Original Research Article

Solving The Hyperbolic Telegraph Equation Using a Modified Adomian Decomposition Method with an Invertible Partial Differential Operator

Abstract

This paper presents an improved technique for solving the hyperbolic telegraph equation using a modified Adomian decomposition method(MADM). This method is based on introducing an invertible partial differential operator. The paper also introduces a special case transformation technique, which simplifies the telegraph equation by rewriting it in the special case form. This technique makes the computation of the iterations easier and faster. The paper demonstrates the efficiency and accuracy of the proposed method for both linear and nonlinear telegraph equations by presenting some numerical examples and comparing the results with other existing methods for solving the telegraph equation. The paper shows that the proposed technique can obtain exact or closed approximate solutions for some cases.

1 Introduction

The telegraph equation is a pair of coupled, linear partial differential equations (PDE) that describe the voltage and current on an electrical transmission line with distance and time. It was originally developed by Oliver Heaviside in 1876 to model the telegraph wires, but it can also be applied to other types of transmission lines, such as radio frequency conductors, telephone lines, power lines, and pulses of direct current. The telegraph equation has many applications in engineering and physics, such as telecommunication signal transmission, microwave propagation, electromagnetic waves, and nerve impulse conduction. Solving the telegraph equation analytically is not always

possible or convenient, especially when the boundary and initial conditions are complicated or nonlinear. Therefore, various numerical and approximate methods have been proposed and used to obtain solutions of the telegraph equation[3, 4, 6, 8, 9, 11, 13, 14, 16], such as finite difference methods, Runge-Kutta methods, perturbation methods, homotopy methods, and Adomian decomposition method (ADM). Among these methods, ADM is a popular and powerful technique that can handle linear and nonlinear problems without linearization or discretization. It provides the solution as an infinite series that converges rapidly and has easily computable components. However, ADM also has some limitations and drawbacks, such as the limited choice of acceptable linear operators and initial approximations, the difficulty of integrating higher order deformation equations, and the need of using the so-called Adomian polynomials. To overcome these challenges and improve the accuracy and efficiency of ADM, several modifications have been proposed by various researchers. One of them is the modified Adomian decomposition method (MADM), which was introduced by Hasan and Zhu in 2009[5].This modification is based on developing a new invertible differential operator and was introduced for solving second-order ordinary differential equations with constant coefficients . The main objective of this paper is to use the differential operator introduced in This method has some advantages over ADM, such as simplifying the calculation process, giving exact solutions for some equations by using only a few iterations, and with this method only the initial conditions are needed for finding the solution.
We will apply MADM to solve some telegraph equations with different initial conditions. We will compare the results with exact solutions or numerical solutions obtained by other methods.

2 Analysis Of The Method

In this work, we consider the general hyperbolic telegraph equation of the form

$$\frac{\partial^2 u}{\partial t^2} \;+\; a\,\frac{\partial u}{\partial t} \;+\; b\,u \;= c\,\frac{\partial^2 u}{\partial x^2} \;+\; f(x,t) + F(u(x,t)), \tag{2.1}$$

with initial conditions as follows:

$$u(x,0) = g_1(x) \quad , \quad u_t(x,0) = g_2(x), \tag{2.2}$$

where a,b and c are constants related to the inductance, capacitance and conductance of the cable respectively[15], f, g_1, g_2 are known functions, and the unknowing function u can be voltage or current through the wire at position x and time t, and F(u(x,t))represents the nonlinear terms [11].

Under the transformation $a = \alpha + \beta$ and $b = \alpha\beta$ Eq.(2.1) becomes

$$\frac{\partial^2 u}{\partial t^2} \;+\; (\alpha+\beta)\,\frac{\partial u}{\partial t} \;+\; \alpha\beta\,u \;= c\,\frac{\partial^2 u}{\partial x^2} \;+\; f(x,t) + F(u(x,t)) \tag{2.3}$$

We propose the new differential operator $L_t(.)$ as follows

$$L_t(.) \;=\; e^{-\alpha t}\,\frac{\partial}{\partial t}\,e^{-(\beta-\alpha)t}\,\frac{\partial}{\partial t}\,e^{\beta t}\,(.). \tag{2.4}$$

Applying this operator to u results in

$$\begin{aligned}
L_t\,(u) \;&=\; e^{-\alpha t}\,\frac{\partial}{\partial t}\,e^{-(\beta-\alpha)t}\,\frac{\partial}{\partial t}\,[e^{\beta t}u]\\
&=\; e^{-\alpha t}\,\frac{\partial}{\partial t}\,e^{-(\beta-\alpha)t}\,[\,u_t\,e^{\beta t}\;+\;\beta\,u\,e^{\beta t}\,]\\
&=\; e^{-\alpha t}\,\frac{\partial}{\partial t}\,[\,u_t\,e^{\alpha t}\;+\;\beta\,u\,e^{\alpha t}\,]\\
&=\; e^{-\alpha t}\,[\,u_{tt}\,e^{\alpha t}\;+\;\alpha\,u_t\,e^{\alpha t}\;+\;\beta\,u_t\,e^{\alpha t}\;+\;\alpha\beta\,u\,e^{\alpha t}\,]\\
&=\; u_{tt}\;+\;(\alpha+\beta)\,u_t\;+\;\alpha\beta\,u\,.
\end{aligned}$$

