Original Research Article

Pointwise Clique-Safe Domination in Graphs

Abstract

Let $G = (V(G), E(G))$ be any finite, undirected, simple graph. The clique centrality of a vertex $x \in V(G)$, denoted by $\omega_G(x)$, is the maximum size of a clique in G containing x. A set $D \subseteq V(G)$ is introduced in this paper as a pointwise clique-safe dominating set of G if for every vertex $y \in D^c$ there exists a vertex $x \in D$ such that $xy \in E(G)$ where $\omega_{\langle D\rangle_G}(x) \geq \omega_{\langle D^c\rangle_G}(y)$. The smallest cardinality of such a pointwise clique-safe dominating set of G is called the pointwise clique-safe domination number of G, denoted by $\gamma_{pcs}(G)$. This study aims to generate some observable properties of the parameter and to evaluate the minimum pointwise clique-safe dominating sets of some special families of graphs such as the complete graph K_n, fan graph F_n, wheel graph W_n and complete bipartite $K_{m,n}$ as well as graphs obtained under the mycielski operation.

Keywords: clique-safe domination, pointwise clique-safe domination number, clique centrality

2020 Mathematics Subject Classification: 05C69, 05C75

1 Introduction

The study of games and recreational mathematics partly led to the investigation of domination in graphs. There was an attempt by De Jaenisch [8] to determine the number of queens required to cover an $n \times n$ chess board. This is just one of the domination-related problems that were introduced from more or less a century before the formal study of domination in graphs. In 1962, Claude Berge [2] introduced the coefficient of external stability which is known today as domination number. In the same year, Oystein Ore [13] introduced the terms dominating set and domination number. From there, numerous studies have been done on domination in graphs. These studies include total domination, weakly connected domination, clique domination and many more.

Let $G = (V(G), E(G))$ be any finite, undirected, simple graph. A nonempty subset D of $V(G)$ is a dominating set of G if for every vertex $y \in D^c$, there exists $x \in D$ such that $xy \in E(G)$. The smallest cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. Any dominating set of G of cardinality equal to $\gamma(G)$ is called a minimum dominating set of G or a γ-set of G.

Example 1.1. Consider graph G in Figure 1. Let $D = \{v_1\}$. Observe that every vertex in $V(G) \setminus D$ is adjacent to v_1. Hence, D is a dominating set of G and subsequently $\gamma(G) = 1$.

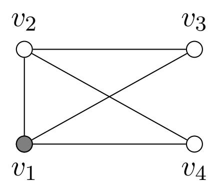


Figure 1: The graph G

In 1988, Cozzens and Kelleher [5] introduced the dominating cliques in graphs, where a clique in G is a subset $W \subseteq V(G)$ such that the subgraph $\langle W \rangle_G$ induced by W in G is complete. They defined a clique dominating set as a set of vertices that dominates G and induces a complete subgraph of G. They also characterized the classes of graphs containing some dominating sets that induce complete subgraphs. In [7], Canoy and Daniel characterized the clique dominating sets in the join, corona, composition and cartesian product of graphs.

In [6], Eballe and Liwat introduced the clique-safe domination in graphs which is also related to this study. They defined the clique-safe dominating set in graphs and give parameters for the clique-safe domination numbers of the path and cycle graphs, where a clique-safe dominating set $D \subseteq V(G)$ is called a clique-safe dominating set in G if the size of the largest clique in $\langle D \rangle_G$ is at least as large as the size of the largest clique in $\langle D \rangle_G$.

Tan and Cabahug [15] characterized the safe sets in graphs, where a safe set of G is a nonempty $S \subseteq V(G)$ such that for every component A of $\langle S \rangle_G$ and every component B of $\langle V(G) \setminus S \rangle_G$ adjacent to A, it holds that $|A| \geq |B|$. They also present a new method of computing the minimum cardinality of a safe set of the path graph and cycle graph using simple modular arithmetic. On the other hand, Madriaga and Eballe [12] introduced the clique centrality of a vertex $v \in V(G)$, denoted by $\omega_G(v)$, as the maximum size of a clique in G containing vertex v.

