Maximum Linear Forest of Graphs Resulting from Some Binary Operations

Original Article

Abstract

For a connected nontrivial graph G, the maximum linear forest of G is the linear forest having maximum number of edges. The number of edges in a maximum linear forest is denoted by $\ell(G)$. In this paper we determine the maximum linear forest of the join and union of nontrivial connected graphs G and H, denoted by $G + H$ and $G \cup H$, respectively.

Keywords: maximum linar forest, join of graphs, union of graphs

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1 Introduction

Graph Theory is a branch of discrete mathematics that is being distinguished by its geometric approach to the study of various objects. The foundation of Graph Theory's basic concept and idea was pioneered by a Swiss mathematician Leonhard Paul Euler (1707-1782). He settled a famous unsolved problem of his day called the Königsberg Bridge Problem. The problem's answer remained indeterminable until Euler was able to prove that the walk is impossible by drawing a picture consists of "dots" and "line-segments" representing the landmasses and the bridges that connected them.

After the Euler's work, subsequent discoveries of graph theory by some researchers had their roots in the physical world. It was in the year 1847 when the concept of a $Tree$, a connected graph without cycles, was introduced by a Physicist, Gustav Kirchhoff (1824-1887). A decade after, trees were used by Arthur Cayley (1821-1895), James Joseph Sylvester (1806-1897), Georg Polya (1887-1985), and others, to solve problems involving enumeration of certain chemical molecules [7]. Since then, there had been many rediscoveries contributed in the field of Graph Theory. This study is anchored on the concept of trees specifically the parameter of maximum linear forest.

2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definitions in [1], [7], [8], [10], [11], [12].

Definition 2.1. [8] A graph is *acyclic* if it has no cycles. A *tree* is a connected acyclic graph. Any graph without cycles is a *forest*; thus, the components of a forest are trees.

Definition 2.2. [6] A *linear forest* (F) is an acyclic subgraph of a graph where the degree of any vertex is at most two. Equivalently, a linear forest is a vertex disjoint union of paths.

Definition 2.3. A *maximum linear forest* in G is a linear forest in G with maximum number of edges. The number of edges in maximum linear forest of graph G is denoted by $\ell(G)$.

Corollary 2.1. *[3] Every connected graph contains a spanning tree.*

Corollary 2.2. *[9] If G is a forest with n vertices and k components, then G has $n-k$ edges.*

3 Main Results

3.1 Maximum Linear Forest of Graphs

Proposition 3.1. *Let G be a connected nontrivial graph of order n and G has no vertices adjacent to at least 3 end-vertices. Then G has a linear forest F of order n, i.e., $|V(F)| = n$.*

Proof. Let G be a connected nontrivial graph such that $|V(G)| = n$ and G has no vertices adjacent to at least 3 end-vertices. Then by Corollary 2.1, G contains a spanning tree T^*. Naturally, $|V(T^*)| = n$. Now, we remove degrees of T^* until we arrive at a collection of vertex disjoint paths. The resulting graph is a linear forest having n vertices, i.e., $|V(F)| = n$. □

Proposition 3.2. *Let F' and F'' be two linear forests in a connected nontrivial graph G of order n such that $|V(F')| = p$ and $|V(F'')| = m$, where $m < p$. Then $M(F') \leq M(F'')$.*

Proof. Let G of order n be a connected nontrivial graph and let F' and F'' be linear forest in G such that $|V(F')| = p$ and $|V(F'')| = m$, where $m < p$. We determine the edge count or edge number of F' and F'' using Corollary 2.2, where the number of components of a forest is subtracted from the total number of vertices. We consider the following cases:

Case 1: F' and F'' has k components.

$M(F') = p - k$, and

$M(F'') = m - k$

since $m < p$, this implies that $M(F') < M(F'')$.

Case 2: F' has r components and F'' has s components, where $r < s$.

$M(F') = p - r$, and

$M(F'') = m - s$

since $m < p$ and $r < s$, this implies that $M(F') < M(F'')$.

Case 3: F' has r components and F'' has s components, where $s < r$.

$M(F') = p - r$, and

$M(F'') = m - s$

since $m < p$ and $s < r$, this implies that $M(F') = M(F'')$.

