

# Robust Ratio Estimation with an Application to Covid-19 Data from Louisiana

## ABSTRACT

The efficiency of the traditional ratio estimator decreases with the presence of outliers or when the underlying distribution is not normal. We consider the error term is from the Laplace distribution and propose a new robust ratio estimator that utilizes Lloyd's estimator. We derive the mean square error of the proposed estimator and compare its performance with some other existing estimators' performances via simulations. We use the proposed estimator to estimate Covid-19 cases and deaths in Louisiana and demonstrate its performance.

*Keywords:* Generalized least square estimator, modified maximum likelihood, robust ratio estimator, auxiliary variable, Covid-19

## 1. INTRODUCTION

The ratio estimator improves the precision of the sample mean  $\bar{y}$  when the study variable is linearly correlated with an auxiliary variable whose values are known for each unit of the population (Cochran, 1940). The ratio estimator is a good choice to estimate the population mean  $\bar{Y}$  when the study variable has a Normal distribution (Cochran, 1977). Let the relationship between the study variable  $y_j$ , and the auxiliary variable  $z_j$  in the population is given as

$$y_j = \beta z_j + h(z_j)e_j, \tag{1.1}$$

where  $h(z_j)$  is a function of  $z_j$ ,  $e_j$  is an *iid* random variable with mean zero and variance  $\sigma_e^2$ , and  $g(e_j)$  is the probability density function of  $e_j$  ( $j = 1, 2, \dots, N$ ). We refer to the assumed model in Eq. (1.1) as the *true model*. The accuracy of the estimates and statistical inferences depend on the accuracy of the assumptions made on the true model. The Gaussian assumption on  $g(e_j)$  provides the well-known ratio estimator,

$$\bar{y}_1 = \left( \frac{\bar{y}}{\bar{z}} \right) \bar{Z}, \tag{1.2}$$

with the mean square error (MSE)

$$MSE(\bar{y}_1) = \text{var}(\bar{y}) - 2R \text{cov}(\bar{y}, \bar{z}) + R^2 \text{var}(\bar{z}), \tag{1.3}$$

where  $\bar{z} = \sum_{j=1}^n z_j / n$ ,  $\bar{Z} = \sum_{j=1}^N z_j / N$ ,  $\text{var}(\bar{z}) = [(1-f)/n] S_z^2$ ,  $\text{var}(\bar{y}) = [(1-f)/n] S_y^2$ ,  $\text{cov}(\bar{y}, \bar{z}) = \frac{1-f}{n} S_{zy}$ ,  $f = n/N$  and  $R = \bar{Y} / \bar{Z}$ . The traditional ratio estimator in Eq. (1.2) is more efficient than the sample mean  $\bar{y}$  if

$\rho > V_z / 2V_y$ , where  $\rho$  is the population correlation coefficient between  $z$  and  $y$ ,  $V_z = S_z / \bar{Z}$ ,  $V_y = S_y / \bar{Y}$ ,  $S_z$  and  $S_y$  are the population standard deviations of  $z$  and  $y$ , respectively. There are real-life situations, however, where assumptions on a true model might be violated. For example, model misspecification occurs when one assumes a specific theoretical model for the population under study, whereas in reality a different model describes it better. The estimates might not be reliable when there is misspecification in the model. Contamination occurs when a few values of the sample data are extreme. If these outliers have the potential to be influential points, ignoring them or proceeding with standard procedures can lead to a seriously biased inference. When there are model violations as such, one needs to modify the estimation procedures and robustify the estimators. This study explores a novel estimation method in simple random sampling (SRS) to maintain the quality of the statistical inferences based on the true model when there is misspecification or contamination in the data.

In survey sampling, robust estimation of the mean in the presence of outliers under normality has been discussed by several authors. Farrel and Barrera (2006); Kadilar et al. (2007) and Subzar et al. (2019) utilized the M-estimation technique to make ratio-type estimators robust. Oral and Kadilar (2011 a and b) and Oral and Oral (2011) integrated Tiku's Modified Maximum Likelihood Estimator (MMLE) into various ratio-type estimators, and studied their properties under misspecification and contamination by following the model (1.1) in which  $g(e_j)$  was assumed to be from a long tailed symmetric (LTS) family. LTS distribution is a symmetrical distribution with a shape parameter  $p$ , and its kurtosis changes from  $\infty$ , 9, 5, to 4.2 for  $p=2.5$ , 3, 4, and 5, respectively; when  $p$  tends to  $\infty$ , it approaches to a Normal distribution. Oral and Kadilar (2011 a and b) and Oral and Oral (2011) showed that, when  $g(e_j)$  follows a LTS distribution or data has outliers, their estimators provide more efficient estimates than the traditional ratio estimator; see also Tiku and Bhasin (1982) and Tiku and Vellaisamy's (1996). More recently, Azaz et al. (2019) highlighted several problems with MMLE's weight function, and suggested using the Generalized Least Squares Estimation (GLSE) instead of MMLE for the robustification process in SRS. They showed that when  $g(e_j)$  follows a LTS distribution, integrating GLSE into the traditional ratio estimator yields more efficient ratio estimators with respect to the traditional, and MMLE integrated ratio estimator.

Specifically, Oral and Oral (2011) proposed the following robust ratio estimator

$$\bar{y}_{oo} = \frac{\hat{\mu}_y}{\bar{Z}} \bar{Z}$$

where  $\hat{\mu}_y$ , which is a weighted mean, is estimated using the MMLE from the LTS distribution. Azaz et al. (2019) showed that although the above estimator works very well for large samples, for very small sample sizes it does not assign small enough weights to the extreme values. Thus, they modified this estimator such that the weights would be smaller for extreme values and improved the robustness. Later, Sanaullah et al. replaced the  $\hat{\mu}_y$  above with its Best Linear Unbiased Estimator (BLUE) of  $\mu_y$ , and worked properties of their proposed estimator.

Although the LTS family provides a flexible symmetric family of distributions for modeling, in many real-life applications the Laplace distribution might provide a better choice as it generally has a sharper peak depending on its scale parameter. As an example, Purdom and Holmes (2005) showed that the error distribution for gene expression data from microarray experiments can be fitted nicely with the Laplace distribution. Bottai and Zhang (2010) used Laplace distribution to model survival time in patients with small cell lung cancer. Biological processes and health-related data every so often reveal heavy-tailed distributions with a sharp peak. In such cases, Laplace distribution might be the natural choice. In fact, statistical models and applications based on Laplace distribution have been rapidly developed in the recent years: Yang (2014) provided a robust mean change point estimation in linear regression assuming that the errors follow the Laplace distribution. Song et al. (2014) proposed a robust estimation for mixture linear regression models under the assumption that the errors are from the Laplace distribution. More recently, Lu and Chang (2020) proposed a robust algorithm for multiphase regression models to deal with data drawn from heavy-tailed distributions. Thus, in this study, we extend the work of Azaz et al. (2019) to the case where the error term in Eq. (1.1) is from the Laplace distribution. We integrate the GLSE into the traditional ratio estimator, and study its properties and robustness under both misspecification and contamination models assuming that the true model's  $g(e_j)$  is characterized by the Laplace distribution. We also use the publicly available Covid-19 data for Louisiana, and by estimating **i)** the average number of deaths using the average number of cases as the auxiliary information, **ii)** fourth wave's average number of deaths using the third wave's average number of deaths, **iii)** fourth wave's average number of cases using third wave's average number of cases, we demonstrate that the proposed estimator is superior to the traditional ratio estimator.