So, under this operator the left hand side of Eq.(2.3) becomes $L_t u$ so the telegraph Eq.(2.3) can be written as

$$L_t u = c\,u_{xx} + f(x,t)\;+F(u(x,t)). \tag{2.5}$$

The inverse operator L_t^{-1} is therefore considered a two-fold integral operator, as below,

$$L_t^{-1}(.) = e^{-\beta t}\int_0^t e^{(\beta-\alpha)t}\int_0^t e^{\alpha t}(.)\;dtdt\,. \tag{2.6}$$

Applying L_t^{-1} to the left hand side of the Eq.(2.3)

$$L_t^{-1}(u_{tt}+(\alpha+\beta)u_t+\alpha\beta u) = e^{-\beta t}\int_0^t e^{(\beta-\alpha)t}\int_0^t e^{\alpha t}(u_{tt}+(\alpha+\beta)u_t+\alpha\beta u)\;dtdt$$

$$= e^{-\beta t}\int_0^t e^{(\beta-\alpha)t}[e^{\alpha t}u_t + \beta e^{\alpha t}u - u_t(x,0) - \beta u(x,0)]\;dt$$

$$= e^{-\beta t}\int_0^t [e^{\beta t}u_t + \beta e^{\beta t}u - e^{(\beta-\alpha)t}u_t(x,0) - \beta e^{(\beta-\alpha)t}u(x,0)]\;dt$$

$$= e^{-\beta t}[e^{\beta t}u - u_(x,0) - \frac{1}{\beta-\alpha}e^{(\beta-\alpha)t}u_t(x,0) - \frac{1}{\beta-\alpha}\beta e^{(\beta-\alpha)t}u(x,0) + \frac{1}{\beta-\alpha}u_t(x,0) + \frac{\beta}{\beta-\alpha}u(x,0)]$$

$$= u - \frac{1}{\beta-\alpha}e^{-\alpha t}u_t(x,0) - \frac{\beta}{\beta-\alpha}e^{-\alpha t}u(x,0) + \frac{1}{\beta-\alpha}e^{-\beta t}u_t(x,0) + \frac{\alpha}{\beta-\alpha}e^{-\beta t}u(x,0)\,.$$

Operating with L_t^{-1} on Eq.(2.5) we get

$$\begin{aligned}
u = \frac{1}{\beta-\alpha}e^{-\alpha t}u_t(x,0) + \frac{\beta}{\beta-\alpha}e^{-\alpha t}u(x,0) - \frac{1}{\beta-\alpha}e^{-\beta t}u_t(x,0) - \frac{\alpha}{\beta-\alpha}e^{-\beta t}u(x,0)\\
+cL_t^{-1}(u_{xx}) + L_t^{-1}(f(x,t))\;+L_t^{-1}(F(u(x,t))).
\end{aligned}$$

Substituting the initial conditions (2.2) in the above equation, then

$$u = \frac{1}{\beta-\alpha}\, e^{-\alpha t}\, g_2(x) + \frac{\beta}{\beta-\alpha}\, e^{-\alpha t}\, g_1(x) - \frac{1}{\beta-\alpha}\, e^{-\beta t}\, g_2(x) - \frac{\alpha}{\beta-\alpha}\, e^{-\beta t}\, g_1(x) + c\, L_t^{-1}(u_{xx}) + L_t^{-1}(f(x,t)) + L_t^{-1}(F(u(x,t))). \tag{2.7}$$

The ADM suggest that the unknown linear function the u may be represented by the decomposition series

$$\sum_{k=0}^{\infty} u_k,$$

where the components $u_k, k \geq 0$ can be computed recursively, and the non-linear term $F(u(x,t))$ can be expressed by an infinite series of the so-called Adomian polynomials A_k given in the form

$$F(u(x,t)) = \sum_{k=0}^{\infty} A_k(u_0, u_1, u_2, ..., u_k),$$

where the Adomian polynomials A_k can be evaluated by using the following expression

$$A_k = \frac{1}{k!}\frac{d^k}{d\lambda^k}\left[F\left(\sum_{i=0}^{k}\lambda^i u_i\right)\right]_{\lambda=0}, \quad k = 0, 1, 2, ... \tag{2.8}$$

Therefore the solution in a series form is

$$\sum_{k=0}^{\infty} u_k = \frac{1}{\beta-\alpha}e^{-\alpha t}g_2(x) + \frac{\beta}{\beta-\alpha}e^{-\alpha t}g_1(x) - \frac{1}{\beta-\alpha}e^{-\beta t}g_2(x) - \frac{\alpha}{\beta-\alpha}e^{-\beta t}g_1(x) + L_t^{-1}(f(x,t)) + cL_t^{-1}(\sum_{k=0}^{\infty}(u_k)_{xx}) + L_t^{-1}(\sum_{k=0}^{\infty}(A_k).$$

Through using ADM, the components $u_k(x,t)$ can be determined as

$$u_0 = \frac{1}{\beta-\alpha}\, e^{-\alpha t}\, g_2(x) + \frac{\beta}{\beta-\alpha}\, e^{-\alpha t}\, g_1(x) - \frac{1}{\beta-\alpha}\, e^{-\beta t}\, g_2(x) - \frac{\alpha}{\beta-\alpha}\, e^{-\beta t}\, g_1(x) + L_t^{-1}(f(x,t))$$

and

$$u_{k+1} = c\, L_t^{-1}\, (\, (u_k)_{xx}\,) + L_t^{-1} A_k, \quad k = 0, 1, 2, 3, \ldots.$$

Once we have determined the components of $u(x,t)$, the solution in a series form is established by summing up these iterations. This series could provide the exact solution in a closed form.