A dominating set $D \subseteq V(G)$ is introduced in this paper as a *pointwise clique-safe dominating set* of G if for every vertex $y \in V(G) \setminus D = D^c$ there exists a vertex $x \in D$ such that $xy \in E(G)$ where $\omega_{\langle D \rangle_G}(x) \geq \omega_{\langle D^c \rangle_G}(y)$. The minimum cardinality obtainable from among all pointwise clique-safe dominating sets of G is referred to as the pointwise clique-safe domination number of G, denoted by $\gamma_{pcs}(G)$. Any pointwise clique-safe dominating set D of G such that $|D| = \gamma_{pcs}(G)$ is called a minimum pointwise clique-safe dominating set of G or a γ_{pcs}-set of G.

Example 1.2. Consider the path P_5 in Figure 2. Let $D = \{v_2, v_4\}$. Observe that D dominates P_5 and that the $\langle D \rangle_{P_5} = \overline{K}_2$, $\langle D^c \rangle_{P_5} = \overline{K}_3$. It can be seen in the diagram that $\omega_{\langle D \rangle_G}(v_2) = \omega_{\langle D \rangle_G}(v_4) = 1$, $\omega_{\langle D^c \rangle_G}(v_1) = \omega_{\langle D^c \rangle_G}(v_3) = \omega_{\langle D^c \rangle_G}(v_5) = 1$. Clearly, D is a pointwise clique-safe dominating set of P_5 and that $\gamma_{pcs}(P_5) = 2$.

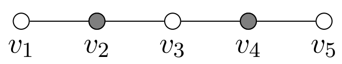


Fig. 2. Path consideration

This study investigates the concept of pointwise clique-safe domination in graphs. It aims to generate some general properties of pointwise clique-safe domination in graphs as well as evaluate the minimum pointwise clique-safe dominating sets of some special families of graphs such as the complete graph K_n, fan graph F_n, wheel graph W_n, and complete bipartite $K_{m,n}$, as well as graphs obtained under the mycielski operation. As a consequence, the pointwise clique-safe domination numbers of those aforementioned graphs are obtained.

Throughout this paper, every graph is considered in the context of being simple, finite, and undirected. Other terminologies not specifically defined in this paper may be found in [3].

2 Basic Properties

Some general results on Pointwise Clique-safe Domination in Graphs

Below is our working definition of the pointwise clique-safe dominating set of a graph G:

Definition 2.1. A set $D \subseteq V(G)$ is a *pointwise clique-safe dominating set* of G if D is a dominating set of G and for every vertex $y \in D^c$ there exists a vertex $x \in D$ such that $xy \in E(G)$ where $\omega_{\langle D \rangle_G}(x) \geq \omega_{\langle D^c \rangle_G}(y)$.

Theorem 2.1. For any graph G, the set $D = V(G)$ is a pointwise clique-safe dominating set of G. As a consequence, $\gamma_{pcs}(G) \leq n$.

Proof. Observe that the set $D = V(G)$ dominates G. Since D^c is empty, the set $D = V(G)$ is a pointwise clique-safe dominating set of G. This implies that $\gamma_{pcs}(G) \leq n$. □

Our next result provides some bounds for the pointwise clique-safe domination number of G, where these bounds can be observed to be sharp.

Theorem 2.2. *For any graph G of order n, $1 \leq \gamma_{pcs}(G) \leq n$, where both bounds are sharp.*

Proof. By Theorem 2.1, $\gamma_{pcs}(G) \leq n$. But it is also obvious that $\gamma_{pcs}(G) \geq 1$. If G is the star graph S_{n-1} of order n, then $\gamma_{pcs}(G) = 1$. On the other hand, if G is the null graph $\overline{K}_n$ of order n, then $\gamma_{pcs}(G) = n$. □

Theorem 2.3. *Let G be a nontrivial connected graph. Then $\gamma_{pcs}(G) = 2$ if and only if one of the following holds:*

a.) *There exists a dominating set D of G containing two elements such that $\langle D \rangle_G = K_2$ and that $\omega_{\langle D^c \rangle}(y) \leq 2$ for every $y \in D^c$;*

b.) *There exists a dominating set D of G containing two elements such that $\langle D \rangle_G = \overline{K}_2$ and that $\omega_{\langle D^c \rangle}(y) = 1$ for every $y \in D^c$, which means that $\langle D^c \rangle_G$ is a null graph.*