## 2. GLSE OF THE MEAN AND VARIANCE FOR THE LAPLACE DISTRIBUTION

Following Lloyd (1952)'s GLS estimation, let  $y_1, y_2, \dots, y_n$  be a SRS from the Laplace distribution

$$f(y) : L(\mu_y, \sigma_y) = \frac{1}{2\sigma} \text{Exp}\left[-\left(\frac{|y - \mu_y|}{\sigma_y}\right)\right] \quad -\infty \leq y \leq \infty, \quad (2.1)$$

where  $\mu_y$  and  $\sigma_y$  are the location and scale parameters, respectively. Let  $y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(n)}$  be the order statistics from the sample above, and  $z_{[1]} \leq z_{[2]} \leq \dots \leq z_{[n]}$  be their concomitants in the true model (1.1). Let

$V_{(j)} = \frac{y_{(j)} - \mu_y}{\sigma_y}$  ( $j=1, 2, \dots, n$ ) be the standardized variate in (2.1). Let the means, variances and covariances of the

order statistics  $V_{(j)}$  be  $t_{(j)}$ ,  $\omega_{jj}$  and  $\omega_{ji}$ , respectively. Further, let  $\mathbf{y}' = (y_{(1)}, y_{(2)}, \dots, y_{(n)})$ ,  $\mathbf{t}' = (t_{(1)}, t_{(2)}, \dots, t_{(n)})$ ,  $\mathbf{1}' = (1, 1, \dots, 1)$  and  $\mathbf{\Omega} = \omega_{ji}$  for  $j, i = 1, 2, \dots, n$ . The BLUE of  $\mu_y$  and  $\sigma_y$  for  $L(\mu_y, \sigma_y)$  are given by,

$$\hat{\mu}_y^* = \frac{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{y}}{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}} \quad (\text{or } \hat{\mu}_y^* = \sum_{j=1}^n \phi_j y_{(j)}), \quad (2.2)$$

and

$$\hat{\sigma}_y^* = \frac{\mathbf{t}'\mathbf{\Omega}^{-1}\mathbf{y}}{\mathbf{t}'\mathbf{\Omega}^{-1}\mathbf{t}}, \quad (2.3)$$

where  $\phi_j = \sum_{i=1}^n v_{ji} / \sum_{i=1}^n \sum_{j=1}^n v_{ji}$ ;  $v_{ji}$  are the elements of the inverse matrix  $\mathbf{\Omega}$ , and their variances are

$$\text{var}(\hat{\mu}_y^*) = \frac{\sigma_y^2}{\mathbf{1}'\mathbf{\Omega}^{-1}\mathbf{1}}, \text{ and } \text{var}(\hat{\sigma}_y^*) = \frac{\sigma_y^2}{\mathbf{t}'\mathbf{\Omega}^{-1}\mathbf{t}} \quad (2.4)$$

(Govindarajulu, 1966). The exact values of  $t_{(j)}$  and  $\mathbf{\Omega}$  were tabulated by Govindarajulu (1966) for  $n \leq 20$ . For large sample sizes the elements of  $t_{(j)}$  may be calculated by the formula

$$\int_{-\infty}^{t_{(j)}} f(v)dv = j / (n+1), \quad f(v) = \frac{1}{2} \text{exp}(-|v|), \quad -\infty < v < \infty. \quad (2.5)$$

and the elements of  $\mathbf{\Omega}$  are determined from the equation

$$\omega_{ij} \cong p_i(1 - p_j) / ((n+2)f(u_i)f(u_j)), \quad (2.6)$$

where  $p_i = i / (n+1)$ ,  $p_j = j / (n+1)$ ,  $F(u_i) = p_i$  and  $f(v) = F'(v)$ ; see, David and Nagaraja, (2004). Using  $t_{(j)}$  and  $\mathbf{\Omega}$ , one may get the solutions for Eq. (2.2)-(2.4).

To evaluate the weight function of GLSE, we calculate the values of the coefficients  $\phi_j$  for  $n = 5, 10, 12$  and  $15$  and present them in Figure 1. As can be seen from the Figure, the function  $\phi_j$  allocates higher weights to the central observations and lower weights to the extreme observations, so the extreme values get minimum weights, and the effects of the outliers are minimized.

### 3. PROPOSED ROBUST RATIO ESTIMATOR

Assuming that the true model (1.1) follows the Laplace distribution given in (2.1) with  $E(y_j | z_j) = \mu_j = \beta z_j$  and  $\text{var}(y_j | z_j) = \sigma_e^2$  ( $j = 1, 2, \dots, n$ ), we propose the robust ratio estimator

$$\bar{y}_p = \frac{\hat{\mu}_y^*}{\bar{z}} \bar{Z}, \tag{3.1}$$

where  $\hat{\mu}_y^*$  is the BLUE given in (2.2). The approximate MSE of (3.1) under the true model (1.1) can be derived as follows. Let

$$\bar{y}_p - \bar{Y} = \frac{\hat{\mu}_y^*}{\bar{z}} \bar{Z} - \bar{Y}, \text{ or } \bar{y}_p - \bar{Y} = \bar{Z}(\hat{R} - R),$$

where

$\hat{R} = \frac{\hat{\mu}_y^*}{\bar{z}}$  and  $R = \frac{\bar{Y}}{\bar{Z}}$ . If we denote  $g(\bar{z}, \hat{\mu}_y^*) = \hat{R}$  and  $g(\bar{Z}, \bar{Y}) = R$ , by applying the Taylor series approximation to  $\hat{R} - R$  around  $(\bar{Z}, \bar{Y})$  we get

$$\begin{aligned} g(\bar{z}, \hat{\mu}_y^*) &\cong g(\bar{Z}, \bar{Y}) + (\bar{z} - \bar{Z}) \left. \frac{\partial g(\bar{z}, \hat{\mu}_y^*)}{\partial \bar{z}} \right|_{\substack{\bar{z}=\bar{Z} \\ \hat{\mu}_y^*=\bar{Y}}} + (\hat{\mu}_y^* - \bar{Y}) \left. \frac{\partial g(\bar{z}, \hat{\mu}_y^*)}{\partial \hat{\mu}_y^*} \right|_{\substack{\bar{z}=\bar{Z} \\ \hat{\mu}_y^*=\bar{Y}}}, \\ \hat{R} - R &\cong (\hat{\mu}_y^* - \bar{Y}) \frac{1}{\bar{Z}} - (\bar{z} - \bar{Z}) \frac{\bar{Y}}{\bar{Z}^2}, \\ \bar{Z}(\hat{R} - R) &\cong (\hat{\mu}_y^* - \bar{Y}) - (\bar{z} - \bar{Z}) R, \\ \bar{Z}^2 E(\hat{R} - R)^2 &\cong E(\hat{\mu}_y^* - \bar{Y})^2 + E(\bar{z} - \bar{Z})^2 R^2 - 2R(E(\hat{\mu}_y^* - \bar{Y})(\bar{z} - \bar{Z})). \end{aligned} \tag{3.2}$$

