Restrained dr-Power Dominating Sets in Graphs

Original Article

Abstract

Consider a nontrivial connected graph G. In this context, a set R that is not empty and a subset of V (G) is referred to as a restrained dr-power dominating set of G. This means that the induced subgraph of the complement of R in G does not contain any isolated vertex and qualifies as a dr-power dominating set of G. To determine the restrained dr-power domination number of G, denoted as $\gamma^*_{rpw}(G)$, we look at the minimum cardinality of a restrained dr-power dominating set.

This study presents significant insights into the restrained dr-power dominating set of a graph G. It provides concrete realizations and exact values for the restrained dr-power domination number within specific graph classes, such as path and cycle graphs, as well as in the context of join and corona operations. Additionally, characterizations of the restrained dr-power dominating set in the join and corona of graphs are demonstrated.

Keywords: restrained domination, dr-power domination, restrained dr-power domination.

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1 Introduction

Constant monitoring of power systems is a crucial responsibility for electrical power companies, involving the observation of specific variables. To achieve this, Phase Measurement Units (PMUs) are strategically placed at selected locations within the system. However, given the high cost associated with PMUs, it becomes essential to minimize their quantity to ensure cost-effectiveness.

Let's consider a graph G = (V,E) that represents an electrical power system. In this representation, each vertex corresponds to an electrical node, while the edges symbolize transmission lines connecting two nodes. A PMU is responsible for measuring the state variables of the vertex where it is positioned, as well as the incident edges and their respective endpoint vertices. These observed vertices play a significant role in the power system monitoring process.

The concept of the power dominating set (PDS) in the context of electric power networks is introduced in a research article by Haynes et al. [5]. The authors establish the NP-completeness of the PDS problem, even when considering bipartite graphs or chordal graphs as restrictions. However, they also present a linear algorithm that effectively solves the PDS for trees, along with its associated theoretical properties. In [10], Cabahug Jr. I.S. and Canoy, S.R., introduce a variation of power dominating set called directly-remotely (dr)-power dominating set. This variation is in between the power dominating set and the dominating set in a given connected nontrivial graph G. They have shown exact values in some classes of graphs and in some binary operations as well as its corresponding characterizations. The variations of the dr-power dominating sets can also be found in [9, 11].

In this study we focuses on a novel variation of the dr-power dominating set known as the restrained dr-power dominating set. We provide insights into its realizations, exact values, and characterizations within specific graph classes, as well as its implications in the context of join and corona operations.

2 Some Definitions

The definitions and its corresponding examples that are not in this study can be found in [2, 5, 9, 10, 11]. Below are the definition of the main concept.

Definition 2.1. [2] Let G = (V,E) be a simple graph. A subset $S \subseteq V$ is a restrained dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S as well as another vertex in $V - S$, i.e, S is a dominating set and $\langle V - S \rangle$ has no isolated vertex. The restrained domination number of G, denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G. A subset S of V (G) with cardinality $\gamma_r(G)$ is called γ_r-set of G.

Definition 2.2. [10] Let G = (V,E) be a simple graph and let $\emptyset \neq P \subseteq V(G)$. An edge e = uv of G is directly observed by P if $u \in P$ or $v \in P$. A vertex u of G is directly observed if u is incident to a directly observed edge. And edge $e' = xy$ is remotely observed by P if $x, y \notin P$ and x, y are directly observed vertices or at least one of x and y is incident to k edges where $k-1$ of these edges are directly observed by P. A non-directly observed vertex u of G which is incident to a remotely observed edge is called remotely observed vertex. Let $O^P_V(G)$ be the set of all directly observed and remotely observed vertices and O^P_E

(G)(G) be the set of all directly observed and remotely observed edges. Then $P \subset V(G)$ is a dr-power dominating set of G if $O^P_V(G) = V(G)$ and $O^P_E(G) = E(G)$.

The minimum cardinality of a dr-power dominating set is called the dr-power domination number of G and is denoted by $\gamma^*_{pw}(G)$ is called a γ^*_{pw}-set G.

