
Restrained dr-Power Dominating Sets in Graphs

Abstract

Let G be a simple graph. A set $P \subseteq V(G)$ is called restrained dr -power dominating set of G if $\langle V - P \rangle$ has no isolated vertex and a dr -power dominating set of G . The minimum cardinality of a restrained dr -power dominating set, denoted by $\gamma_{r_{pw}}^*(G)$, is the restrained dr -power domination number of G . This parameter is at most equal to the restrained domination number of a graph G . It is shown that any positive integers a and b where $a \leq b$, there exist a graph G such that the restrained dr -power domination number is equal to a and restrained domination number is equal to b . Restrained dr -power dominating sets in the join and corona of graphs are also characterized.

Keywords: restrained domination, dr -power domination, restrained dr -power domination.

2020 Mathematics Subject Classification: 05C35

1 Introduction

An important task of electrical power companies is to constantly monitor their power system as defined by a set of variables. These variables can be monitored by placing Phase Measurement Units (PMUs) at selected locations in the system. However, due to the high cost of the PMUs, it is desirable to minimize their number to be cost effective.

Let $G = (V, E)$ be a graph representing an electrical power system, where a vertex represents an electrical node and an edge represents a transmission line joining two electrical nodes. A PMU measures the state variables for vertex at which it is placed and its incident edges and their end vertices. These vertices are said to be observed. Considering the power system monitoring as a variation of the dominating set problem, the dr -power dominating set is defined.

Let $G = (V, E)$ be a simple graph. A subset $S \subseteq V$ is a restrained dominating set of G if every vertex in $V - S$ is adjacent to a vertex in S as well as another vertex in $V - S$, i.e, S is a dominating set and $\langle V - S \rangle$ has no isolated vertex. The restrained domination number of G , denoted by $\gamma_r(G)$, is the minimum cardinality of a restrained dominating set of G . A subset S of $V(G)$ with cardinality $\gamma_r(G)$ is called γ_r -set of G .

Let $G = (V, E)$ be a simple graph and let $\emptyset \neq P \subseteq V(G)$. An edge $e = uv$ of G is directly observed by P if $u \in P$ or $v \in P$. A vertex u of G is directly observed if u is incident to a directly observed edge. And edge $e' = xy$ is remotely observed by P if $x, y \notin P$ and x, y are directly observed vertices or at least one of x and y is incident to k edges where $k - 1$ of these

edges are directly observed by P . A non-directly observed vertex u of G which is incident to a remotely observed edge is called remotely observed vertex. Let $O_V^P(G)$ be the set of all directly observed and remotely observed vertices and $O_E^P(G)$ be the set of all directly observed and remotely observed edges. Then $P \subset V(G)$ is a dr -power dominating set of G if $O_V^P(G) = V(G)$ and $O_E^P(G) = E(G)$. The minimum cardinality of a dr -power dominating set is called the dr -power domination number of G and is denoted by $\gamma_{pw}^*(G)$ is called a γ_{pw}^* -set G . A dr -power dominating set D is said to be restrained dr -power dominating set if the induced subgraph $\langle V - D \rangle$ has no isolated vertex. The restrained dr -power domination number of a graph G , denoted by $\gamma_{rpw}^*(G)$, is the minimum cardinality of a restrained dr -power dominating set in G . A subset D of $V(G)$ with cardinality $\gamma_{rpw}^*(G)$ is called γ_{rpw}^* -set of G . This parameter lies between the dr -power dominating number and restrained domination number. Moreover, there exist a connected graph G such that $\gamma_{rpw}^*(G) \leq \gamma_r(G)$.

2 Main Results

Each restrained dominating set is a restrained dr -power dominating sets and each restrained dr -power dominating set is a dr -power dominating set. Thus the next statement is immediate from this observation.

Remark 2.1. For a graph G with no isolated vertex, $\gamma_{pw}^*(G) \leq \gamma_{rpw}^*(G) \leq \gamma_r(G)$.

The next result is the restrained dr -power domination number of the path and cycle graphs.

Proposition 2.1. *Let n be a positive integer. Then,*

$$\gamma_{rpw}^*(P_n) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n-1}{4} & \text{if } n \equiv 1 \pmod{4} \\ \frac{n+2}{4} & \text{if } n \equiv 2 \pmod{4} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

for $n \geq 5$. On one hand,

$$\gamma_{rpw}^*(C_n) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \pmod{4} \\ \frac{n+3}{4} & \text{if } n \equiv 1 \pmod{4} \\ \frac{n+2}{4} & \text{if } n \equiv 2 \pmod{4} \\ \frac{n+1}{4} & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

for $n \geq 4$.