Thus, from (3.2) the MSE of the proposed estimator can be written as

$$MSE(\bar{y}_p) \cong \text{var}(\hat{\mu}_y^*) + R^2 \text{var}(\bar{z}) - 2R \text{cov}(\hat{\mu}_y^*, \bar{z}). \tag{3.3}$$

The variance  $\text{var}(\hat{\mu}_y^*)$  in Eq. (3.3) can be written as  $\text{var}(\hat{\mu}_y^*) = \sigma_y^2 \phi' \Omega \phi$ , where  $\phi$  is the vector consisting of the elements of the coefficients  $\phi_j$ ,  $\text{var}(\bar{z})$  is the same as given above, and  $\text{cov}(\hat{\mu}_y^*, \bar{z})$  can be given as

$$\text{cov}(\hat{\mu}_y^*, \bar{z}) = \text{cov}\left(\sum_{j=1}^n \phi_j y_{(j)}, \sum_{j=1}^n z_j / n\right).$$

Since  $\sum_{j=1}^n z_j = \sum_{j=1}^n z_{[j]}$ , we can write

$$\text{cov}(\hat{\mu}_y^*, \bar{z}) = \text{cov}\left(\sum_{j=1}^n \phi_j y_{(j)}, \sum_{i=1}^n z_{[i]} / n\right),$$

or, alternatively

$$\text{cov}(\hat{\mu}_y^*, \bar{z}) = \text{cov}\left(\sum_{j=1}^n \phi_j [\mu_y + \sigma_y V_{(j)}], \sum_{j=1}^n [\mu_z + \sigma_z U_{[j]}] / n\right), \quad (3.4)$$

where  $V_{(j)} = \frac{y_{(j)} - \mu_y}{\sigma_y}$ ,  $U_{[j]} = \frac{z_{[j]} - \mu_z}{\sigma_z}$ , and the covariance between  $V_{(j)}$  and  $U_{[j]}$  can be given as

$\text{cov}(V_{(j)}, U_{[j]}) = \rho \sigma_z \sigma_y \Omega$ ; see Wang (2008). The expression in Eq. (3.4) may also be written by using a matrix notation,

$$\text{cov}(\hat{\mu}_y^*, \bar{z}) = \rho \sigma_z \sigma_y \phi' \Omega \delta, \quad (3.5)$$

where  $\delta$  is the  $n \times 1$  vector with elements  $\frac{1}{n}$  and  $\phi' = (\phi_1, \phi_2, \dots, \phi_n)$ . Finally, the expression of  $MSE(\bar{y}_p)$  is obtained as,

$$MSE(\bar{y}_p) \cong \sigma_y^2 \phi' \Omega \phi + R^2 \text{var}(\bar{z}) - 2R \rho \sigma_z \sigma_y \phi' \Omega \delta. \quad (3.6)$$

#### 4. EFFICIENCY COMPARISONS

Suppose that the underlying super-population, i.e. the true model, is from (2.1). The GLS estimator  $\hat{\mu}_y^*$  in (2.2) is calculated from the order statistics  $y_{(1)} \leq y_{(2)} \leq \dots \leq y_{(n)}$  for the random sample of size  $n$ . In order to show that the proposed estimator is more efficient than the traditional sample mean, we first establish the conditions where GLSE  $\hat{\mu}_y^*$  is more efficient than  $\bar{y}$ , which we denote by  $\bar{y}_0$  and solve the inequality below

$$E(\bar{y}_0 - \bar{Y})^2 > E(\hat{\mu}_y^* - \bar{Y})^2. \quad (4.1)$$

Here,  $E(\bar{y}_0 - \bar{Y})^2 = \frac{\sigma_y^2}{n} (1 - f)$  and

$$\begin{aligned} E(\hat{\mu}_y^* - \bar{Y})^2 &= E[(\hat{\mu}_y^* - \mu_y) - (\bar{Y} - \mu_y)]^2, \\ &= E[(\hat{\mu}_y^* - \mu_y)^2 + (\bar{Y} - \mu_y)^2 - 2(\hat{\mu}_y^* - \mu_y)(\bar{Y} - \mu_y)], \\ &= \text{var}(\hat{\mu}_y^*) - 2\text{cov}(\hat{\mu}_y^*, \bar{Y}) + \text{var}(\bar{Y}), \end{aligned} \quad (4.2)$$

where  $\text{cov}(\hat{\mu}_y^*, \bar{Y})$  can be obtained by using the identity  $\sum_{j=1}^N Y_j = \sum_{j=1}^n y_j + \sum_{j=1}^{N-n} Y_j$  as

$$\text{cov}(\hat{\mu}_y^*, \bar{Y}) = \frac{n}{N} \text{cov}(\hat{\mu}_y^*, \bar{y}_0). \quad (4.3)$$

Since  $Y_1, Y_2, \dots, Y_N$  are independently and identically distributed random variables with

$$\text{var}(\bar{Y}) = \frac{\sigma_y^2}{N}, \quad (4.4)$$

integrating (4.3) and (4.4) into (4.2) provides

$$E(\hat{\mu}_y^* - \bar{Y})^2 = \text{var}(\hat{\mu}_y^*) - 2 \frac{n}{N} \text{cov}(\hat{\mu}_y^*, \bar{y}_0) + \frac{\sigma_y^2}{N}. \quad (4.5)$$

Thus, the inequality in (4.1) becomes

$$\frac{\sigma_y^2}{n} > \text{var}(\hat{\mu}_y^*) + 2 \frac{n}{N} \left( \frac{\sigma_y^2}{n} - \text{cov}(\hat{\mu}_y^*, \bar{y}_0) \right). \quad (4.6)$$

It can be showed that inequality (4.6) always satisfies, and hence  $\hat{\mu}_y^*$  is always more efficient than  $\bar{y}_0$ . For demonstration, the results from a simulation study are given below for sample sizes  $n = 10, 20, 30$  and  $500$ .

$n$	$\text{var}(\bar{y}_0)$	$\text{var}(\hat{\mu}_y^*)$
10	0.2040	0.1460
20	0.0988	0.0662
30	0.0672	0.0425
500	0.0042	0.0021

Note that, the exact expressions of  $\text{var}(\hat{\mu}_y^*)$  and  $\text{cov}(\hat{\mu}_y^*, \bar{y}_0)$  are  $\text{var}(\hat{\mu}_y^*) = \phi' \mathbf{\Omega} \phi \sigma_y^2$  and  $\text{cov}(\hat{\mu}_y^*, \bar{y}_0) = \phi' \mathbf{\Omega} \omega \sigma_y^2$ .