Proof. Suppose that $\gamma_{pcs}(G) = 2$. This means that the cardinality of any minimum pointwise clique-safe dominating set D of G is 2. We consider two cases:

i. Suppose $D = \{a, b\}$ where $ab \in E(G)$. Clearly, D is dominating in G where $\langle D \rangle_G = K_2$. Moreover, it is necessary that $\omega_{\langle D^c \rangle}(y) \leq 2$ for every $y \in D^c$. This proves part (a) above.

ii. Suppose $D = \{a, b\}$ where $ab \notin E(G)$. Clearly, D is dominating in G where $\langle D \rangle_G = \overline{K}_2$. Furthermore, it is necessary that $\omega_{\langle D^c \rangle}(y) = 1$ for every $y \in D^c$, which means that $\langle D^c \rangle_G$ is a null graph. This proves part (b) above.

The converse is straightforward. □

3 Pointwise Clique-safe Domination in Special Graphs

The following definitions are for some special graphs considered in this study:

Definition 3.1. [3] A graph G is said to be *complete* if every pair of distinct vertices in it are adjacent. A complete graph of order n is denoted by K_n.

Definition 3.2. [3] A fan graph F_n is a graph of order $n \geq 3$ which is obtained by joining a new vertex to all the vertices of the path P_{n-1}.

Definition 3.3. [3] A wheel graph W_n is a graph of order $n \geq 4$ which is obtained by joining a new vertex to all the vertices of the cycle C_{n-1}.

Definition 3.4. [3] A graph G is called a *bipartite graph* if the vertex-set $V(G)$ of G can be partitioned into two nonempty subsets V_1 and V_2, called *partite sets* of G, such that every edge in G joins a vertex in V_1 with a vertex in V_2. If each vertex in V_1 is adjacent to every vertex in V_2, then G is called a *complete bipartite graph*; in this case, $G = K_{m,n}$ if $|V_1| = m$ and $|V_2| = n$. A *star* of order $n + 1$ is the complete bipartite graph $K_{1,n}$.

Illustration 3.1. Figure 3 below shows the complete graph K_5, fan graph F_5, wheel graph W_5, and the complete bipartite graph $K_{5,4}$. Moreover, vertices a and b are called the root vertices of F_5 and W_5, respectively.

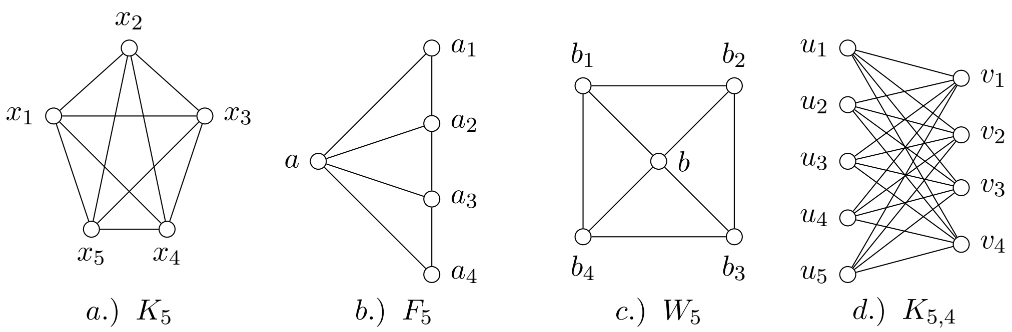


Fig. 3. complete graph K5, fan graph F5, wheel graph W5, and the complete bipartite graph K5,4

Theorem 3.2. *Let K_n be a complete graph of order n. A set $D \subseteq V(K_n)$ is a pointwise clique-safe dominating set of K_n if and only if D contains at least half of the vertices of K_n.*

Proof. Let $D \subseteq V(K_n)$ such that $|D| \geq \frac{n}{2}$. If $x \in D$, then x dominates K_n. If $D^c = \emptyset$, then D is a pointwise clique-safe dominating set of K_n. So suppose $D^c \neq \emptyset$. Let $y \in D^c$. Then for any $x \in D$ we have $\omega_{\langle D \rangle_{K_n}}(x) = |D| \geq n - |D| = \omega_{\langle D^c \rangle_{K_n}}(y)$. This means that D is a pointwise clique-safe dominating set of K_n. On the other hand, if $D \subseteq V(K_n)$ such that $|D| < \frac{n}{2}$, then a similar argument can be applied to show that D is not a pointwise clique-safe dominating set of K_n. □

Corollary 3.3. *The pointwise clique-safe domination number of the complete graph K_n is given by* $\gamma_{pcs}(K_n) = \lceil \frac{n}{2} \rceil$.