Theorem 2.1. *Let G be a connected nontrivial graph. Then the following statements are equivalent;*

- (i) $\gamma_{rpw}^*(G) = \gamma_r(G)$.
- (ii) G has a γ_{rpw}^* -set which is a dominating set.
- (iii) G has a γ_{rpw}^* -set R such that every vertex is directly observed by R .

Proof: Suppose that $\gamma_{rpw}^*(G) = \gamma_r(G)$ and let R be a γ_r -set of G . Then R is a restrained dr -power dominating set and by assumption is a γ_{rpw}^* -set of G . Thus, (i) implies (ii). Clearly, (ii) implies (iii). Let R_0 , be a γ_{rpw}^* -set that satisfies the condition in (iii) and let $a \in V(G) - R_0$. Then, by assumption, a is directly observed by R_0 . Thus, there exist $b \in R_0$ such that $ab \in E(G)$. Thus, R_0 is a restrained dominating set of G and $\gamma_r(G) \leq |R_0| = \gamma_{rpw}^*(G)$. By Remark 1, $\gamma_{rpw}^*(G) \leq \gamma_r(G)$. Thus, $\gamma_{rpw}^*(G) = \gamma_r(G)$. Therefore, (iii) implies (i). This completes the proof. \square

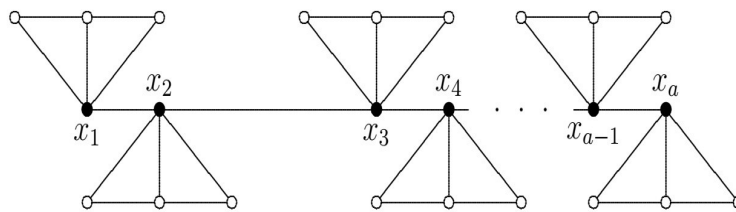
Theorem 2.2. Let $a, b \in \mathbb{Z}^+$ such that $a \leq b$. Then there exist a connected nontrivial graph G such that $\gamma_{r_{pw}}^*(G) = a$ and $\gamma_r(G) = b$.

Proof: Consider the following cases:

Case 1: $a = b$

Let G be the graph shown in Fig.1. Clearly, the set $R = \{x_i | i = 1, 2, \dots, a-1, a\}$ is a $\gamma_{r_{pw}}^*$ -set and a γ_r -set of G . Thus, $\gamma_{r_{pw}}^*(G) = \gamma_r(G) = |R| = a = b$.

Figure 1: Model diagram 1



Case 2: $a < b$

Let G be the graph in Fig.2. The set $R_o = \{x_i | i = 1, 2, \dots, a-1, a\}$ is a $\gamma_{r_{pw}}^*$ -set of G . This implies that $\gamma_{r_{pw}}^*(G) = \gamma_r(G) = |R_o| = a$. Now, observed that in Fig.3 the set $F = \{x_i | i = 1, 2, \dots, a-1\} \cup \{y_i | i = 1, 2, \dots, b-a+1\}$ is a γ_r -set of G . Thus, $\gamma_r(G) = |F| = (a-1) + (b-a+1) = b$. Therefore, $a < b$.

Figure 2: Model diagram 2

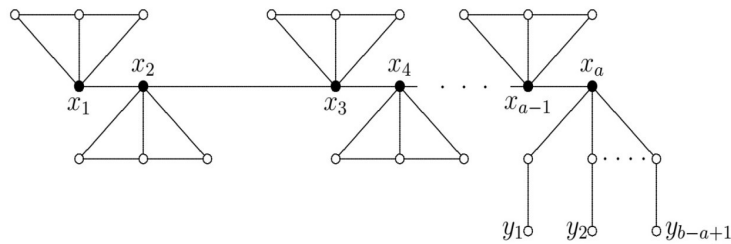
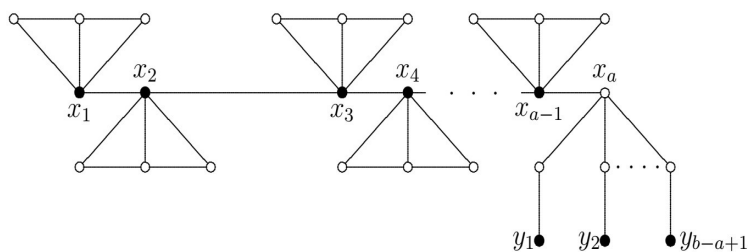


Figure 3: Model diagram 3



This proves the assertion. □

Theorem 3 implies that we can actually fix the values of a and set the values of b as large as possible. Thus the next result follows.