The solution $u(x,t)$ can be approximated by the truncated series:

$$\phi_k = \sum_{m=0}^{k-1} u_m, \quad \lim_{k\to\infty} \phi_k = u(x,t).$$

3 Special Case : $\alpha = \beta$

In this case the telegraph equation has the form

$$\frac{\partial^2 u}{\partial t^2} + 2\alpha \frac{\partial u}{\partial t} + \alpha^2 u = c\, \frac{\partial^2 u}{\partial x^2} + f(x,t) + F(u(x,t)), \tag{3.1}$$

with initial conditions as follows:

$$u(x,0) = g_1(x) \quad , \quad u_t(x,0) = g_2(x). \tag{3.2}$$

So the differential operator becomes

$$L_t(.) \;=\; e^{-\alpha t}\, \frac{\partial}{\partial t}\, \frac{\partial}{\partial t}\, e^{\alpha t}\, (.). \tag{3.3}$$

And the inverse operator is

$$L_t^{-1}(.) = e^{-\alpha t} \int_0^t \int_0^t e^{\alpha t} (.)\, dtdt. \tag{3.4}$$

Applying L_t^{-1} of (3.4) to the left-hand side of Eq.(3.1) we find

$$\begin{aligned} L_t^{-1}[\, u_{tt} \;+\; 2\alpha\, u_t \;+\; \alpha^2\, u] &= e^{-\alpha t} \int_0^t \int_0^t e^{\alpha t} (u_{tt} + 2\alpha u_t + \alpha^2 u)\, dtdt \\ &= e^{-\alpha t} \int_0^t (e^{\alpha t} u_t + \alpha e^{\alpha t} u - u_t(x,0) - \alpha u(x,0))\, dt \\ &= u - te^{-\alpha t} u_t(x,0) - e^{-\alpha t} u(x,0) - \alpha t e^{-\alpha t} u(x,0). \end{aligned} \tag{3.5}$$

In an operator form Eq.(3.1) is written as

$$L_t(u) = c\, \frac{\partial^2 u}{\partial x^2} + f(x,t) + F(u(x,t)) \tag{3.6}$$

Operating with L_t^{-1} on Eq.(3.6) results in

$$\begin{aligned} u =& te^{-\alpha t} u_t(x,0) + e^{-\alpha t} u(x,0) + \alpha t e^{-\alpha t} u(x,0) + L_t^{-1} f(x,t) \\ &+ c\, L_t^{-1} u_{xx} + L_t^{-1} F(u(x,t)) \end{aligned} \tag{3.7}$$

Substituting the initial conditions (3.2) in Eq.(3.7) we get

$$\begin{aligned} u =& te^{-\alpha t}\, g_2(x) + e^{-\alpha t}\, g_1(x) + \alpha t\, e^{-\alpha t}\, g_1(x) + L_t^{-1} f(x,t) \\ &+ c\, L_t^{-1} u_{xx} + L_t^{-1} F(u(x,t)) \end{aligned} \tag{3.8}$$

By the ADM the solution u is considered as an infinite series $\sum_{k=0}^{\infty} u_k$, so Eq.(3.8) becomes

$$\begin{aligned} \sum_{k=0}^{\infty} u_k =& te^{-\alpha t}\, g_2(x) + e^{-\alpha t}\, g_1(x) + \alpha t\, e^{-\alpha t}\, g_1(x) + L_t^{-1} f(x,t) \\ &+ c\, L_t^{-1} (\sum_{k=0}^{\infty} u_k)_{xx} + L_t^{-1} \sum_{k=0}^{\infty} A_k. \end{aligned} \tag{3.9}$$

The components of u is given by

$$u_0 = te^{-\alpha t}\, g_2(x) + e^{-\alpha t}\, g_1(x) + \alpha t\, e^{-\alpha t}\, g_1(x) + L_t^{-1} f(x,t)$$
$$u_{k+1} = L_t^{-1}((u_k)_{xx}) + L_t^{-1} A_k, \quad k = 0, 1, 2, \ldots.$$

4 Special Case Transformation Technique

Throughout our research we found that when we solve equations of the special case type (i.e when $\alpha = \beta$) by using our modified method, in most of the cases we can figure out the exact solution after just a few iterations. So we propose that we convert the problem to the special case before solving it, by a way of rewriting the telegraph equation such that the coefficient of the dependent variable u is equal to the square of the half of the coefficient of its first derivative u_t as follows :

Suppose the telegraph equation has the form

$$u_{tt} + au_t + bu = cu_{xx} + f(x,t) + F(u(x,t)). \tag{4.1}$$

First we rewrite the Eq.(4.1) by adding $(\frac{a}{2})^2 u$ to its both sides as follows:

$$u_{tt} + 2 * (\frac{a}{2})u_t + bu + (\frac{a}{2})^2 u = (\frac{a}{2})^2 u + cu_{xx} + f(x,t) + F(u(x,t))$$
$$u_{tt} + 2(\frac{a}{2})u_t + (\frac{a}{2})^2 u = [(\frac{a}{2})^2 - b]u + cu_{xx} + f(x,t) + F(u(x,t)).$$

We put $\frac{a}{2} = \alpha$ and $(\frac{a}{2})^2 - b = \gamma$ then Eq.(4.1) will take the form

$$u_{tt} + 2\alpha\, u_t + \alpha^2\, u = \gamma\, u + c\, u_{xx} + f(x,t) + F(u(x,t)). \tag{4.2}$$

It is clear that Eq.(4.2) is still equivalent to the original Eq.(4.1) [7].

