}}, x_{h_{k+1}})$ and $s_k = M(x_{g_k}, x_{g_k}, x_{h_k})$, we have

$0 \leq \lim_{k\to\infty} sup\, \xi(\mathcal{S}(x_{g_{k+1}}, x_{g_{k+1}}, x_{h_{k+1}}), M_s(x_{g_k}, x_{h_k})) < 0$, a contradiction. Thus $\{x_h\}$ is a Cauchy sequence.

Case 2. $s > 1$.

Then by Lemma 1.5 there exists $\epsilon > 0$ and sequence of positive integers $\{h_k\}$ and $\{g_k\}$ such that $h_k > g_k \geq k$ satisfying

$$\mathcal{S}\left(x_{g_k}, x_{g_k}, x_{h_k}\right) \geq \epsilon \tag{2.4}$$

Let us take the smallest h_k satisfying (2.4), then we have $h_k > g_k \geq k$ with $\mathcal{S}\left(x_{g_k}, x_{g_k}, x_{h_k}\right) \geq \in$ and $\mathcal{S}\left(x_{g_k}, x_{g_k}, x_{h_{k-1}}\right) < \epsilon$ satisfying (i)-(iv) of lemma 1.5.

$\epsilon \leq \mathcal{S}\left(x_{g_k}, x_{g_k}, x_{h_k}\right) \leq M_s(x_{g_k}, x_{h_k})$

$$= \max\{\mathcal{S}(x_{g_k}, x_{g_k}, x_{h_k}), \mathcal{S}(x_{g_k}, x_{g_k}, Tx_{g_k}), \mathcal{S}(x_{h_k}, x_{h_k}, Tx_{h_k}), \frac{\mathcal{S}\left(x_{g_k}, x_{g_k}, Tx_{h_k}\right)+\mathcal{S}(x_{g_k}, x_{g_k}, Tx_{h_k})}{2s} \qquad (2.5)$$

Taking $h \to \infty$ in (2.5) and using (i)-(iv) of lemma 1.5, we have

$\epsilon \leq \lim_{k\to\infty} sup\, M_s(x_{g_k}, x_{h_k}) \leq max\{s\epsilon, 0, \frac{s^2\epsilon+s^2\epsilon}{2s}\} = s\epsilon.$

By Lemma 1.12 we have $\beta(x_{g_k}, x_{h_k}) \geq \gamma((x_{g_k}, x_{h_k})$. Hence from (1.3) we have

$0 \leq \xi(s^4\, \mathcal{S}(Tx_{g_k}, Tx_{g_k}, Tx_{h_k}), M_T(x_{g_k}, x_{h_k}))$.

Now we have

$0 \leq \lim_{k\to\infty} sup\, \xi(s^4\, \mathcal{S}(Tx_{g_k}, Tx_{g_k}, Tx_{h_k}), M_T(x_{g_k}, x_{h_k}))$

$\leq \lim_{k\to\infty} \sup[M_T(x_{g_k}, x_{h_k}) - s^4\, \mathcal{S}(Tx_{g_k}, Tx_{g_k}, Tx_{h_k})]$

$\leq \lim_{k\to\infty} \sup M_T(x_{g_k}, x_{h_k}) - s^4 \lim_{k\to\infty} inf\; \mathcal{S}(Tx_{g_k}, Tx_{g_k}, Tx_{h_k})$

$\leq s\epsilon - s^4(\frac{\epsilon}{s^2}) < 0$, a contradiction. So we conclude that $\{x_h\}$ is a Cauchy sequence in $(X, \mathcal{S})$.

Since X is a S-metric space then, there exists $x^1 \in X$ such that $\lim_{h\to\infty} x_h = x^1$. Since T is a β-γ-continuous and $\beta(x_h, x_{h+1}) \geq \gamma(x_h, x_{h+1})$ for all $h \in N$, we have $x^1 = \lim_{n\to\infty} x_{h+1} = \lim_{h\to\infty} Tx_h = T \lim_{h\to\infty} x_h = Tx^1$. Hence T has a fixed point.