Definition 2.3. A dr-power dominating set R is said to be restrained dr-power dominating set if the induced subgraph <V −R> has no isolated vertex. The restrained dr-power domination number of a graph G, denoted by $\gamma^*_{rpw}(G)$, is the minimum cardinality of a restrained dr-power dominating set in G. A subset D of V (G) with cardinality $\gamma^*_{rpw}(G)$ is called γ^*_{rpw}-set of G. This parameter lies between the dr-power dominating number and restrained domination number. Moreover, there exist a connected graph G such that $\gamma^*_{rpw}(G) \leq \gamma_r(G)$.

3 Main Results

From the definition, we can easily observed that every restrained dominating set can be considered as a restrained dr-power dominating set, and similarly, every restrained dr-power dominating set can be viewed as a dr-power dominating set. Consequently, the following statement directly follows from this observation.

Remark 3.1. For a graph G with no isolated vertex, $\gamma^*_{pw}(G) \leq \gamma^*_{rpw}(G) \leq \gamma_r(G)$.

The subsequent result corresponds to the restrained dr-power domination number of path and cycle graphs, which can be effortlessly confirmed arithmetically.

Theorem 3.1. Let $n \in \mathbb{Z}_+$. Then,

$$\gamma^*_{rpw}(P_n) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \bmod 4 \\ \frac{n-1}{4} & \text{if } n \equiv 1 \bmod 4 \\ \frac{n+2}{4} & \text{if } n \equiv 2 \bmod 4 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

for $n \geq 5$. On the otherhand,

$$\gamma^*_{rpw}(C_n) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \bmod 4 \\ \frac{n+3}{4} & \text{if } n \equiv 1 \bmod 4 \\ \frac{n+2}{4} & \text{if } n \equiv 2 \bmod 4 \\ \frac{n+1}{4} & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

for $n \geq 4$.

Theorem 3.2. For a connected nontrivial graph G, the following statements are equivalent:;

(i) γ^*

$_{rpw}(G) = \gamma_r(G)$.
(ii) G has a γ^*_{rpw}-set which is a dominating set.
(iii) G has a γ^*_{rpw}-set R such that every vertex is directly observed by R.

Proof:

Suppose that $\gamma^*_{rpw}(G) = \gamma_r(G)$, let R be a γ_r-set of G. It follows that R is both a restrained drpower dominating set and a γ_{rpw}-set of G, based on the given assumption. Hence, statement (i) implies statement (ii). Clearly, statement (ii) implies statement (iii). Now, consider a γ^*_{rpw}-set R_0 that satisfies the condition in statement (iii), and let $a \in V(G)$ that does not belong to R_0. By assumption, a is directly observed by R_0, implying the existence of a vertex $b \in R_0$ such that ab is an edge in G. Thus, R_0 is a restrained dominating set of G, and we have $\gamma_r(G) \le |R_0| = \gamma^*_{rpw}(G)$. By Remark 3.1, we can deduce that $\gamma^*_{rpw}(G) \le \gamma_r(G)$. Hence, $\gamma^*_{rpw}(G) = \gamma_r(G)$. Therefore, statement (iii) implies statement (i). □

Theorem 3.3. Let $a, b \in Z_+$ such that $a \le b$. Then there exist a connected nontrivial graph G such that $\gamma^*_{rpw}(G) = a$ and $\gamma_r(G) = b$.

Proof: Consider the following cases:

Case 1: $a = b$

Let G be the graph shown in Figure 1. Clearly, the set $R = \{x_i | i = 1, 2, \ldots, a-1, a\}$ is a γ^*_{rpw}-set and a γ_r-set of G. Thus, $\gamma^*_{rpw}(G) = \gamma_r(G) = |R| = a = b$.

Case 2: $a < b$

Let G be the graph in Figure 2. The set $R_o = \{x_i | i = 1, 2, \ldots, a-1, a\}$ is a γ^*_{rpw}-set of G.
This implies that $\gamma^*_{rpw}(G) = \gamma_r(G) = |R| = a$. Now, observed that in Figure 3 the set $F = \{x_i | i = 1, 2, \ldots, a-1\} \cup \{y_i | i = 1, 2, \ldots, b-a+1\}$ is a γ_r-set of G. Thus, $\gamma_r(G) = |F| = (a-1)+(b-a+1) = b$.
Therefore, $a < b$.

This proves the assertion. □

Theorem 3 implies that we can actually fix the values of a and set the values of b as large as possible. Thus the next result follows.

Corollary 3.4. The difference $\gamma^*_{rpw} - \gamma_r$ can be made arbitrarily large.