Now the differential operator $L_t(.)$ becomes

$$L_t(.) = e^{-\alpha t}\, \frac{\partial^2}{\partial t^2}\, e^{\alpha t}\, (.). \tag{4.3}$$

The inverse operator L_t^{-1} is therefore considered a two-fold integral operator, as below

$$L_t^{-1}(.) = e^{-\alpha t} \int_0^t \int_0^t e^{\alpha t}(.)\, dtdt, \tag{4.4}$$

and the solution u according to (3.8) is

$$\begin{aligned} u =& te^{-\alpha t} g_2(x) + e^{-\alpha t} g_1(x) + \alpha t e^{-\alpha t} g_1(x) + \gamma L_t^{-1} u \\ &+ cL_t^{-1}(u_{xx}) + L_t^{-1} f(x,t) + L_t^{-1} F(u(x,t)). \end{aligned} \tag{4.5}$$

It is obvious that the iterations made by this technique is easier to calculate because u_0 and L_t^{-1} only contain the exponential function with α and the solution converges faster as demonstrated by the following numerical examples.

5 Numerical Examples

In this section, we present some numerical examples to illustrate the application of the MADM to solve the telegraph equation in different cases with different initial conditions.We compared the MADM with the exact solutions. We also show the convergence and accuracy of the MADM by computing the absolute error for some examples.

Example 1:

Consider the following one-dimensional telegraph equation [3]

$$u_{tt} + 2\ u_t + 2\ u = u_{xx} + xe^{-t}, \quad x \in [0, 1], \tag{5.1}$$

with initial conditions

$$u(x, 0) = x \quad , \quad u_t(x, 0) = -x.$$

Here we have $\alpha + \beta = 2$ and $\alpha\beta = 2$
$\Rightarrow \alpha = 1 + i$ and $\beta = 1 - i$. Substituting in (2.6) then the inverse operator is

$$L_t^{-1}(.) = e^{-(1-i)t} \int_0^t e^{(-2i)t} \int_0^t e^{(1+i)t}(.)\ dtdt. \tag{5.2}$$

Using (2.7), the solution u is given by

$$u = \frac{x}{2i}e^{-(1+i)t} - \frac{1-i}{2i}xe^{-(1+i)t} - \frac{x}{2i}e^{-(1-i)t} + \frac{1+i}{2i}xe^{-(1-i)t} + L_t^{-1}(u_{xx}) + L_t^{-1}(xe^{-t})$$

$$u = \frac{x}{2}e^{-(1+i)t} + \frac{x}{2}e^{-(1-i)t} + L_t^{-1}(u_{xx}) + L_t^{-1}(xe^{-t}).$$

So we have

$$\sum_{k=0}^{\infty} u_k = \frac{x}{2}e^{-(1+i)t} + \frac{x}{2}e^{-(1-i)t} + L_t^{-1}(xe^{-t}) + L_t^{-1}(\sum_{k=0}^{\infty}(u_k)_{xx}).$$

By the decomposition method we get the following recurrence relations

$$u_0 = \frac{x}{2}e^{-(1+i)t} + \frac{x}{2}e^{-(1-i)t} + L_t^{-1}(xe^{-t}),$$

$$u_{k+1} = L_t^{-1}(\sum_{k=0}^{\infty}(u_k)_{xx}) \quad \forall k \geq 0.$$

$$
\begin{aligned}
L_t^{-1}(xe^{-t}) &= e^{-(1-i)t}\int_0^t e^{(-2i)t}\int_0^t e^{(1+i)t}(xe^{-t})dtdt \\
&= xe^{-(1-i)t}\int_0^t e^{(-2i)t}\int_0^t e^{it}dtdt \\
&= xe^{-(1-i)t}\int_0^t e^{(-2i)t}[\frac{1}{i}e^{it}-\frac{1}{i}]dt \\
&= xe^{-(1-i)t}\int_0^t [\frac{1}{i}e^{-it}-\frac{1}{i}e^{(-2i)t}]dt \\
&= xe^{-(1-i)t}[e^{-it}-\frac{e^{(-2i)t}}{2}-\frac{1}{2}] \\
&= xe^{-t}-\frac{x}{2}e^{-(1+i)t}-\frac{x}{2}e^{-(1-i)t}.
\end{aligned}
$$

The components of $u(x,t)$ is given by

$$
\begin{aligned}
u_0 &= \frac{x}{2}e^{-(1+i)t}+\frac{x}{2}e^{-(1-i)t}+xe^{-t}-\frac{x}{2}e^{-(1+i)t}-\frac{x}{2}e^{-(1-i)t} \\
&= xe^{-t}, \\
u_1 &= L_t^{-1}(u_0)_{xx}=0, \\
&\vdots \\
u_k &= 0 \quad \forall k=1,2,3,....
\end{aligned}
$$

Therefore the exact solution is

$$u=u_0=xe^{-t}.$$

Example 2

Consider the linear telegraph equation [16]

$$\frac{\partial^2 u}{\partial t^2}+4\frac{\partial u}{\partial t}+4u=\frac{\partial^2 u}{\partial x^2}, \tag{5.3}$$

with initial conditions

$$u(x,0)=1+e^{2x} \qquad , \qquad u_t(x,0)=-2.$$

Here we have $\alpha=\beta$.
Substituting $\alpha=2$ in Eq.(3.3) we get

$$\Rightarrow L_t(.)=e^{-2t}\frac{\partial^2}{\partial t^2}e^{2t}(.). \tag{5.4}$$