In the following theorem, we replace the β-γ-continuity of T by another condition.

Theorem 2.2. Let $(X, \mathcal{S})$ be a S-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\beta, \gamma : X * X \to [0, \infty)$ be mappings.

Suppose that the following conditions are satisfied:

I. T is a generalized β-γ-Z-contraction with respect to ξ,
II. T is a triangular β-orbital admissible mapping with respect to γ,
III. there exists $x_1 \in X$ such that $\beta(x_1, Tx_1) \geq \gamma(x_1, Tx_1)$,
IV. if $\{x_h\}$ is a sequence in X such that $\beta(x_h, x_{h+1}) \geq \gamma(x_h, x_{h+1})$ for all $h \in N$ and $x_h \to x_1 \in X$ as $h \to \infty$, then there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x^1) \geq \gamma(x_{h_p}, x^1)$ for all $p \in N$.

Then $\{T^n x_1\}$ converges to an element x^1 of X and x^1 is a fixed point of T.

Proof. By using similar arguments as in the proof of Theorem 2.1, we obtain that the sequence $\{x_h\}$ defined by $x_{h+1} = Tx_h$ converges to $x^1 \in X$ and $\beta(x_h, x_{h+1}) \geq \gamma(x_h, x_{h+1})$ for all $p \in N$.

By (iv), there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p},\ x^1) \geq \gamma(x_{h_p},\ x^1)$ for all $p \in N$. Hence from (1.3) we have

$$0 \leq \xi(s^4\, \mathcal{S}(Tx_{h_p}, Tx_{h_p}, Tx^1), M_T(x_{h_p}, x^1)) = \xi(s^4\, \mathcal{S}(Tx_{h_p+1}, Tx_{h_{p+1}}, Tx^1), M_T(x_{h_p}, x^1))$$
$$< M_T(x_{h_p}, x^1) - s^4\, \mathcal{S}(Tx_{h_p+1}, Tx_{h_{p+1}}, Tx^1), \quad (2.8)$$

Which implies that $s^4\mathcal{S}(Tx_{h_p+1}, Tx_{h_{p+1}}, Tx^1) < M_T(x_{h_p}, x^1)$.

Now, we have

$$s\mathcal{S}(Tx_{h_p+1}, Tx_{h_{p+1}}, Tx^1) \leq s^4\mathcal{S}(Tx_{h_p+1}, Tx_{h_{p+1}}, Tx^1) < M_T(x_{h_p}, x^1) \quad (2.9)$$

And

$$\mathcal{S}(x^1, x^1, Tx^1) \leq\ M_T(x_{h_p}, x^1)\ = \max\{\mathcal{S}(x_{h_p}, x_{h_p}, x^1), \mathcal{S}(x_{h_p}, x_{h_p}, Tx_{h_p}), \mathcal{S}(x^1, x^1, Tx^1),$$
$$\frac{\mathcal{S}\left(x_{h_p}, x_{h_p}, Tx^1\right) + \mathcal{S}(Tx_{h_p}, Tx_{h_p}, x^1)}{2}\}$$
$$\leq \max\{\mathcal{S}(x_{h_p}, x_{h_p}, x^1), \mathcal{S}(x_{h_p}, x_{h_p}, Tx_{h_p}), \mathcal{S}(x^1, x^1, Tx^1)$$
$$\frac{\mathcal{S}\left(x_{h_p}, x_{h_p}, x^1\right) + \mathcal{S}\left(x^1, x^1, Tx^1\right) + \mathcal{S}(Tx_{h_p}, Tx_{h_p}, x^1)}{2}\},$$

On taking limits as $h \to \infty$ we have

$$\mathcal{S}(x^1, x^1, Tx^1) \leq \lim_{p\to\infty} M_T(x_{h_p}, x^1) \leq \mathcal{S}(x^1, x^1, Tx^1).$$