Proof. This is immediate from Theorem 3.2. □

Theorem 3.4. *Let F_n be a fan graph of order $n \geq 3$. Let $V(F_n) = \{a, a_1, a_2, ..., a_{n-1}\}$ with a as the root vertex. If a set $D \subseteq V(F_n)$ contains the root vertex of F_n and at least one of the a_i then D is a pointwise clique-safe dominating set of F_n.*

Proof. Notice that if $D \subseteq V(F_n)$ contains the root vertex a of F_n and at least one vertex a_i for some $i = 1, 2, 3, ..., n-1$, then $\{a\}$ is a dominating set of F_n and $\omega_{\langle D \rangle_{F_n}}(a) \geq 2 \geq \omega_{\langle D^c \rangle_{F_n}}(y) = 1$ or 2 for every $y \in D^c$. Hence, D is a pointwise clique-safe dominating set of F_n. □

Corollary 3.5. *The pointwise clique-safe domination number of the fan graph F_n of order $n \geq 3$ is given by $\gamma_{pcs}(F_n) = 2$.*

Proof. Notice that there is no singleton set $D \subseteq V(F_n)$ that is a pointwise clique-safe dominating set in F_n. This means that $\gamma_{pcs}(F_n) \geq 2$. But by Theorem 3.4, $\gamma_{pcs}(F_n) \leq 2$. Combining the two inequalities, we obtain $\gamma_{pcs}(F_n) = 2$. □

Theorem 3.6. *Let W_n be a wheel graph of order $n \geq 4$. Let $V(W_n) = \{b, b_1, b_2, ..., b_{n-1}\}$ with b as the root vertex of W_n. If a set $D \subseteq V(W_n)$ contains the root vertex of W_n and at least one of the b_i, then D is a pointwise clique-safe dominating set of W_n.*

Proof. Notice that if $D \subseteq V(W_n)$ contains the root vertex b of W_n and at least one vertex b_i for some $i = 1, 2, 3, ..., n-1$, then $\{b\}$ is a dominating set of W_n and $\omega_{\langle D \rangle_{W_n}}(b) \geq 2 \geq \omega_{\langle D^c \rangle_{W_n}}(y) = 1$ or 2 for every $y \in D^c$. Hence, D is a pointwise clique-safe dominating set of W_n. □

Corollary 3.7. *The pointwise clique-safe domination number of the wheel graph W_n of order $n \geq 4$ is given by $\gamma_{pcs}(W_n) = 2$.*

Proof. This is exactly analogous to the proof of Corollary 3.5, using Theorem 3.6 instead. □

Theorem 3.8. *Let $K_{m,n}$ be a complete bipartite graph with partite sets A and B such that $|A| = m$ and $|B| = n$. Let $D \subseteq V(K_{m,n})$. Then D is a pointwise clique-safe dominating set of $K_{m,n}$ if and only if one of the following conditions hold:*

- *a.)* $D = A$;
- *b.)* $D = B$;
- *c.)* $D = C \cup E$, *where* $\emptyset \neq C \subseteq A$, $\emptyset \neq E \subseteq B$.

Proof. Observe that each partite set of $K_{m,n}$ is a dominating set such that $\omega_{\langle A \rangle_{K_{m,n}}}(x) = 1 = \omega_{\langle B \rangle_{K_{m,n}}}(y)$ for every $x \in A$ and $y \in B$. Hence, A and B are pointwise clique-safe dominating sets of $K_{m,n}$. This shows parts (a) and (b). For part (c), let $D = C \cup E$, where $\emptyset \neq C \subseteq A$, $\emptyset \neq E \subseteq B$. If $D^c = \emptyset$, then we are done. So suppose $D^c \neq \emptyset$. Let $y \in D^c$, $u_1 \in C$, and $u_2 \in E$. If $y \in A$, then $u_2 y \in E(K_{m,n})$ and $\omega_{\langle D \rangle_{K_{m,n}}}(u_2) = 2 \geq \omega_{\langle D^c \rangle_{K_{m,n}}}(y) = 1$ or 2. On the other hand, if $y \in B$, then $u_1 y \in E(K_{m,n})$ and $\omega_{\langle D \rangle_{K_{m,n}}}(u_1) = 2 \geq \omega_{\langle D^c \rangle_{K_{m,n}}}(y) = 1$ or 2. In either case, D is a pointwise clique-safe dominating set of $K_{m,n}$. This shows part (c).
The converse is straightforward. □