The inverse operator L_t^{-1} is

$$L_t^{-1}(.) = e^{-2t}\int_0^t\int_0^t e^{2t}(.)dtdt. \tag{5.5}$$

Then by Eq.(3.8) we get

$$u = e^{-2t} + e^{2x-2t}(1+2t) + L_t^{-1}\frac{\partial^2 u}{\partial x^2}$$

By the decomposition method we get the following recurrence relations

$$\begin{aligned} u_0 &= e^{-2t} + e^{2x-2t}(1+2t), \\ u_{k+1} &= L_t^{-1}\frac{\partial^2 u_k}{\partial x^2}, \quad \forall k = 0,1,2,3,... \end{aligned}$$

The components of the solution $u(x,t)$ is given by

$$\begin{aligned} u_0(x,t) =& e^{-2t} + e^{2x-2t}(1+2t), \\ u_1(x,t) =& e^{-2t}\int_0^t\int_0^t e^{2t}(4e^{2x-2t}(1+2t))\,dtdt = e^{2x-2t}\left(\frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\right), \\ u_2(x,t) =& e^{-2t}\int_0^t\int_0^t e^{2t}(4e^{2x-2t}\left(\frac{(2t)^2}{2!} + \frac{(2t)^3}{3!}\right))\,dtdt = e^{2x-2t}\left(\frac{(2t)^4}{4!} + \frac{(2t)^5}{5!}\right), \\ &\vdots \\ u_k(x,t) =& e^{2x-2t}\left(\frac{(2t)^{2k}}{(2k)!} + \frac{(2t)^{2k+1}}{(2k+1)!}\right). \end{aligned}$$

The solution $u(x,t)$ in a series form is given by

$$\begin{aligned} u(x,t) =& u_0(x,t) + u_1(x,t) + u_2(x,t).... \\ =& e^{-2t} + e^{2x-2t}\left(1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \frac{(2t)^5}{5!} + ...\right). \end{aligned}$$

Which gives the exact solution

$$u(x,t) = e^{-2t} + e^{2x-2t}e^{2t} = e^{-2t} + e^{2x}.$$

Example 3

Consider the telegraph equation [4]

$$u_{tt} + u_t + u = u_{xx} + x^2 + t - 1, \tag{5.6}$$

with initial conditions

$$u(x,0) = x^2 \quad , \quad u_t(x,0) = 1.$$

Here we have $\alpha+\beta=1$ and $\alpha\beta=1$,

$$\Rightarrow \quad \alpha=\frac{1}{2}-\frac{i\sqrt{3}}{2} \quad , \quad \beta=\frac{1}{2}+\frac{i\sqrt{3}}{2}.$$

So, by Eq.(2.6) the inverse operator is given by

$$L_t^{-1}(.)=e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t}\int_0^t e^{(i\sqrt{3})t}\int_0^t e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t}(.)dtdt.$$

Substituting α,β and the initial conditions in (2.7) we get

$$u=e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t}\left(\frac{1}{i\sqrt{3}}+(\frac{1}{2}-\frac{i\sqrt{3}}{6})x^2\right)+e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t}\left(\frac{-1}{i\sqrt{3}}+(\frac{1}{2}+\frac{i\sqrt{3}}{6})x^2\right)+L_t^{-1}(x^2+t-1)+L_t^{-1}u_{xx}$$

$$\begin{aligned}L_t^{-1}(x^2+t-1)=&e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t}\int_0^t e^{(i\sqrt{3})t}\int_0^t e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t}\left(x^2+t-1\right)dtdt\\=&x^2+t-2+e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t}\left((\frac{-1}{2}+\frac{i\sqrt{3}}{6})x^2+1\right)+e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t}\left((\frac{-1}{2}-\frac{i\sqrt{3}}{6})x^2+1\right)\end{aligned}$$

$$u=x^2+t-2+e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t}\left(\frac{1}{i\sqrt{3}}+1\right)+e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t}\left(\frac{-1}{i\sqrt{3}}+1\right)+L_t^{-1}u_{xx}.$$

So by ADM, we have the following recurrence relations

$$\begin{aligned}u_0&=x^2+t-2+e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t}\left(\frac{1}{i\sqrt{3}}+1\right)+e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t}\left(\frac{-1}{i\sqrt{3}}+1\right),\\u_{k+1}&=L_t^{-1}\left(u_k\right)_{xx}, \quad k=0,1,2,3,...\end{aligned}$$

Therefore

$$\begin{aligned}u_1=&L_t^{-1}(u_0)_{xx}=L_t^{-1}(2)=e^{-(\frac{1}{2}-\frac{i\sqrt{3}}{2})t}\left(\frac{-1}{i\sqrt{3}}-1\right)+e^{-(\frac{1}{2}+\frac{i\sqrt{3}}{2})t}\left(\frac{1}{i\sqrt{3}}-1\right)+2,\\u_2=&L_t^{-1}(u_1)_{xx}=L_t^{-1}(0)=0,\\&\vdots\\u_k=&0 \quad ,k=2,3,4,...\end{aligned}$$

$$\Rightarrow u=u_0+u_1+u_2+...=x^2+t.$$

Which is the exact solution.

Example 4

We will solve this example first by using the general case, and then we will convert it to the special case to see how we can figure out the exact solution easily.