Corollary 2.3. *The difference $\gamma_{r_{pw}}^* - \gamma_r$ can be made arbitrarily large.*

The join $G+H$ of two graphs G and H is the graph with $V(G+H) = V(G) \cup V(H)$ and $E(G+H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in E(H)\}$.

Theorem 2.4. *Let G and H be connected nontrivial graphs. Then $\emptyset \neq R \subseteq V(G+H)$ is a restrained dr-power dominating set of $G+H$ if and only if one of the following statements holds:*

- (i) $R \subseteq V(G)$ such that R is a restrained dominating set of G .
- (ii) $R \subseteq V(H)$ such that R is a restrained dominating set of H .
- (iii) $R = R_1 \cup R_2$ where $\emptyset \neq R_1 \subseteq V(G)$ and $\emptyset \neq R_2 \subseteq V(H)$.

Proof:

Let G and H be connected nontrivial graphs and $\emptyset \neq R \subseteq V(G+H)$ be a restrained dr-power dominating set of $G+H$. Suppose $R \subseteq V(G)$ and suppose further that R is not a dominating set. Then there exist a vertex $x \in V(G)$ such that $x \notin N[R]$. Let $y \in V(H)$. Then $y \in N[R]$ implying that y is directly observed by R . Thus, the edge $xy \in E(G+H)$ is not observed by R . A contradiction to the assumption. Thus, R is a dominating set. Consequently, R is a restrained dominating set of G . Similar argument can also be apply in (ii). If $R \cap V(G) \neq \emptyset$ and $R \cap V(H) \neq \emptyset$ then (iii) holds. The converse immediately follows. □

Corollary 2.5. *Let G and H be connected nontrivial graphs. Then*

$$\gamma_{r_{pw}}^*(G+H) = \begin{cases} 1, & \text{if } \gamma_r(G) = 1, \gamma_r(H) = 1 \\ 2, & \text{otherwise.} \end{cases}$$

The corona of two graphs G and H , denoted by $G \circ H$, is define to be the graph obtained by taking one copy of G which has p vertices and p copies of H , and then joining the i th vertex of G to every vertex in the i th copy of H . The subgraph H_v of the corona $G \circ H$ is a copy of H that is joined to a vertex $v \in V(G)$.

Theorem 2.6. *Let G and H be nontrivial connected graphs. Then $\emptyset \neq R \subseteq V(G \circ H)$ is a restrained dr-power dominating set if and only if*

$$R = X \cup \left(\bigcup_{v \in X} Y_v \right) \cup \left(\bigcup_{v \in X} Z_v \right)$$

where $X \subseteq V(G), Y_v \subseteq V(H^v)$ for each $v \in X$ and $Z_x \subseteq V(H)$ is a restrained dr-power dominating set of H^v for each $v \notin X$.

Proof:

Let $R \subseteq V(G \circ H)$ be a restrained dr-power dominating set of $G \circ H$. Let $X = V(G) \cap R$ and let $v \in X$. Set $Y_v = V(H^v) \cap R$. Clearly, $Y_v \subseteq V(H^v)$ for each $v \in X$. Next, let $v \notin X$ and set $Z_v = V(H^v) \cap R$. Suppose Z_v is not a dominating set of H^v . Then there exist $y \in V(H^v)$ such that $y \notin N_{(H^v)}[Z_v]$. Let $w \in N_{(H^v)}(y) - Z_v$. Since R is a dr-power dominating set of $G \circ H$, y must be remotely observed vertex of R . Also, by assumption, none of the edges incident to y is directly observed by P . Furthermore, since $v, w \notin R$, the edge vw is remotely observed by R . This implies that vy is not observed by R , contrary to our assumption that R is a dr-power dominating set of $G \circ H$. Thus Z_v is a dominating set of $G \circ H$. Consequently, Z_v is a restrained dominating set of $G \circ H$. Therefore,

$$R = X \cup \left(\bigcup_{v \in X} Y_v \right) \cup \left(\bigcup_{v \in X} Z_v \right)$$

For the converse, suppose $R = X \cup \left(\bigcup_{v \in X} Y_v \right) \cup \left(\bigcup_{v \in X} Z_v \right)$ as described. Clearly, $O_V^R(G \circ H) = V(G \circ H)$ and $O_E^R(G \circ H) = E(G \circ H)$. Therefore, R is a restrained dr-power dominating set of $G \circ H$.

Corollary 2.7. Let G and H be nontrivial connected graphs with $|V(G)| = n$. Then,

$$\gamma_{r_{pw}}^*(G \circ H) = n$$

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