Corollary 3.9. *The pointwise clique-safe domination number of the complete bipartite graph $K_{m,n}$ is given by*

$$\gamma_{pcs}(K_{m,n}) = \begin{cases} 1 & \text{if either } m = 1 \text{ or } n = 1 \\ 2 & \text{if both } m, n \geq 2 \end{cases} \tag{3.1}$$

Proof. This is a direct consequence of Theorem 3.8. □

4 Pointwise Clique-safe Domination in Graphs Under the Mycielski Operation

Below is the definition of the specific unary operation considered in this study.

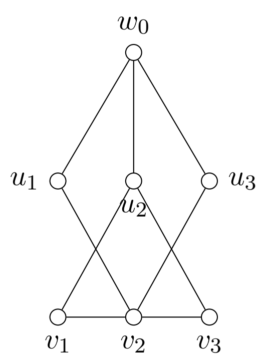
Definition 4.1. [9] Consider a graph G with $V(G) = \{v_1, v_2, v_3, ..., v_n\}$. To obtain the *Mycielski of* G, denoted by $\mu(G)$, the following steps are applied:

(i) Consider graph G as our initial graph.

(i) Add a set of new vertices $U = \{u_1, u_2, u_3, ..., u_n\}$ and add edges from vertex u_i of U to the vertices v_j in $V(G)$ if the corresponding vertices v_i and v_j are adjacent in G.

(ii) Add another new vertex w_0 and add edges joining w_0 to each element in U.

Illustration 4.1. Figure 4 below shows the Mycielski graph $\mu(P_3)$ of P_3.



$a.)$ $\mu(P_3)$

Fig. 4. Mycielski graph μ(P3) of P3

Remark 4.2. *One may verify that if G is of order n and size m, then the Mycielski graph $\mu(G)$ is of order $2n + 1$ and size $3m + n$.*

Lemma 4.3. *If D is a pointwise clique-safe dominating set of $\mu(G)$, then $V(G) \cap D$ is a pointwise clique-safe dominating set of G.*

Proof. Let D be a pointwise clique-safe dominating set of $\mu(G)$, and let $W = V(G) \cap D$ and $C = V(G) \setminus D$. Suppose W does not dominates G. This means that for every vertex $y \in C$ not dominated by W, there exists $u_i \in D$ such that $yu_i \in E(\mu(G))$. But there is no guarantee that

$\omega_{\langle D\rangle_{\mu(G)}}(u_i) \geq \omega_{\langle C\rangle_G}(y)$, especially if $\omega_{\langle C\rangle_G}(y) \geq 3$. Hence, there must exist $x \in W$ such that $xy \in E(G)$ and $\omega_{\langle W\rangle_G}(x) \geq \omega_{\langle C\rangle_G}(y)$. Therefore, W must be a pointwise clique-safe dominating set of G.

□

Theorem 4.4. *Let G be a connected nontrivial graph of order n and $\mu(G)$ be the Mycielski graph of G with W as the pointwise clique-safe dominating set of G. If $D \subseteq V(\mu(G))$ satisfies any of the conditions below then D is a pointwise clique-safe dominating set of $\mu(G)$.*

a.) $D = V(G) \cup \{u_i\}$ *for some* $i = 1, 2, ..., n$;

b.) $D = V(G) \cup \{w_0\}$;

c.) $D = W \cup \{w_0\} \cup \{u_i : v_i \in W\}$;

d.) $D = W \cup \{w_0\} \cup \{u_i : v_i \notin W\}$;

Proof. For part (a), we let $D \subseteq V(\mu(G)) = V(G) \cup \{u_i\}$ for some $i = 1, 2, ..., n$. Observe that D dominates $\mu(G)$. Now for every $y \in D^c$, there exists $x \in D$ such that $\omega_{\langle D^c\rangle_{\mu(G)}}(y) = 2 \leq \omega_{\langle D\rangle_{\mu(G)}}(x)$. Hence, D is a pointwise clique-safe dominating set of $\mu(G)$. The same argument can be used to prove part (b), in which for every $y \in D^c$, there exists $x \in D$ such that $\omega_{\langle D^c\rangle_{\mu(G)}}(y) = 1 \leq \omega_{\langle D\rangle_{\mu(G)}}(x)$.