Consider the following tow-dimensional telegraph equation [8]

$$u_{tt} + 3u_t + 2u = u_{xx} + u_{yy}, \tag{5.7}$$

with initial conditions

$$u(x, y, 0) = e^{x+y} \quad , \quad u_t(x, y, 0) = -3e^{x+y}.$$

Here we have $\alpha + \beta = 3$, and $\alpha\beta = 2 \quad \Rightarrow \alpha = 1, \quad \beta = 2.$

Substituting α, β and the given initial conditions in (2.7) we get

$$u = e^{x+y}(2e^{-2t} - e^{-t}) + L_t^{-t}[u_{xx} + u_{yy}], \tag{5.8}$$

where

$$L_t^{-t}(.) = e^{-2t} \int_0^t e^t \int_0^t e^t(.)dtdt.$$

So

$$u_0 = e^{x+y}(2e^{-2t} - e^{-t}),$$

and

$$u_{k+1} = L_t^{-t}[(u_k)_{xx} + (u_k)_{yy}].$$

$$\begin{aligned}
u_1 =& e^{x+y-t}(6 - 2t) + e^{x+y-2t}(-6 - 4t),\\
u_2 =& e^{x+y-t}(-2t^2 + 16t - 36) + e^{x+y-2t}(4t^2 + 20t + 36),\\
u_3 =& e^{x+y-t}(-\frac{4}{3}t^3 + 20t^2 - 112t + 240) + e^{x+y-2t}(-\frac{8}{3}t^3 - 28t^2 - 128t - 240),\\
u_4 =& e^{x+y-t}(-\frac{2}{3}t^4 + 16t^3 - 160t^2 + 800t - 1680) + e^{x+y-2t}(\frac{4}{3}t^4 + 24t^3 + 200t^2 + 880t + 1680),\\
u_5 =& e^{x+y-t}(-\frac{4}{15}t^5 + \frac{28}{3}t^4 - 144t^3 + 1232t^2 - 5824t + 12096) + e^{x+y-2t}(-\frac{8}{15}t^5 - \frac{44}{3}t^4 - 192t^3\\
&- 1456t^2 - 6272t - 12096),\\
&\vdots
\end{aligned}$$

Therefor the solution in a series form is

$u = u_1 + u_2 + u_3 + u_4 + ...$

$$\begin{aligned}
u =& -\frac{4}{15}e^{x+y-t}(-\frac{159375}{4} + \frac{38415}{2}t - \frac{8175}{2}t^2 + 485t^3 - \frac{65}{2}t^4 + t^5 + ...)\\
&+ \frac{1}{15}e^{x+y-2t}(-159360 - 82560t - 19200t^2 - 2560t^3 - 200t^4 - 8t^5 + ...).
\end{aligned}$$

Now we will solve this problem by the special case transformation technique . First we rewrite the equation as

$$\frac{\partial^2 u}{\partial t^2} + 2(\frac{3}{2})\frac{\partial u}{\partial t} + (\frac{3}{2})^2 u = \frac{1}{4}u + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Substituting $\alpha = \frac{3}{2}$ and using the given initial conditions in Eq.(3.3) we get

$$\Rightarrow L_t(.) = e^{-\frac{3}{2}t}\ \frac{\partial^2}{\partial t^2}\ e^{\frac{3}{2}t}(.). \quad (5.9)$$

The inverse operator L_t^{-1} is

$$L_t^{-1}(.) = e^{-\frac{3}{2}t}\int_0^t\int_0^t e^{\frac{3}{2}t}(.)\ dtdt. \quad (5.10)$$

Substituting α, β and the initial conditions in Eq.(3.8) we get

$$u = -3te^{x+y-(\frac{3}{2})t} + e^{x+y-(\frac{3}{2})t} + \frac{3}{2}te^{x+y-(\frac{3}{2})t} + L_t^{-1}\frac{1}{4}u + L_t^{-1}\frac{\partial^2 u}{\partial x^2} + L_t^{-1}\frac{\partial^2 u}{\partial y^2}$$

$u = e^{x+y-(\frac{3}{2})t}[1-\frac{3}{2}t] + L_t^{-1}\frac{1}{4}u + L_t^{-1}\frac{\partial^2 u}{\partial x^2} + L_t^{-1}\frac{\partial^2 u}{\partial y^2}.$

By the decomposition method we have the following recurrence relations

$$\begin{aligned} u_0 &= e^{x+y-(\frac{3}{2})t}[1-\frac{3}{2}t], \\ u_k &= L_t^{-1}\frac{1}{4}(u_{k-1}) + L_t^{-1}(\frac{\partial^2 u_{k-1}}{\partial x^2})\ + L_t^{-1}(\frac{\partial^2 u_{k-1}}{\partial y^2}) \\ &= \frac{9}{4}L_t^{-1}(u_{k-1}), \forall\ k = 0, 1, 2, 3, \ldots \end{aligned}$$

Therefore the components of the solution $u(x, y, t)$ is

$$\begin{aligned} u_0 &= e^{x+y-(\frac{3}{2})t}(1-\frac{3}{2}t), \\ u_1 &= \frac{9}{4}L_t^{-1}u_0 = \frac{9}{4}L_t^{-1}(e^{x+y-\frac{3}{2}t}(1-\frac{3}{2}t)) = \frac{9}{4}e^{-\frac{3}{2}t}\int_0^t\int_0^t e^{\frac{3}{2}t}e^{x+y-\frac{3}{2}t}(1-\frac{3}{2}t)dtdt \\ &= \frac{4}{9}e^{x+y-\frac{3}{2}t}\left(\frac{t^2}{2!} - \frac{3t^3}{2*3!}\right) = e^{x+y-\frac{3}{2}t}\left(\frac{(-\frac{3t}{2})^2}{2!} - \frac{(-\frac{3t}{2})^3}{3!}\right), \\ u_2 &= e^{x+y-\frac{3}{2}t}\left(\frac{(-\frac{3t}{2})^4}{4!} - \frac{(-\frac{3t}{2})^5}{5!}\right), \\ &\vdots \\ u_k &= e^{x+y-\frac{3}{2}t}\left(\frac{(-\frac{3t}{2})^{2k}}{(2k)!} - \frac{(-\frac{3t}{2})^{(2k+1)}}{(2k+1)!}\right). \end{aligned}$$