Figure 1:

Figure 2:

Figure 3:

Theorem 3.5. Let G and H be connected nontrivial graphs. Then $\emptyset \ne R \subseteq V(G + H)$ is a restrained dr-power dominating set of $G + H$ if and only if one of the following statements holds:
(i) $R \subseteq V(G)$ such that R is a restrained dominating set of G.
(ii) $R \subseteq V(H)$ such that R is a restrained dominating set of H.
(iii) $R = R_1 \cup R_2$ where $\emptyset \ne R_1 \subseteq V(G)$ and $\emptyset \ne R_2 \subseteq V(H)$.

Proof:

Let G and H be connected nontrivial graphs and $\emptyset \ne R \subseteq V(G + H)$ be a restrained dr-power dominating set of $G + H$. Suppose $R \subseteq V(G)$ and suppose further that R is not a dominating set. Then there exist a vertex $x \in V(G)$ such that $x \notin N[R]$. Let $y \in V(H)$. Then $y \in N[R]$ implying that y is directly observed by R. Thus, the edge $xy \in E(G+H)$ is not observed by R. A contradiction to the assumption. Thus, R is a dominating set. Consequently, R is a restrained dominating set of G. Similar argument can also be apply in (ii). If $R \cap V(G) \neq \emptyset$ and $R \cap V(H) \neq \emptyset$ then (iii) holds. The converse immediately follows. □

Corollary 3.6. Let G and H be connected nontrivial graphs. Then

$$\gamma^*_{rpw}(G + H) = \begin{cases} 1, & \text{if } \gamma_r(G) = 1, \gamma_r(H) = 1 \\ 2, & \text{otherwise.} \end{cases}$$

Theorem 3.7. Let G and H be nontrivial connected graphs. Then $\emptyset \neq R \subseteq V(G \circ H)$ is a restrained dr-power dominating set if and only if

$$R = X \bigcup \left(\bigcup_{v \in X} Y_v \right) \bigcup \left(\bigcup_{v \in X} Z_v \right)$$

where $X \subseteq V(G)$, $Y_v \subseteq V(H_v)$ for each $v \in X$ and $Z_x \subseteq V(H)$ is a restrained dr-power dominating set of H_v for each $v \notin X$.

Proof:

Let $R \subseteq V(G \circ H)$ be a restrained dr-power dominating set of $G \circ H$. Let $X = V(G) \cap R$ and let $v \in X$. Set $Y_v = V(H_v) \cap R$. Clearly, $Y_v \subseteq V(H_v)$ for each $v \in X$. Next, let $v \notin X$ and set $Z_v = V(H_v) \cap R$. Suppose Z_v is not a dominating set of H_v. Then there exist $y \in V(H_v)$ such that $y \notin N_{H_v}[Z_v]$. Let $w \in N_{H_v}(y) - Z_v$. Since R is a dr-power dominating set of $G \circ H$, y must be remotely observed vertex of R. Also, by assumption, none of the edges incident to y is directly observed by P. Furthermore, since v, $w \notin R$, the edge vw is remotely observed by R. This implies that vy is not observed by R, contrary to our assumption that R is a dr-power dominating set of $G \circ H$. Thus Z_v is a dominating set of $G \circ H$. Consequently, Z_v is a restrained dominating set of $G \circ H$. Therefore,

$$R = X \bigcup \left(\bigcup_{v \in X} Y_v \right) \bigcup \left(\bigcup_{v \in X} Z_v \right)$$

For the converse, suppose $R = X \bigcup \left(\bigcup_{v \in X} Y_v \right) \bigcup \left(\bigcup_{v \in X} Z_v \right)$ as described. Clearly, $O_R^{V(G \circ H)} = V(G \circ H)$ and O_R

$E(G \circ H) = E(G \circ H)$. Therefore, R is a restrained dr-power dominating set of $G \circ H$.

Corollary 3.8. Let G and H be nontrivial connected graphs with $|V(G)| = n$. Then, $\gamma^*_{rpw}(G \circ H) = n$.

4 Conclusion

This study determined the exact values of restrained dr-power domination number of the path and cycle graphs and explore the binary operations of graphs, such as join and corona, to derive their precise values. Additionally, we address realization problems that involve the restrained dr-power domination number and its related domination parameters.

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Competing Interests

The authors declare that they have no competing interests.

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