By the preceding cases, $M(F') \leq M(F'')$. □

Proposition 3.3. *Let $\mathscr{F}_m$ be a collection of linear forests of a connected nontrivial graph G such that $|V(\mathscr{F})| = m$ for all $F^i \in \mathscr{F}_m$, where i denotes the number of components of the linear forest F^i. Then for F^i, $F^j \in \mathscr{F}_m$ with $i < j$,*

$$M(F^i) > M(F^j).$$

Proof. Let G be a connected nontrivial graph and let $\mathscr{F}_m$ be a collection of linear forests of G of order n with the property that if $F^i \in \mathscr{F}_m$, then $|V(F^i)| = m$, with $mleqn$ and i denotes the number of components of F^i, $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

Now, consider two elements F^i and F^j in $\mathscr{F}_m$, with $\lfloor \frac{n}{2} \rfloor$. With this and by Corollary 2.2, $M(F^i) = m - i > m - j = M(F^j)$. □

Corollary 3.1. *Let G be a connected nontrivial graph of order n and suppose that the set $\mathscr{F}_n$ of all linear forests of G with n vertices is nonempty. Then a linear forest in $\mathscr{F}_n$ having the least number of components is a maximum linear forest in G.*

Proof. If $\mathscr{F}_n$ is nonempty, then G has a linear forest of order n, by Proposition 3.2, each linear forest in $\mathscr{F}_n$ has more edges than the linear forest of G not in $\mathscr{F}_n$. Conclusion follows from Corollary 3.1. □

Remark 3.1. Let F_m^i be a collection of all linear forests in a connected nontrivial graph G such that for each $F \in F_m^i$, $|V(F)| = m$ and F has exactly i components. Then for any two $F_1, F_2 \in F_m^i$,

$$M(F_1) = M(F_2).$$

3.2 Maximum Linear Forest of Special Graphs (Paths and Cycle)

Remark 3.2. Since paths and cycles are connected nontrivial graphs, in view of Proposition 3.3, we have:

(i) A maximum linear forest of a connected nontrivial graph G with $\Delta(G) = 2$ contains all vertices of G.

(ii) A maximum linear forest of a connected nontrivial graph G with $\Delta(G) = 2$ has two components.

From this point on, we employ notations given in the following definition.

Definition 3.1. Let G be a connected nontrivial graph of order n with maximum degree, $\Delta(G) = 2$. We adopt the following:

(i) $C_G^i = \{S_1, S_2, S_3, ..., S_i\}$ be a partition of $V(G)$ where each S_k, $1 \leq k \leq i$, contains at least two adjacent vertices which forms a path P_k in G. (Note also that i, $2 \leq i \leq \lfloor \frac{n}{2} \rfloor$, indicates the number of components.)

(ii) $F_G^i = \cup_{k=1}^i P_k$ is linear forest form by the disjoint union of P_k, as defined in (i).

Remark 3.3. Each C_G^i and F_G^i is not unique. We can choose any vertex disjoint sets in G as long as the vertices we choose are adjacent.

Theorem 3.2. *If G is a connected nontrivial graph of order n with $\Delta(G) = 2$. Then G has a linear forest F of order n and $\ell(G) = n - 2$.*

Proof. Suppose G is a connected nontrivial graph of order n with $\Delta(G) = 2$. In view of Remark 3.2 (i), a maximum linear forest of G must contain n vertices. Consider the linear forests F_G^2 and F_G^k where $k > 2$, such that $|V(F_G^2)| = |V(F_G^k)| = n$. By Corollary 2.2, we have

$$M(F_G^2) = n - 2 > n - k = M(F_G^k).$$

Thus, F_G^2 is a maximum linear forest in G where $\ell(G) = M(F_G^2) = n - 2$. □

Remark 3.4. The number of edges of a maximum linear forest in P_n and C_n with $\Delta(G) = 2$ is given by:

$$\ell(P_n) = \ell(C_n) = n - 2, \text{ where } n \geq 4.$$

3.3 Maximum Linear Forest Resulting from Some Binary Operations

3.3.1 Maximum Linear Forest in the Union of Graphs

Lemma 3.3. *Let G and H be connected nontrivial graphs of order n and m respectively, where $\Delta(G) = \Delta(H) = 2$ and let $G \cup H$ of order $n+m$ be the union of G and H. Then the linear forest F of order $n+m$ with two components is a maximum linear forest in $G \cup H$.*

Proof. Let G of order n and H of order m be connected nontrivial graphs where $\Delta(G) = \Delta(H) = 2$ and let $G \cup H$ of order $n+m$ be the union of G and H. Note that in $G \cup H$, G and H are disconnected components. Hence, we consider the following cases:

Case 1: $G = P_n$ and $H = P_m$, i.e., $G \cup H = P_n \cup P_m$

By definition of linear forest, since $G \cup H$ is a disjoint union of two paths as its components, equivalently, $G \cup H$ is a linear forest itself, i.e., $F^2_{G\cup H}$ of order $n+m$. This means $G \cup H$ contains maximum number of vertices and with least number of components. By Corollary 3.1, $F^2_{G\cup H}$ is a maximum linear forestin $G \cup H$.