For large samples, the proposed estimator in (3.1) has smaller MSE with respect to the MSE of the traditional ratio estimator in (1.2), when  $y$  is from Laplace distribution. To show that  $\bar{y}_p$  is always more efficient than  $\bar{y}_1$  for large samples, we write

$$MSE(\bar{y}_1) = E(\bar{y}_1^2) - \bar{Y}^2 \text{ and } MSE(\bar{y}_p) = E(\bar{y}_p^2) - \bar{Y}^2. \quad (4.7)$$

Considering two correlated random variables  $y$  and  $z$ , we can write  $E(r(z)h(y)) = E(r(z)E(h(y) | z))$  and we obtain the expected values in Eq. (4.7) as

$$E(\bar{y}_1^2) = E\left(\left(\frac{\bar{Z}}{\bar{z}}\right)^2 \bar{y}_0^2\right) = \bar{Z} E\left(\left(\frac{1}{\bar{z}}\right)^2 E(\bar{y}_0^2 | z)\right), \quad (4.8)$$

and

$$E(\bar{y}_p^2) = E\left(\left(\frac{\bar{Z}}{\bar{z}}\right)^2 (\hat{\mu}_y^*)^2\right) = \bar{Z} E\left(\left(\frac{1}{\bar{z}}\right)^2 E((\hat{\mu}_y^*)^2 | z)\right). \quad (4.9)$$

Since  $E(\bar{y}_0^2) > E(\hat{\mu}_y^*)^2$  from (4.1), we have  $E(\bar{y}_0^2 | z) > E((\hat{\mu}_y^*)^2 | z)$ ; this follows from the fact that, on a probability space  $(\Psi, \xi, P)$ ,  $P(E(y) \leq E(z)) = 1$  implies  $P(E(y|H) \leq E(z|H)) = 1$  for any  $H \subset \xi$  (Loeve, 1977). Consequently,

$$\left(\frac{1}{\bar{z}^2}\right) E(\bar{y}^2 | z) > \left(\frac{1}{\bar{z}^2}\right) E((\hat{\mu}_y^*)^2 | z)$$

and

$$\bar{Z}^2 E\left(\left(\frac{1}{\bar{z}^2}\right) E(\bar{y}^2 | z)\right) > \bar{Z}^2 E\left(\left(\frac{1}{\bar{z}^2}\right) E((\hat{\mu}_y^*)^2 | z)\right);$$

therefore, it is clear from (4.7)-(4.9) that  $MSE(\bar{y}_1) > MSE(\bar{y}_p)$  for large sample sizes. The proposed estimator  $\bar{y}_p$  performs better than  $\bar{y}_1$  for small samples, if

$$MSE(\bar{y}_p) < MSE(\bar{y}_1), \text{ or}$$

$$\text{cov}(\bar{y}_0, \bar{z}) < C_1, \text{ where } C_1 = \frac{1}{2R} [\text{var}(\bar{y}_0) - \text{var}(\hat{\mu}_y^*)] + \text{cov}(\hat{\mu}_y^*, \bar{z}). \tag{4.10}$$

### 5. ROBUSTNESS OF THE PROPOSED ESTIMATOR

To evaluate the robustness properties of the proposed estimator, we conduct an extensive simulation study as follows. We consider a population from the model (1.1), and generate  $e_j$  and  $z_j$  independently, where the random error  $e_j$  is from  $L(0, \sigma_e^2)$  and  $z_j$  is from  $U(0,1)$  ( $1 \leq j \leq N$ ). Let  $\Pi_N$  represent the super-population of size  $N$  consisting of  $(z_1, y_1), (z_2, y_2), \dots, (z_N, y_N)$ . To assure that the correlation coefficient  $\rho$  is sufficiently high, the values of the parameter  $\beta$  in the model (1.1) are chosen such that the correlation coefficient is 0.60. The value of  $\beta$  which satisfies this condition is determined by  $\beta^2 = \text{var}(e)\rho^2 / (1 - \rho^2)\text{var}(z)$ ; see Oral and Kadilar (2011 a) and Oral and Oral (2011). To calculate the MSE of the estimators in (1.2) and (3.1), one has to calculate the  $\bar{y}_0, \bar{y}_1$  and  $\bar{y}_p$  from all  ${}_N C_n$  possible samples of size  $n$  from  $\Pi_N$ . We consider the values  $N = 500$  and  $n = 10, 30, 70$  and 100. Since  ${}_N C_n$  is extremely large, we choose  $M=50,000$  possible simple random samples of size  $n$  which then give 50,000 values for each estimator, i.e.,  $\bar{y}_0, \bar{y}_1$  and  $\bar{y}_p$ . In calculating  $\bar{y}_p$  using  $n=10, 30, 70$  and 100, we also need to calculate the coefficients  $\phi_j$  and integrate them into Eq. (3.1). To compare the efficiencies, we calculate

the values of the MSEs of each estimator from the expressions  $MSE(\bar{y}_0) = \sum_{i=1}^M (\bar{y}_0 - \bar{Y})^2 / M$ ,

$MSE(\bar{y}_1) = \sum_{i=1}^M (\bar{y}_1 - \bar{Y})^2 / M$  and  $MSE(\bar{y}_p) = \sum_{i=1}^M (\bar{y}_p - \bar{Y})^2 / M$ , under the true model and under eleven different

contamination or misspecification models. The description of each model is given below

**True model:** All  $N$  observations are from  $L(0,1)$  and no outlier is present.

**Dixon’s outlier model-I:**  $N - N_0$  observations are from  $L(0,1)$  and  $N_0$  (we do not know which) are from  $L(0,d)$ , where  $N_0$  is calculated from the formula  $\left\lceil \left[ \frac{N}{10} + \frac{1}{2} \right] \right\rceil$ .

**Dixon’s outlier model-II:**  $N - N_0$  observations are from  $L(0,1)$  and  $N_0$  (we do not know which) are from  $N(0,d)$ , where  $N_0$  is calculated from the formula  $\left\lceil \left[ \frac{N}{10} + \frac{1}{2} \right] \right\rceil$ .

**Tatum’s, (1997) localized scale disturbances model-I:** A proportion  $1 - \phi$  of observations are from a population  $L(0,1)$  and a proportion  $\phi$  of the observations are from  $L(0,d)$ .

**Tatum’s, (1997) localized scale disturbances model-II:** A proportion  $1 - \phi$  of observations are from a population  $L(0,1)$  and a proportion  $\phi$  of the observations are from  $N(0,d)$ .

**Amiri and Allahyari’s (2012) single step shift in the scale model-I:** the first  $N(1 - \phi)$  observations are from  $L(0,1)$  and last  $N(\phi)$  observations are from  $L(0,d)$ .

**Amiri and Allahyari's (2012) single step shift in the scale model-II:** the first  $N(1-\varphi)$  observations are from  $L(0,1)$  and last  $N(\varphi)$  observations are from  $N(0,d)$ .