Therefore $\lim_{p\to\infty} M_T(x_{h_p}, x^1) = \mathcal{S}(x^1, x^1, Tx^1)$

From (2.9) we have

$$c\ \leq s\, \mathcal{S}(x^1, x^1, Tx_{h_p}) + s\mathcal{S}(x_{h_p}, x_{h_p}, Tx^1) \leq s\, \mathcal{S}(x^1, x^1, Tx_{h_p}) + M_T(x_{h_p}, x^1) \quad (2.10)$$

On taking limit as $p \to \infty$ on (2.10), we have

$$\mathcal{S}(x^1, x^1, Tx^1) \leq s \lim_{p\to\infty} \mathcal{S}(x_{h_{p+1}}, x_{h_{p+1}}, Tx^1) \leq\ \mathcal{S}(x^1, x^1, Tx^1). \quad (2.11)$$

Hence we have

$$\lim_{p\to\infty} \mathcal{S}(x_{h_{p+1}}, x_{h_{p+1}}, Tx^1) = \frac{1}{s}\ \mathcal{S}(x^1, x^1, Tx^1). \quad (2.12)$$

Suppose $x^1 \neq Tx^1$. Now we choose $c_p = s\, \mathcal{S}(x_{h_{p+1}}, x_{h_{p+1}}, Tx^1)$ and $d_p = M_T(x_{h_p}, x^1)$ from property (ξ_3), it follows that

$0 \leq \lim_{p\to\infty} sup\xi(s^4\, \mathcal{S}(Tx_{h_p}, Tx_{h_p}, Tx^1), M_T(x_{h_p}, x^1) < 0$, a contradiction. Hence $Tx^1 = x^1$.

Therefore T has a fixed point.

Theorem 2.3. In addition to the hypotheses of Theorem 2.1(Theorem 2.2), assume the following condition (A): for all $x \neq y \in X$, there exists $w \in X$ such that $\beta(x,w) \geq \gamma(y,w)$, $\beta(y,w) \geq \gamma(x,w)$ and $\beta(w,Tw) \geq \gamma(w,Tw)$. Then T has a unique fixed point.

Proof. Suppose that z^1 and y^1 are two fixed points of T with $z^1 \neq y^1$. Then by our assumption, there exists $w \in X$ such that $\beta(z^1,w) \geq \gamma(z^1,w)$, $\beta(y^1,w) \geq \gamma(y^1,w)$ and $\beta(w,Tw) \geq \gamma(w,Tw)$ so that condition (iii) of Theorem 2.1(Theorem 2.2) holds with $x_1 = w$, also. Now, by applying Theorem 2.1(Theorem 2.2), we deduce that $\{T^h w\}$ converges to a fixed point x^1 of T and hence the sequence is $\{\mathcal{S}(x^1,x^1,T^h w)\}$ is bounded.

Now, since $\mathcal{S}(z^1,z^1,T^h w) \leq s\,[\mathcal{S}(z^1,z^1,x^1) + \mathcal{S}(x^1,x^1,T^h w)]$, we have the sequence $\mathcal{S}(z^1,z^1,T^h w)$ is bounded. Therefore there exists a subsequence $\{\mathcal{S}(z^1,z^1,T^{h_p} w)\}$ of $\mathcal{S}(z^1,z^1,T^h w)$ such that $\lim_{h\to\infty} \mathcal{S}(z^1,z^1,T^{h_p} w) = r$, for some non-negative real r.