Case 2: $G = P_n$ and $H = C_m$, i.e., $G \cup H = P_n \cup C_m$

Observe that $G \cup H$ is a disjoint union of a path and a cycle. Hence we consider $F^i_{G\cup H}$, $2 \leq i \leq \left\lfloor \frac{n+m}{2} \right\rfloor$ as a linear forest in $G \cup H$ (see Definition 3.1). But note that since we want to construct a linear forest with least components, we leave P_n as it is and we remove at least one edge in C_m, i.e., $P_m = C_m - e$. Hence, as we remove one edge in C_m, it now becomes a path. Consequently, we have $F^2_{G\cup H} = P_n \cup P_m$ which is of order $n+m$ with 2 components and thus, a maximum linear forest of $G \cup H$.

Case 3: $G = C_n$ and $H = C_m$, i.e., $G \cup H = C_n \cup C_m$

Observe that $G \cup H$ is a disjoint union of two cycles. Hence we consider $F^i_{G\cup H}$, $2 \leq i \leq \left\lfloor \frac{n+m}{2} \right\rfloor$ as a linear forest in $G \cup H$ (see Definition 3.1). To construct a linear forest with least components, we remove at least one edge from each cycle in $G \cup H$, i.e., $P_n = C_n - e$ and $P_m = C_m - e$. Now, this results to a linear forest in $G \cup H$ having 2 components, $F^2_{G\cup H} = P_n \cup P_m$ of order $n+m$ for which is the maximum linear forest in $G \cup H$.

By the preceding cases, a linear forest of order $n+m$ with two components among all linear forests that can be form is a maximum linear forest of $G \cup H$. □

Corollary 3.4. *If G and H are connected nontrivial graphs of order n and m, respectively, with $\Delta(G)\Delta(H) = 2$, then*

$$\ell(G \cup H) = \begin{cases} n+m & \text{if } G = P_n \text{ and } H = P_m, \\ n+m-1 & \text{if } G = P_n \text{ and } H = C_m, \\ n+m-2 & \text{if } G = P_n \text{ and } H = P_m. \end{cases}$$

Proof. In view of the proof of Lemma 3.3, conclusion follows from each specified cases. □

3.3.2 Maximum Linear Forest of Join of Graphs

Lemma 3.5. *Let G and H be connected nontrivial graphs of order n and m respectively, where $\Delta(G) = \Delta(H) = 2$, and let $G + H$ of order $n+m$ be the join of G and H. Then a linear forest F of order $n+m$ with two components is a maximum linear forest in $G + H$.*

Proof. Observe that G and H are either path or cycle; hence, G and H both have spanning paths P_n and P_m. These paths form a linear forest $F^2_{G+H} = P_n \cup P_m$ of order $n+m$ with 2 components. By Corollary 3.1, F^2_{G+H} is a maximum linear forest in $G + H$. □

Proposition 3.4. *If G and H are connected nontrivial graphs of order n and m, respectively, with $\Delta(G) = 2$ and $\Delta(H) = 2$. Then in the join $G + H$, $\ell(G + H) = n + m - 2$.*

Proof. Let G of order n and H of order m be connected nontrivial graphs where $\Delta(G) = \Delta(H) = 2$, for which $G = (V_n, E_n)$ and $H = (V_m, E_m)$, and let $G + H$ of order $n + m$ be the join of G and H. By Lemma 3.5, we can find a maximum linear forest of order $n+m$ in $G+H$ with least components. Suppose F^2_{G+H} is a maximum linear forest in $G + H$ and let $r = n + m$. By Theorem 3.2,

$$\begin{aligned} \ell(G+H) &= M(F^2_{g+H}) &= r - 2 \\ & &= n + m - 2 \end{aligned}$$

Thus, $\ell(G + H) = n + m - 2$. □

4 Conclusion and Recommendation

For simple graph G, the maximum linear forest of G is the linear forest having maximum number of edges among all formed linear forest, the number of edges in a maximum linear forest is denoted by $\ell(G)$. In this article, the maximum linear forest is investigated under the binary operations union and join. For future research, it would be interesting to determine the number of maximum linear forests in graphs under some binary operations including union, join, composition, tensor product, normal product, dot product and the strong product of graphs.

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Competing Interests

The authors declare that they have no competing interests.

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