**Misspecification model:**  $L(0,d=1.07)$ ,

where  $\varphi(=0.10,0.15)$  denotes the percentage, and  $d(=3,4,5)$  refers to the extremity of the contamination. Realize that the first model, i.e., the true model, is given for the sake of comparisons, and all other models are chosen as its plausible alternatives. In order to make direct comparisons between these models, the generated  $e_j$ 's ( $j=1,2,3,\dots,N$ ) were standardized to have the same variance as that of the true model. The simulated values of the MSEs and their corresponding relative efficiencies  $E_{0,h}$  are given in Tables 1 and 2, respectively, where

$E_{0,h} = \frac{MSE(\bar{y}_0)}{MSE(\bar{y}_h)}$  for  $h=1, p$ , where  $E_{0,1}$  denotes the relative efficiency of the traditional ratio estimator and  $E_{0,p}$  denotes the relative efficiency of proposed robust ratio estimator.

It can be seen from Table 1 that, the MSE of the proposed robust ratio estimator increases as the percentage of contamination increases (from a low value of  $\varphi=0.10$  to a comparatively high value of  $\varphi=0.15$ ), which is

expected, but the same increase in the MSE of the traditional ratio estimator is larger compared to the proposed estimator. Besides, for a given sample size, the MSE values of the proposed estimator stays the same compared to the true model demonstrating its robustness to plausible deviations from the assumed Laplace distribution. Tables 1 and 2 show that, the MSEs of the proposed robust ratio estimator are smaller compared to MSEs of the traditional ratio estimator under all types of model violations. We conclude that the proposed robust ratio estimator is both robust and more efficient than the traditional ratio estimator when the error term is from Laplace distribution.

## 6. COVID-19 IN LOUISIANA: ESTIMATING THE MORTALITY AND CASE NUMBERS

The first case of Covid-19 was identified in Wuhan, China in December 2019. Since then, it has spread worldwide causing an ongoing pandemic. It has affected all nations profoundly, causing issues from crippling economies to mental health problems (Baldwin and Werner di Mauro, 2020; Roy et al., 2021; O'Connor et al., 2021; Liu et al, 2022). Consequently, accurately estimating the Covid-19 infections and deaths has been crucial for epidemiologists, public health workers, and federal governments to be able to make public policies and combat the disease. Epidemiologists use infectious disease models to estimate cases and deaths. However, with novel infectious diseases, such as Covid-19, the initial estimates regarding the cases and deaths can be erroneous (Biggs and Littlejohn, 2021). Besides, due to the evolving nature of Covid-19, it has been challenging to predict cases and deaths for different waves caused by different mutations. For example, while the Omicron variant was found to be more transmissible than the Delta variant, it was also observed to be less severe (Mahase, 2022). Some of these infectious disease models, such as the agent-based model, have been criticized for their unrealistic assumptions. Therefore, health researchers might want to consider utilizing more simplistic approaches in estimating the cases or deaths, especially when there are too many unknowns at the beginning of pandemics.

Louisiana (LA) state reported its first case of Covid-19 in March 2020 and immediately became one of the hot spots of the pandemic in the U.S. (KC et al., 2020). Two years after the first reported case, in March 2022, there were 1,229,511 total confirmed Covid-19 cases, and 16,862 total confirmed deaths from Covid-19 in the LA state. As of September 2022 these numbers increased to a total of 1,435,368 confirmed cases and a total of 17,885 deaths\*. Considering that the number of deaths due to Covid-19 is related to the number of confirmed cases, we analyze the publicly available data reported by New York Times\*\*. More specifically, we consider the newly confirmed Covid-19 daily cases and deaths in LA. As doctors and epidemiologists are interested more in average numbers than the raw numbers, we consider the average number of confirmed cases and average number of deaths. Our analyses include three different situations:

### Case-I

We estimate the average number of Covid-19 deaths in LA between 7/1/2021 and 8/14/2021, using the average number of confirmed cases for the same time period as the auxiliary information.

**Case-II**

Considering LA's third wave's (between 11/5/2021 and 3/5/2021) average number of deaths as the auxiliary information, we estimate LA's fourth wave's (between 6/1/2021 and 9/29/2021) average number of deaths.

**Case-III**

Considering LA's third wave's average number of confirmed Covid-19 cases as the auxiliary information, we estimate LA's fourth wave's average number of confirmed Covid-19 cases.

See Figure 2 for the first five waves of Covid-19 in LA; scatter plots of the variables described for each situation above is given in Figure 3. Reviewing several normality tests, we conclude that none of the residuals are normally distributed (Table 3); the normal probability plots also confirms non-normality as well as the existence of several outliers (Figure 4). On the other hand, Laplace distribution fits very well to each case, see Figure 5. Goodness of fit tests from Laplace distribution also provide evidence that we can model the residuals using the Laplace distribution, see Table 4.

To evaluate the performance of the proposed robust ratio estimator with its competitors, we report the mean estimates and the MSEs given in (1.3) and (3.6) along with the relative efficiencies  $E_{0,h} = \text{var}(\bar{y}_0) / \text{MSE}(\bar{y}_h)$  where  $h = 1, p$ . The results are presented in Table 5.

From Table 5, it can be seen that the MSE of the traditional ratio estimator is inflated. The MSE of the proposed robust ratio estimator is stable in the presence of outliers. Besides, the proposed robust ratio estimator is more efficient compared to the traditional ratio estimator. This result is expected since the condition (3.10) is satisfied for all cases studied (see, Table 4). We conclude that the proposed estimator based on GLSE is a better estimator than its counterparts.

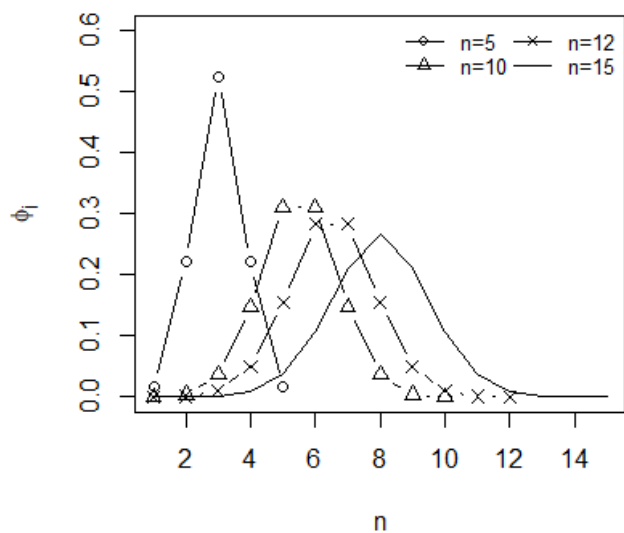
**7. CONCLUDING REMARKS**

Traditional ratio estimator becomes unstable and inefficient when outliers are present in the data, or if the underlying distribution is different than Normal. To robustify and increase the efficiency of the traditional ratio estimator, we utilize the GLS estimation, which provides robust estimators by assigning smaller weights to the outliers and larger weights to the central observations, and by assuming that the underlying population is a Laplace distribution, we propose a new robust ratio estimator. We study the efficiency and robustness properties of the proposed ratio estimator via a simulation study. We apply the proposed robust estimator to the Covid-19 data and show that it is more efficient than the mean and traditional ratio estimator.