Now, we have

$$\mathcal{S}(z^1,z^1,T^{h_p} w) \leq M_T(z^1,T^{h_p} w)$$

$$= \max\{ \mathcal{S}(z^1,z^1,T^{h_p} w), \mathcal{S}(z^1,z^1,Tz^1), \mathcal{S}(T^{h_p} w,T^{h_p} w,T^{h_{p+1}} w), \frac{\mathcal{S}\left(z^1,z^1,T^{h_{p+1}} w\right)+\mathcal{S}(Tz^1,Tz^1,T^{h_p} w)}{2s}$$

$$= \max\{ \mathcal{S}\left(z^1,z^1,T^{h_p} w\right), \mathcal{S}(T^{h_p} w,T^{h_p} w,T^{h_{p+1}} w), \frac{\mathcal{S}\left(z^1,z^1,T^{h_{p+1}} w\right)+\mathcal{S}(z^1,z^1,T^{h_p} w)}{2s}\} \tag{2.13}$$

$$\leq \max\{ \mathcal{S}\left(z^1,z^1,T^{h_p} w\right), \mathcal{S}(T^{h_p} w,T^{h_p} w,T^{h_{p+1}} w),$$

$$\frac{s\left[\mathcal{S}\left(z^1,z^1,T^{h_p}\right)+\mathcal{S}\left(T^{h_p} w,T^{h_p} w,T^{h_{p+1}} w\right)\right]+\mathcal{S}\left(z^1,z^1,T^{h_p} w\right)}{2s}\}$$

On taking limits as $p \to \infty$ we have $\lim_{h\to\infty} M_T(z^1,T^{h_p} w) = r$.

Now we show that $r = 0$. Suppose $r > 0$.

Since T is triangular β-orbital admissible with respect to γ, we have $\beta(w,T^h w) \geq \gamma(w,T^h w)$ and hence $\beta(z^1,T^h w) \geq \gamma(z^1,T^h w)$ and $\beta(y^1,T^h w) \geq \gamma(y^1,T^h w)$ for all $h \in N$.

Now from (1.3) we have $\xi(s^4\, \mathcal{S}\left(z^1,z^1,T^{h_{p+1}} w\right), M_T(z^1,T^{h_p} w) \geq 0$.

Hence, we have $s^4\, \mathcal{S}\left(z^1,z^1,T^{h_{p+1}} w\right) \leq M_T(z^1,T^{h_p} w)$ which implies that

$$s\, \mathcal{S}\left(z^1,z^1,T^{h_{p+1}} w\right) \leq s^3\, \mathcal{S}\left(z^1,z^1,T^{h_{p+1}} w\right) \leq M_T\left(z^1,T^{h_p} w\right).$$

Now, we have

$$\mathcal{S}\left(z^1,z^1,T^{h_p} w\right) \leq s\, \mathcal{S}\left(z^1,z^1,T^{h_{p+1}} w\right) + s\, \mathcal{S}\left(T^{h_{p+1}} w,T^{h_{p+1}} w,T^{h_p} w\right)$$
$$\leq M_T\left(z^1,T^{h_p} w\right) + s\, \mathcal{S}\left(z^1,z^1,T^{h_p} w\right)$$

On taking limits as $p \to \infty$ we have

$\lim_{h\to\infty} s\,\mathcal{S}(z^1, z^1, T^{h_{p+1}}w) = r$.

Now, by choosing $t_p = s\,\mathcal{S}(z^1, z^1, T^{h_{p+1}}w)$ and $s_p = M_T(z^1, T^{h_p}w)$, from property (ξ_3), it follows that

$0 \leq \lim_{p\to\infty} sup\xi(s^4\,\mathcal{S}(z^1, z^1, T^{h_{p+1}}w), M_T(z^1, T^{h_p}w)) < 0$,

a contradiction. Hence $r = 0$. Hence $T^{h_p}w \to z^1$ as $h \to \infty$. Therefore $z^1 = x^1$.

Similarly we can prove that $y^1 = x^1$.

Thus it follows that $z^1 = y^1$, a contradiction. Hence T has a unique fixed point.

2. A fixed point result in partially ordered S-metric spaces

Definition 3.1. Let $(X, \succcurlyeq)$ be a partially ordered set. If there exists a S-metric $\mathcal{S}$ on X with coefficient $s \geq 1$, such that $(X, \mathcal{S})$ is complete, then we say that $(X, \succcurlyeq, \mathcal{S})$ is a partially ordered complete S-metric space with coefficient $s \geq 1$.