For part (c), let $D \subseteq V(\mu(G)) = W \cup \{w_0\} \cup \{u_i\}$ for $i = 1, 2, ..., n$ such that $u_i : v_i \in W$. Note that every vertex $y \in D^c \cap V(G)$ is already pointwise clique-safe dominated by D. Moreover, for every vertex $u_i \in D^c$, there exists $x \in D \cap V(G)$ such that $xu_i \in V(\mu(G))$ and $\omega_{\langle D\rangle_{\mu(G)}}(x) \geq \omega_{\langle D^c\rangle_{\mu(G)}}(u_i)$. This proves part (c). Now, for part (d), let $D \subseteq V(\mu(G)) = W \cup \{w_0\} \cup \{u_i : v_i \notin W\}$. Note that every vertex $y \in D^c \cap V(G)$ is already pointwise clique-safe dominated by D. Moreover, for every vertex $u_i \in D^c$, there exists $x \in D \cap V(G)$ such that $xu_i \in V(\mu(G))$ and $\omega_{\langle D\rangle_{\mu(G)}}(x) \geq \omega_{\langle D^c\rangle_{\mu(G)}}(u_i)$. This proves part (d). □

Since we cannot yet make general characterization for the pointwise clique-safe dominating set of the mycielski of any graph, we will just characterize the mycielski of some special families of graphs.The next result gives the pointwise clique-safe domination of the mycielski graph of the complete graph $\mu(K_n)$.

Theorem 4.5. *Let $\mu(K_n)$ be the mycielski graph of the complete graph K_n and W be the pointwise clique-safe dominating set of K_n. Then $D \subseteq \mu(K_n)$ is the pointwise clique-safe dominating set of $\mu(K_n)$ if and only if any of the following holds:*

i.) *If n is even,* $D = \gamma_{pcs} - set$ *of* $K_n \cup \{u_i : v_i \in W\}$ *for some* $i = 1, 2, ..., n$;

ii.) *If n is odd,* $D = \gamma_{pcs} - set$ *of* $K_n \cup \{u_i\}$ *for some* $i = 1, 2, ..., n$.

Proof. Suppose $D \subseteq \mu(K_n)$ is the pointwise clique-safe dominating set of $\mu(K_n)$. By Lemma 4.3, D contains W. Since D dominates $\mu(K_n)$, D must contain a u_i as a dominating vertex of u_0. Now, if K_n is of even order, by Corollary 3.3, W and W^c have the same order which implies that their vertices have the same clique centrality. This means that we must choose a vertex u_i that can increase the clique centrality of the vertices of W and that u_i happens to be a vertex that does not corresponds to any element of W. Hence, $D = \gamma_{pcs} - set$ of $K_n \cup \{u_i : v_i \in W\}$ for some $i = 1, 2, ..., n$.
Now, suppose n is odd. By Corollary 3.3, $|W| > |W^{c|}$. This means that we need not to increase

the clique centrality of the every element of W. Hence, we can choose any u_i as an element of D. The converse is straightforward. □

The previous result asserts that in both cases, the smallest cardinality of D is equal. Now, the next result provides the pointwise clique-safe domination number of the mycielski graph of the complete graph $\mu(K_n)$.

Corollary 4.6. *The pointwise clique-safe domination number of the mycielski graph of the complete graph $\mu(K_n)$ is given by*

$$\gamma_{pcs}(\mu(K_n)) = \gamma_{pcs}(K_n) + 1.$$

Proof. This is a direct consequence of Theorem 4.5. □

5 Conclusion

In this article, the concept of pointwise clique-safe domination is introduced and its corresponding pointwise clique-safe domination number is being investigated. Furthermore, the corresponding expressions for the pointwise clique-safe domination number of those mentioned graphs are determined. Finally, the parameter introduced in this paper may be explored further to address some relevant problems as done in [4], [11], [14], [16], [17], [18] , and [19].

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