The solution in a series form is given by

$$u(x,y,t) = e^{x+y-\frac{3}{2}t}[1 - (-\frac{3t}{2}) + \frac{(-\frac{3t}{2})^2}{2!} - \frac{(-\frac{3t}{2})^3}{3!} + \frac{(-\frac{3t}{2})^4}{4!} - \frac{(-\frac{3t}{2})^5}{5!} + ...].$$

Which gives the exact solution

$$u(x,y,t) = e^{x+y-\frac{3}{2}t}e^{-\frac{3}{2}t} = e^{x+y-3t}.$$

We computed the absolute error of each series for this example at various points to compare their convergence rates. The results are presented in Table 1, where the series are truncated at six terms (k=0 to k=5).

Example 5

Consider the telegraph equation [3]

$$u_{tt} + u_t - u = u_{xx}, \quad x \in [0,1], \tag{5.11}$$

with initial conditions

$$u(x,0) = \sin x \quad , \quad u_t(x,0) = -\sin x, \quad t \geq 0.$$

First we rewrite Eq.(5.11) as follows

$$u_{tt} + 2(\frac{1}{2})u_t + \frac{1}{4}u = \frac{5}{4}u + u_{xx}. \tag{5.12}$$

Substituting $\alpha = \frac{1}{2}$ in Eq.(4.3) then

$$L_t(.) = e^{-\frac{t}{2}}\frac{\partial^2}{\partial t^2}e^{\frac{t}{2}}(.),$$

and the inverse operator becomes

$$L_t^{-1}(.) = e^{-\frac{t}{2}}\int_0^t\int_0^t e^{\frac{t}{2}}(.)\ dtdt.$$

Then by Eq.(4.5) we get

$$\begin{aligned} u(x,t) &= -\sin x\ te^{-\frac{t}{2}} + \sin x\ e^{-\frac{t}{2}} + \frac{1}{2}\sin x\ te^{-\frac{t}{2}} + L_t^{-1}(\frac{5}{4}u + u_{xx}) \\ &= \sin x\ e^{-\frac{t}{2}}[1 - \frac{t}{2}] + L_t^{-1}(\frac{5}{4}u + u_{xx}). \end{aligned}$$

$$\sum_{k=0}^{\infty} u_k(x,t) = \sin x\, e^{-\frac{t}{2}}[1-\frac{t}{2}] + L_t^{-1}(\frac{5}{4}\sum_{k=0}^{\infty} u_k + \sum_{k=0}^{\infty}(u_k)_{xx})$$

By the decomposition method

$$u_0(x,t) = \sin x\, e^{-\frac{t}{2}}(1-\frac{t}{2}),$$

$$u_k(x,t) = L_t^{-1}(\frac{5}{4}u_{k-1} + (u_{k-1})_{xx}).$$

So the components of the solution u(x,t) is given by

$$\begin{aligned}
u_0(x,t) &= \sin x\, e^{-\frac{t}{2}}(1-\frac{t}{2}),\\
u_1(x,t) &= L_t^{-1}(\frac{5}{4}u_0 + (u_0)_{xx}) = \frac{1}{4}L_t^{-1}(\sin x\, e^{-\frac{t}{2}}[1-\frac{t}{2}])\\
&= \frac{1}{4}\sin x\, e^{-\frac{t}{2}}\left(\frac{t^2}{2!} - \frac{t^3}{2*3!}\right) = \sin x\, e^{-\frac{t}{2}}\left(\frac{(\frac{t}{2})^2}{2!} - \frac{(\frac{t}{2})^3}{3!}\right),\\
u_2(x,t) &= \frac{1}{4}\frac{1}{4}\sin x\, e^{-\frac{t}{2}}\left(\frac{t^4}{4!} - \frac{t^5}{2*5!}\right) = \sin x\, e^{-\frac{t}{2}}\left(\frac{(\frac{t}{2})^4}{4!} - \frac{(\frac{t}{2})^5}{5!}\right),\\
&\vdots\\
u_k(x,t) &= \sin x\, e^{-\frac{t}{2}}\left(\frac{(\frac{t}{2})^{(2k)}}{(2k)!} - \frac{(\frac{t}{2})^{(2k+1)}}{(2k+1)!}\right).
\end{aligned}$$

The solution in a series form is given by

$$\begin{aligned}
u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots\\
&= \sin x\, e^{-\frac{t}{2}}\left(1 - \frac{t}{2} + \frac{(\frac{t}{2})^2}{2!} - \frac{(\frac{t}{2})^3}{3!} + \frac{(\frac{t}{2})^4}{4!} - \frac{(\frac{t}{2})^5}{5!} + \dots\right).
\end{aligned}$$

Which gives the exact solution

$$\begin{aligned}
u(x,t) &= \sin x\, e^{-\frac{t}{2}}e^{-\frac{t}{2}}\\
&= \sin x\, e^{-t}.
\end{aligned}$$

Example 6

Consider the nonlinear telegraph equation [12]

$$u_{tt} + 2u_t = u_{xx} - u^2 + e^{2x-4t} - e^{x-2t}, \tag{5.13}$$

with initial conditions

$$u(x,0) = e^x, \quad u_t(x,0) = -2e^x.$$

We have

$$\alpha = 2, \quad \beta = 0.$$

Substituting α, β in 2.4 results in

$$L_t(.) = e^{-2t}\frac{\partial}{\partial t}e^{2t}\frac{\partial}{\partial t}(.).$$