Theorem 3.2. Let $(X, \succcurlyeq, \mathcal{S})$ be a partially ordered complete S-metric space with coefficient $s \geq 1$. Let $T : X \to X$ be a selfmap of X. Assume that the following conditions are satisfied:

(i) there exists a simulation mapping ξ such that

$\xi(s^4\mathcal{S}(Tx, Tx, Ty), M_T(x, y)) \geq 0$, for any $x, y \in X$ with $x \succcurlyeq y$,

Where $M_T(x, y) = \max\{\, \mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x,x,Ty)+\mathcal{S}(y,y,Tx)}{2s} \,\}$,

(ii) T is a non-decreasing,

(iii) There exists an $x_1 \in X$ such that $x_1 \succcurlyeq Tx_1$,

(iv) Either T is continuous or $\{x_1\}$ is a decreasing sequence with $x_h \to x^1$ as $h \to \infty$, then there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $x_{h_p} \succcurlyeq x_1$ for all $p \in N$.

Then $\{T^h x_1\}$ converges to an element x^1 of X is a fixed point of T. Further, if for all $x \neq y \in X$, there exists $w \in X$ such that $x \succcurlyeq w$, $y \succcurlyeq w$ and $w \succcurlyeq Tw$, then T has a unique fixed point.

Proof. We define functions $\beta, \gamma : X * X \to [0, \infty)$ by

$$\beta(x, y) = \begin{cases} 3 & if\ x \succcurlyeq y \\ 0 & otherwise \end{cases} \quad and$$

$$\gamma(x, y) = \begin{cases} 1 & if\ \ x \succcurlyeq y \\ 4 & otherwise \end{cases} \quad .$$

Now, for any $x, y \in X$, $\beta(x, y) \geq \gamma(x, y)$ if and only if $x \succcurlyeq y$. By(i), we have

$\xi(s^4\mathcal{S}(Tx, Tx, Ty), M_T(x, y)) \geq 0$. Suppose that $\beta(x, Tx) \geq \gamma(x, Tx)$, then we have

$x \succcurlyeq Tx$. Since T is non-decreasing, we have $Tx \succcurlyeq TTx$ which implies that $\beta(Tx, TTx) \geq \gamma(Tx, TTx)$, hence T is β-orbital admissible with respect to γ.

Further, suppose that $\beta(x, y) \geq \gamma(x, y)$ and $\beta(y, Ty) \geq \gamma(y, Ty)$, so that we have $x \succcurlyeq y$ and $y \succcurlyeq Ty$. It follows that $x \succcurlyeq Ty$ and hence $\beta(x, Ty) \geq \gamma(x, Ty)$. Thus T is triangular β-orbital admissible with respect to γ. Hence T satisfies all the hypotheses of Theorem 2.1 (Theorem 2.2) and T has a fixed point.

Moreover, if for all $x \neq y \in X$, there exists $w \in X$, such that $x \succcurlyeq w$, $y \succcurlyeq w$ and $w \succcurlyeq Tw$, then we have $\beta(x, w) \geq \gamma(x, w)$, $\beta(x, w) \geq \gamma(y, w)$ *and* $\beta(w, Tw) \geq \gamma(w, Tw)$. Hence by Theorem 2.3, T has a unique fixed point.

4. Corollaries

Corollary 4.1. Let $(X, \mathcal{S})$ be a S-metric space. Let $T : X \to X$ and $\beta, \gamma : X * X \to [0, \infty)$ be mappings.