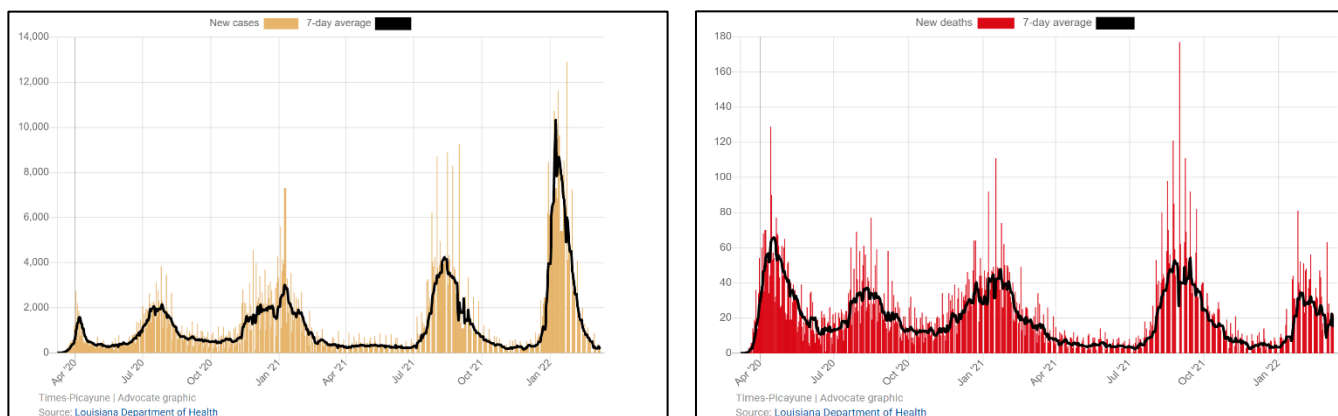
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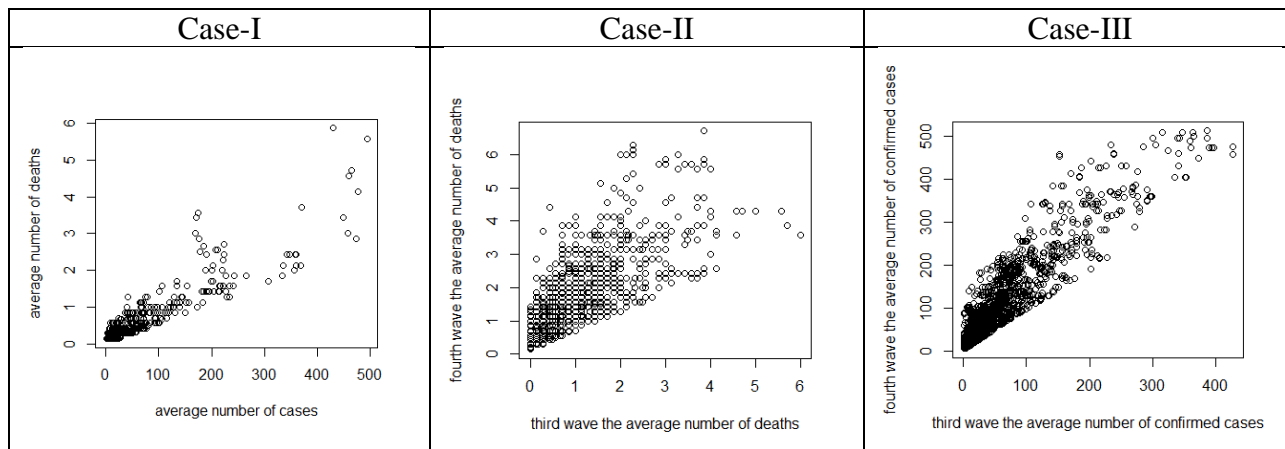
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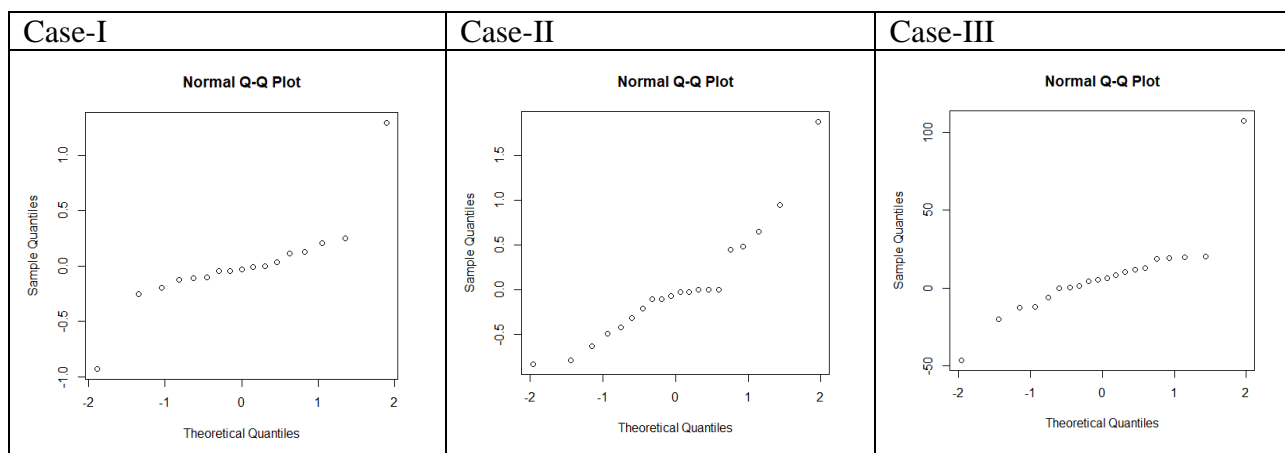
**Figure 1.** The weight function  $\phi_j$  in GLSE to estimate the population mean



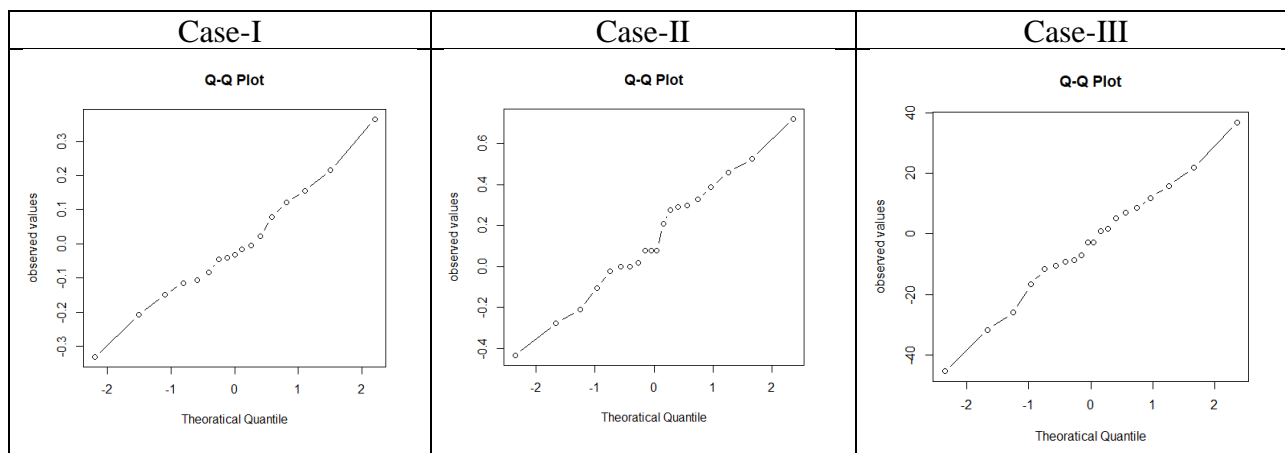
**Figure 2.** Covid-19 cases and deaths in LA; the first five waves.