By 2.6 the inverse integral operator is

$$L_t^{-1} = \int_0^t e^{-2t}\int_0^t e^{2t}(.)dtdt.$$

Therefore by substituting the given initial conditions in 2.7, the solution is given by

$$u(x,t) = e^{x-2t} + L_t^{-1}(e^{2x-4t} - e^{x-2t}) + L_t^{-1}u_{xx} - L_t^{-1}u^2.$$

By the decomposition method the recursive relations are

$$\begin{aligned} u_0 =& e^{x-2t} + L_t^{-1}(e^{2x-4t} - e^{x-2t}),\\ u_{k+1} =& L_t^{-1}(u_k)_{xx} - L_t^{-1}(A_k), \quad k+0,1,2,... \end{aligned}$$

where A_k are the Adomian polynomials for the nonlinear term u^2, and they can be calculated as follows[15]

$$\begin{aligned} A_0 =& u_0^2,\\ A_1 =& 2u_0u_1,\\ A_2 =& 2u_0u_2 + u_1^2,\\ &\vdots \end{aligned}$$

The components u_n of the series solution can be determined as follows

$$\begin{aligned} u_0 &= e^{x-2t} + \int_0^t e^{-2t}\int_0^t e^{2t}(e^{2x-4t} - e^{x-2t})\, dtdt\\ &= \frac{t}{2}e^{x-2t} + \frac{5}{4}e^{x-2t} + \frac{1}{8}e^{2x-4t} - \frac{1}{4}e^{2x-2t} + \frac{1}{8}e^{2x} - \frac{1}{4}e^{2x},\\ u_1 &= L_t^{-1}(u_0)_{xx} - L_t^{-1}(A_0) = \int_0^t e^{-2t}\int_0^t e^{2t}((u_0)_{xx} - u_0^2)\, dtdt,\\ u_2 &= L_t^{-1}(u_1)_{xx} - L_t^{-1}(A_1) = \int_0^t e^{-2t}\int_0^t e^{2t}((u_1)_{xx} - 2u_0u_1)\, dtdt,\\ &\vdots \end{aligned}$$

The solution in series form is

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k = u_0 + u_1 + u_2 + ...\\ &= \frac{t}{2}e^{x-2t} + \frac{5}{4}e^{x-2t} + \frac{1}{8}e^{2x-4t} - \frac{1}{4}e^{2x-2t} + \frac{1}{8}e^{2x} - \frac{1}{4}e^{2x} + ..., \end{aligned}$$

where the exact solution is

$$u(x,t) = e^{x-2t}.$$

The absolute error of this Example 6 is displayed in Table 2. It was computed with three terms of the solution i.e $u_0 + u_1 + u_2$, at $x = 0.01$ for different values of t.

x,y	t	solution by MADM	solution by SMADM	EXACT SOLUTION	AE by MADM	AE by SMADM
0	0.1	0.7408180000	0.7408182207	0.7408182207	2.207×10^{-7}	0
	0.3	0.4065730000	0.4065696597	0.4065696597	3.3403×10^{-6}	0
	0.5	0.2231300000	0.2231301600	0.2231301601	1.601×10^{-7}	1×10^{-10}
0.5	0.1	2.013750000	2.013752708	2.013752707	2.707×10^{-6}	1×10^{-9}
	0.3	1.105170000	1.105170918	1.105170918	9.18×10^{-7}	0
	0.5	0.606540000	0.6065306596	0.6065306597	9.3403×10^{-6}	1×10^{-10}
1	0.1	5.473970000	5.473947392	5.473947392	2.2608×10^{-5}	0
	0.3	3.004170000	3.004166024	3.004166024	3.976×10^{-6}	0
	0.5	1.648730000	1.648721270	1.648721271	8.729×10^{-6}	1×10^{-9}

Table 1: Comparison between exact solution $u = e^{x+y-3t}$ of Example 4 and the solution by MADM and SMADM at different values of x, y and t using 6 iterations(i.e k=5).

t	Exact solution at x=0.01	MADM solution at x=0.01	AE
0.1	0.8269591339	0.8269591350	1.1×10^{-9}
0.2	0.6770568745	0.6770570197	1.452×10^{-7}
0.3	0.5543272847	0.5543296764	2.3917×10^{-6}
0.4	0.4538447953	0.4538606685	15.87×10^{-6}
0.5	0.3715766910	0.3716419421	65.25×10^{-6}
0.6	0.3042212641	0.3044206706	199.41×10^{-6}
0.7	0.2490753046	0.2495734460	498.14×10^{-6}
0.8	0.2039256117	0.2050013396	1.08×10^{-3}
0.9	0.1669601697	0.1690413190	2.08×10^{-3}
1	0.1366954254	0.1403909208	3.70×10^{-3}

Table 2: comparison between the exact solution $u(x,t) = e^{x-2t}$ of Example 6 and the solution obtained by our MADM with three terms of solution $u(x,t) = u_0 + u_1 + u_2$.

6 Conclusions

We proposed an improved technique for solving the hyperbolic telegraph equation using a modified ADM with an invertible partial differential operator. We derived a direct solution formula that converges to the exact or closed approximate solutions. We also introduced a special case transformation technique that simplifies the telegraph equation and makes the computation of the iterations easier and faster than the general one. We demonstrated the accuracy and efficiency of our technique by presenting some numerical examples for linear and nonlinear telegraph equations.

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