Suppose that the following conditions are satisfied:

(i) There exists a simulation mapping ξ such that for any $x, y \in X$, $\beta(x, y) \geq \gamma(x, y)$ implies $\xi(\mathcal{S}(Tx, Tx, Ty), M(x, y)) \geq 0$, where $M(x, y) = \max\{ \mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x,x,Ty)+\mathcal{S}(x,x,Tx)}{2}\}$,

(ii) T is a triangular β-orbital admissible mapping with respect to γ,

(iii) There exists an $x_1 \in X$ such that $\beta(x_1, Tx_1) \geq \gamma(x_1, Tx_1)$, and

(iv) T is an β-γ-continuous mapping, or if $\{x_h\}$ is a sequence in X such that $\beta(x_h, x_{h+1}) \geq \gamma(x_h, x_{h+1})$ for all $h \in N$ and $x_h \to x^1 \in X$ as $h \to \infty$, then there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x^1) \geq \gamma(x_{h_p}, x^1)$ for all $p \in N$.

Then T has a fixed point $x^1 \in X$ and $\{T^h x_1\}$ converges to x^1.

Moreover, if for all $x \neq y \in X$, there exists $w \in X$ such that $\beta(x, w) \geq \gamma(x, w)$, $\beta(y, w) \geq \gamma(y, w)$ and $\beta(w, Tw) \geq \gamma(w, Tw)$, then T has a unique fixed point.

Proof. Follows from Theorem 2.3 by taking $s = 1$.

Remark 4.2. Theorem 1.20 follows as a Corollary to Corollary 4.1 by choosing $\beta(x, y) = \gamma(x, y) = 1$ for all $x, y \in X$,which in turn Theorem 1.20 follows as a Corollary to Theorem 2.3.

Corollary 4.3. Let $(X, \mathcal{S})$ be a S-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\beta, \gamma : X * X \to [0, \infty)$ be mappings. Assume that there exists two continuous function

$\mu, \rho : [0, \infty) \to [0, \infty)$ with $\mu(t) < t \leq \rho(t)$ for all $t > 0$ *and* $\mu(t) = \rho(t) = 0$ if and only if $t = 0$ such that for any $x, y \in X$ with $\beta(x, y) \geq \gamma(x, y)$ implies

$$\rho(s^4 \mathcal{S}(Tx, Tx, Ty) \leq \mu(M_T(x, y)) \quad (4.1)$$

Where $M_T(x, y) = \max\{ \mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x,x,Ty)+\mathcal{S}(y,y,Tx)}{2s} \}$.

Suppose that the following conditions are satisfied:

(i) T is a triangular β-orbital admissible mapping;

(ii) There exists $x_1 \in X$ such that $\beta(x_1, Tx_1) \geq \gamma(x_1, Tx_1)$; and

(iii) Either T is an β-γ-continuous mapping, or if $\{x_h\}$ is a sequence in X such that $\beta(x_h, x_{h+1}) \geq \gamma(x_h, x_{h+1})$ for all $h \in N$ and $x_h \to x^1 \in X$ as $h \to \infty$, then there exists a subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x^1) \geq \gamma(x_{h_p}, x^1)$ for all $p \in N$.

Then $\{T^n x_1\}$ converges to an element x_1 of X and x_1 is a fixed point of T.

Proof. The conclusion of this corollary follows from Theorem 2.1(Theorem 2.2) by taking ξ(t, s) = $\mu(s) = \rho(t)$ for all $t, s \geq 0$.

Corollary 4.4. Let $(X, \mathcal{S})$ be a S-metric space with coefficient $s \geq 1$. Let $T : X \to X$ and $\beta, \gamma : X * X \to [0, \infty)$ be mappings.

Suppose that the following conditions are satisfied:

(i) There exists a simulation mapping ξ such that for any $x, y \in X$ with $\beta(x, y) \geq 1$ implies $\xi(s^4 \mathcal{S}(Tx, Tx, Ty), (M_T(x, y)) \geq 0$, where

$M_T(x, y) = \max\{ \mathcal{S}(x, x, y), \mathcal{S}(x, x, Tx), \mathcal{S}(y, y, Ty), \frac{\mathcal{S}(x,x,Ty)+\mathcal{S}(y,y,Tx)}{2s} \}$.