**Figure 3.** Scatter plot of the variables for Cases I-III



**Figure 4.** The Normal Probability plot of the residuals



**Figure 5.** The Q-Q plot of the residuals where the theoretical quantiles are calculated from Laplace distribution

**Table 1. The values of (1)  $MSE(\bar{y}_o)$ , (2)  $MSE(\bar{y}_1)$  and (3)  $MSE(\bar{y}_p)$**

<i>d</i>	<i>n</i> =10			<i>n</i> =30			<i>n</i> =70			<i>n</i> =100		
	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)	(1)	(2)	(3)
<b>(i) True Model</b>												
	0.2081	0.2427	0.1836	0.0693	0.0665	0.0532	0.0298	0.0271	0.0237	0.0207	0.0191	0.0142
<b>(ii) Dixon's outlier model-I</b>												
3	0.3670	0.3622	0.1876	0.1223	0.1175	0.0610	0.0523	0.0449	0.0255	0.0367	0.0298	0.0179
4	0.5153	0.5015	0.1943	0.1677	0.1749	0.0678	0.0722	0.0618	0.0257	0.0504	0.0383	0.0170
5	0.6776	0.6599	0.1925	0.2297	0.2133	0.0601	0.0983	0.0908	0.0282	0.0672	0.0510	0.0172
<b>(iii) Dixon's outlier model-II</b>												
3	0.7976	0.8599	0.2301	0.2667	0.2564	0.0652	0.1132	0.1022	0.0276	0.0800	0.0661	0.0195
4	1.2591	1.2251	0.2133	0.4193	0.3983	0.0633	0.1807	0.1604	0.0276	0.1265	0.1055	0.0196
5	1.8796	1.9995	0.2394	0.6159	0.5538	0.0606	0.2653	0.2296	0.0273	0.1882	0.1477	0.0183
<b>(iv) Tatum's, (1997) localized scale disturbances model-I using <math>\phi = 0.10</math></b>												
3	0.3672	0.3813	0.1961	0.1220	0.1087	0.0570	0.0525	0.0435	0.0244	0.0365	0.0289	0.0175
4	0.5216	0.6336	0.2380	0.1705	0.1677	0.0632	0.0718	0.0560	0.0234	0.0508	0.0419	0.0184
5	0.6796	0.7160	0.2065	0.2282	0.2171	0.0622	0.0982	0.0859	0.0265	0.0689	0.0514	0.0175
<b>(v) Tatum's, (1997) localized scale disturbances model-II using <math>\phi = 0.10</math></b>												
3	0.7777	0.8149	0.2188	0.2641	0.2552	0.0652	0.1131	0.0948	0.0262	0.0789	0.0623	0.0187
4	1.2660	1.3443	0.2366	0.4213	0.4077	0.0655	0.1815	0.1709	0.0296	0.1286	0.1031	0.0191
5	1.8838	2.3185	0.2735	0.6170	0.6388	0.0705	0.2633	0.2266	0.0268	0.1872	0.1458	0.0183
<b>(vi) Tatum's, (1997) localized scale disturbances model-I using <math>\phi = 0.15</math></b>												
3	0.4508	0.4622	0.2118	0.1497	0.1456	0.0677	0.0638	0.0557	0.0272	0.0451	0.0364	0.0190
4	0.6548	0.7470	0.2513	0.2174	0.2084	0.0675	0.0939	0.0756	0.0262	0.0647	0.0562	0.0214
5	0.9150	1.0337	0.2586	0.3075	0.2852	0.0669	0.1322	0.1132	0.0280	0.0932	0.0799	0.0209
<b>(vii) Tatum's, (1997) localized scale disturbances model-II using <math>\phi = 0.15</math></b>												
3	1.0987	1.2493	0.2919	0.3682	0.3566	0.0731	0.1555	0.1347	0.0299	0.1087	0.1006	0.0244
4	1.7977	1.9547	0.2999	0.5938	0.5759	0.0727	0.2529	0.2201	0.0301	0.1805	0.1464	0.0210
5	2.7223	2.8950	0.3093	0.9140	0.8404	0.0705	0.3889	0.3373	0.0301	0.2716	0.2113	0.0203
<b>(viii) Amiri and Allahyari's (2012) single step shift in the scale model-I using <math>\phi = 0.10</math></b>												
3	0.3656	0.3702	0.1908	0.1222	0.1192	0.0624	0.0525	0.0437	0.0245	0.0368	0.0276	0.0170
4	0.5138	0.5768	0.2169	0.1701	0.1626	0.0629	0.0716	0.0594	0.0250	0.0515	0.0418	0.0185
5	0.6963	0.7531	0.2173	0.2253	0.2218	0.0643	0.0979	0.0852	0.0266	0.0674	0.0510	0.0170
<b>(ix) Amiri and Allahyari's (2012) single step shift in the scale model-II using <math>\phi = 0.10</math></b>												
3	0.8022	0.8517	0.2258	0.2633	0.2725	0.0699	0.1133	0.0981	0.0274	0.0780	0.0633	0.0188
4	1.2643	1.3365	0.2319	0.4212	0.4330	0.0706	0.1828	0.1676	0.0286	0.1256	0.1126	0.0213
5	1.8650	2.0652	0.2460	0.6303	0.6040	0.0654	0.2657	0.2510	0.0254	0.1873	0.1770	0.0228
<b>(x) Amiri and Allahyari's (2012) single step shift in the scale model-I using <math>\phi = 0.15</math></b>												
3	0.4555	0.4725	0.2168	0.1492	0.1409	0.0649	0.1561	0.1243	0.0277	0.0443	0.0342	0.0178
4	0.6450	0.7436	0.2494	0.2195	0.2025	0.0650	0.2587	0.2252	0.0305	0.0662	0.0506	0.0190
5	0.9325	1.0102	0.2527	0.3073	0.2780	0.0645	0.3864	0.3809	0.0347	0.0921	0.0743	0.0197
<b>(xi) Amiri and Allahyari's (2012) single step shift in the scale model-II using <math>\phi = 0.15</math></b>												
3	1.0821	1.3118	0.3134	0.3634	0.3519	0.0722	0.1557	0.1274	0.0283	0.1082	0.0978	0.0238
4	1.7935	1.9424	0.3016	0.5954	0.5821	0.0741	0.2532	0.2311	0.0310	0.1783	0.1472	0.0214
5	2.7065	2.8768	0.3097	0.9056	0.8810	0.0739	0.3904	0.3193	0.0294	0.2724	0.2248	0.0222
<b>(xii) Misspecification Model</b>												
1.07	0.2351	0.2412	0.1923	0.0788	0.0762	0.0570	0.0339	0.0299	0.0258	0.0234	0.0213	0.0197