(ii) T is a triangular β-orbital admissible mapping,

(iii) There exists an $x_1 \in X$ such that $\beta(x_1, Tx_1) \geq 1$, and

(iv) T is an β-continuous mapping, or if $\{x_h\}$ is a sequence in X such that $\beta(x_h, Tx_{h+1}) \geq 1$ for all $h \in X$, and $x_h \to s^1 \in X$ as $h \to \infty$, then there exists subsequence $\{x_{h_p}\}$ of $\{x_h\}$ such that $\beta(x_{h_p}, x^1) \geq 1$ for all $p \in N$.

Then T has a fixed point $x^1 \in X$ and $\{T^h x_1\}$ converges to x^1.

Moreover, if for all $x \neq y \in X$, there exists $w \in X$ such that $\beta(x, w) \geq 1$, $\gamma(y, w) \geq 1$, then T has a unique fixed point.

Proof. Follows from Theorem 2.1(Theorem 2.2) and Theorem 2.3 by taking $\gamma(x, y) = 1$ for all $, y \in X$.

References

[1] A. H. Ansari, S. Chandok, C. Ionescu, Fixed point theorems on metric spaces for weak contractions with auxiliary functions. Journal of Inequalities and Applications (2014), 2014:429.

[2] H. Argoubi, B. Samet, C. Vetro, Nonlinear contractions involving simulation functions in a metric space with a partial order, J. Nonlinear Sci. Appl. **8**(2015), 1082-1094.

[3] G. V. R. Babu, T. M. Dula, Fixed points of almost generalized $(\alpha, \beta) - (\psi, \varphi)$-contractive mappings in b-metric spaces. (Communicated).

[4] G. V. R. Babu, P. D. Sailaja, A Fixed Point Theorem of Generalized Weakly Contractive Maps in Orbitally Complete Metric Spaces. Thai Journal of Mathematics **9**(1) (2011), 1-10.

[5] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales. Fundam. Math. **3**(1922), 133-181. (in French)

[6] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum **9**(2004), 43-53.

[7] P. Chuadchawna, A. Kaewcharoen, S. Plubtieng, Fixed point theorems for generalized $\alpha - \eta$-$\psi -$Gerghty contraction type mappings in $\alpha - \eta$-complete metric spaces. J. Nonlinear Sci. Appl. **9**(2016), 471-485.

[8] M. Geraghty, On contractive mappings. Proc.Amer.Math. Soc. **40**(1973), 604-608.

[9] N. Hussain, M. A. Kutbi, P. Salimi, Fixed Point theory in α-complete metric spaces with applications. Abstr. Appl. Anal., 2014(2014), 11 pages.

[10] N. Hussain, P. Salimi, Suzuki-Wardowski type fixed point theorems for α-GF-contractions. J. Taiwanese **18**(6) (2014), 1879-1895.

[11] E. Karapinar, P. Kumam, P. Salimi, On α- ψ Meir – Keeler contractive mappings. Fixed Point Theory Appl.,2013(2013), 12 pages.

[12] F. Khojasteh, S. Shukla, S. Radenovic, A new approach to the study of fixed point theorems via simulation functions. Filomat **29**(2015), 1189-1194.

[13] M. Olgun, O. Bicer, T. Alyildiz, A new aspect to Picard operators with simulation functions. J. Turk. Math.**40**(2016), 832-837.

[14] O. Popescu, Some new fixed point theorems for α-Geraghty contraction type maps in metric spaces. Fixed Point Theory Appl. 2014(2014), 12 pages.

[15] B.Samet, C. Vetro, P. Vetro, Fixed point theorem for α- ψ-contractive type mappings. Nonlinear Anal. **75**(2012), 2154-2165.

[16] S. Sedghi, N. V. Dung, Fixed Point Theorems on S-Metric Spaces, Mat. Vesnik, **66**(1) (2014), 113-124.

[17] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vesnik, **64**(2012), 258-266.