**Table 2. Relative efficiencies of the estimators with respect to  $\bar{y}_0$**

<i>d</i>	<i>n</i> =10		<i>n</i> =30		<i>n</i> =70		<i>n</i> =100	
	<i>E</i> <sub>0,1</sub>	<i>E</i> <sub>0,<i>p</i></sub>	<i>E</i> <sub>0,1</sub>	<i>E</i> <sub>0,<i>p</i></sub>	<i>E</i> <sub>0,1</sub>	<i>E</i> <sub>0,<i>p</i></sub>	<i>E</i> <sub>0,1</sub>	<i>E</i> <sub>0,<i>p</i></sub>
<b>(i) True Model</b>								
-	0.857	1.133	1.042	1.303	1.100	1.257	1.084	1.458
<b>(ii) Dixon's outlier model-I</b>								
3	1.013	1.956	1.041	2.005	1.165	2.051	1.232	2.050
4	1.028	2.652	0.959	2.473	1.168	2.809	1.316	2.965
5	1.027	3.520	1.077	3.822	1.083	3.486	1.318	3.907
<b>(iii) Dixon's outlier model-II</b>								
3	0.928	3.466	1.040	4.090	1.108	4.101	1.210	4.103
4	1.028	5.903	1.053	6.624	1.127	6.547	1.199	6.454
5	0.940	7.851	1.112	10.163	1.155	9.718	1.274	10.284
<b>(iv) Tatum's, (1997) localized scale disturbances model-I using <math>\varphi = 0.10</math></b>								
3	0.963	1.873	1.122	1.122	1.207	2.152	1.263	2.086
4	0.823	2.192	1.017	1.017	1.282	3.068	1.212	2.761
5	0.949	3.291	1.051	1.051	1.143	3.706	1.340	3.937
<b>(v) Tatum's, (1997) localized scale disturbances model-II using <math>\varphi = 0.10</math></b>								
3	0.954	3.554	1.035	1.035	1.193	4.317	1.266	4.219
4	0.942	5.351	1.033	1.033	1.062	6.132	1.247	6.733
5	0.813	6.888	0.966	0.966	1.162	9.825	1.284	10.230
<b>(vi) Tatum's, (1997) localized scale disturbances model-I using <math>\varphi = 0.15</math></b>								
3	0.975	2.128	1.028	1.028	1.145	2.346	1.239	2.374
4	0.877	2.606	1.043	1.043	1.242	3.584	1.151	3.023
5	0.885	3.538	1.078	1.078	1.168	4.721	1.166	4.459
<b>(vii) Tatum's, (1997) localized scale disturbances model-II using <math>\varphi = 0.15</math></b>								
3	0.825	3.453	1.033	1.033	1.222	5.502	1.106	4.546
4	0.923	5.947	1.023	1.023	1.096	8.168	1.211	8.332
5	0.941	8.739	1.028	1.028	1.223	13.279	1.212	12.270
<b>(viii) Amiri and Allahyari's (2012) single step shift in the scale model-I using <math>\varphi = 0.10</math></b>								
3	0.988	1.916	1.025	1.025	1.201	2.143	1.333	2.165
4	0.891	2.369	1.046	1.046	1.205	2.864	1.232	2.784
5	0.925	3.204	1.016	1.016	1.149	3.680	1.322	3.965
<b>(ix) Amiri and Allahyari's (2012) single step shift in the scale model-II using <math>\varphi = 0.10</math></b>								
3	0.942	3.553	0.966	0.966	1.155	4.135	1.232	4.149
4	0.946	5.452	0.973	0.973	1.091	6.392	1.115	5.897
5	0.903	7.581	1.044	1.044	1.059	10.461	1.058	8.215
<b>(x) Amiri and Allahyari's (2012) single step shift in the scale model-I using <math>\varphi = 0.15</math></b>								
3	0.964	2.101	1.059	1.059	1.256	5.635	1.295	2.489
4	0.867	2.586	1.084	1.084	1.149	8.482	1.308	3.484
5	0.923	3.690	1.105	1.105	1.014	11.135	1.240	4.675
<b>(xi) Amiri and Allahyari's (2012) single step shift in the scale model-II using <math>\varphi = 0.15</math></b>								
3	0.825	3.453	1.033	1.033	1.222	5.502	1.106	4.546
4	0.923	5.947	1.023	1.023	1.096	8.168	1.211	8.332
5	0.941	8.739	1.028	1.028	1.223	13.279	1.212	12.270
<b>(xii) Misspecification Model</b>								
1.07	0.975	1.222	1.034	1.381	1.135	1.316	1.102	1.190

**Table 3. Tests of normality for residuals**

Test	p-value		
	Case-I	Case-II	Case-III
Shapiro Wilk	0.0008	0.0240	0.0003
Anderson Darling	0.0002	0.0340	0.0003
K. Smirnov	0.0274	0.0009	0.0003

**Table 4. Goodness of fit tests for Laplace distribution to assess residuals**

Test	Case-I		Case-II		Case-III	
	Statistic	C. R	Statistic	C. R	Statistic	C. R
Anderson Darling	0.5561	0.9605	0.4423	0.930	0.4131	0.930
Cramer-von Mises	0.0697	0.1370	0.0711	0.131	0.0418	0.131
Watson	0.0629	0.0805	0.0705	0.080	0.0412	0.081

**Table 5. Computational results of for estimating average Covid-19 cases and deaths**

	Case-I	Case-II	Case-III
$N$	946	2662	3116
$n$	17	20	20
$\rho_{zy}$	0.91	0.81	0.93
$R$	0.0104	1.7894	1.8695
$\text{var}(\bar{z})$	353.5508	0.0312	155.2261
$\text{var}(\bar{y})$	0.0273	0.0515	361.4136
$\text{cov}(\bar{y}, \bar{z})$	2.8488	0.0323	220.6684
$R_p$	0.0066	1.1065	1.5735
$\text{var}(\hat{\mu}_y^*)$	0.0249	0.0495	347.2143
$\text{cov}(\hat{\mu}_y^*, \bar{z})$	2.5956	0.0310	211.9987
$\bar{y}_1$	0.4296	0.3652	43.8501
$\bar{y}_p$	0.3601	0.1824	15.1357
$MSE(\bar{y}_1)$	0.0062	0.0358	78.8666
$MSE(\bar{y}_p)$	0.0059	0.0190	64.3833
$E_{0,1}$	4.3786	1.4372	4.5825
$E_{0,p}$	4.6264	2.7072	5.6135