Short Research Article

Simple criteria for $\sqrt[n]{x}$ ($n \in \mathbf{N}$, $n \geq 2$, $x \in \mathbf{R}$) being a rational or an irrational number

Abstract

This paper presents a strong generalization of Euclid's famous result related to $\sqrt{2}$ being an irrational number. In particular, based on the unique prime factorization of integer numbers we obtain very simple criteria which allow us to derive necessary and sufficient conditions for $\sqrt[n]{x}$ ($n \in \mathbf{N}$, $n \geq 2$, $x \in \mathbf{R}$) being rational or irrational.

Keywords: *Number theory, prime factorization, generalization of Euclid's proof, simplification of mathematical proofs,* $\sqrt[n]{x}$ *($n \geq 2$, $x \in$ **R**): rational or irrational number.*

1. Introduction.

Already in ancient times, mathematicians were curious to know for which values $x \in \mathbf{N}$ the root $\sqrt{x}$ is a rational number and for which values of x the root is irrational. In particular, the famous mathematician Euclid has presented his highly renowned proof that $\sqrt{2}$ is an irrational number [1,2,3]. Many centuries later, facts were proven which were significantly more general such as, e.g.: for all primes p it can be shown that $\sqrt{p}$ is irrational or a lot of cases were identified in which $\sqrt[n]{x}$ is irrational [4]. As stated in [5], modern number theory, besides having been influenced by Euclid, is strongly based on the scientific work of C.F. Gauss.

The goal of this paper is even a much more general one as we try to find rather simple (necessary and sufficient) conditions which have to be satisfied to answer the quite demanding question

"In which cases is $\sqrt[n]{x}$ a rational number and when is it irrational? (for $n \geq 2$, $x \in$ **R**)".

This implies that we are looking for proofs being as simple as possible and which cover countably infinite many values of n and uncountably infinite many values of x – indeed an enormous spectrum of different cases. Our proofs often will make use of the well-known fact that integer numbers possess a prime factorization which is unique [6].

Astonishingly, the criteria we will present in this contribution allow us to apply quite elegant tests leading to very quick answers (i.e. answers after a think-time of just a few seconds) regarding the fact whether the n^{th} root of a very large number x is rational or irrational. As arbitrarily chosen examples let us, e.g., consider the numbers $x\_0 = (101)^{10}$, $x\_1 = 2 \bullet (101)^{10}$, and $x\_2 = 5 \bullet (101)^{10}$. Our task will be to decide for these extremely large numbers (possessing more than 20 decimal digits each), for which values of n the root $\sqrt[n]{x}$ is still a rational number. In Section 6 we will shortly come back to this problem and we will apply the results of this paper and will easily answer the question for which values of *n,* the n^{th} root of x_i, $i \in \{0,1,2\}$ results in a rational number.

The rest of this paper will be structured as follows: Sections 2 to 5 will all be concerned deriving criteria which – for different sets of numbers (in particular, natural, rational and real numbers) – allow us to answer the question whether $\sqrt[n]{x}$ is rational for $n \geq 2$ and

- $x \in \{0,1\}$ as well as $x \in \mathbf{R}^-$ (in Section 2)
- $x \in \mathbf{N}$, $x \geq 2$ (in Section 3)
- $x \in \mathbf{Q}^+$, $x \neq 1$ (in Section 4)
- $x \in \mathbf{R}^+ \setminus \mathbf{Q}$ (in Section 5)

Section 6 will summarize some of the main results obtained and shortly discuss their relevance (in particular, in improving mathematical teaching).

2. Criteria for $\sqrt[n]{x}$ being rational ($n \geq 2$, $x \in \{0,1\}$, $x \in \mathbf{R}^-$).

In order to strongly reduce the complexity of Sections 3 – 5, let us start this section with an a priori discussion of the simple cases $x \in \{0,1\}$ and $x \in \mathbf{R}^-$, where $\mathbf{R}^- := \{y \in \mathbf{R} \mid y < 0\}$. All these cases can be solved in a straightforward manner and this allows us to eliminate an extremely large number of cases in each of the Sections 3 – 5 and thus to considerably shorten our search for criteria for $\sqrt[n]{x}$ being irrational.

- Case I: *x=0.*
  $\sqrt[n]{x} = 0 \ \forall n \geq 2$, because $0^n = 0 \ \forall n \in \mathbf{N}$.
- Case II: *x=1.*
  $\sqrt[n]{x} = 1 \ \forall n \geq 2$, because $1^n = 1 \ \forall n \in \mathbf{N}$.
- Case III: $x \in \boldsymbol{R}^-$ *and n even.*
  As n is even $\exists\, m \in \mathbf{N}$: $n=2m$. Assume $\exists\, y \in \mathbf{R}$: $y \neq 0$ such that $\sqrt[n]{x} = y \Leftrightarrow x = y^n = (y^m)^2 > 0$. The last inequation holds because $y \in \mathbf{R} \Rightarrow y^m \in \mathbf{R}$ and each square of a real number which is unequal 0 is strictly positive. So, we have obtained the contradiction to our initial assumption $x<0$ and we can conclude $\sqrt[n]{x}$ is not a real number $\forall\, x \in \mathbf{R}^-$.
- Case IV: $x \in \boldsymbol{R}^-$ *and n odd.*
  Let x^* denote $x^* = -x$. Then, evidently $x^*>0$.
  Thus, $\sqrt[n]{x} = \sqrt[n]{-x*} = \sqrt[n]{(-1)x*} = \sqrt[n]{-1} \bullet \sqrt[n]{x*} = -\sqrt[n]{x*}$.
  If we now would assume, e.g., $n=3$ and $x^*=8$ (i.e. $x= -8$) some mathematicians would recommend to conclude $-\sqrt[n]{x*} = -\sqrt[3]{8} = -2$, which seems evident to most people. Other mathematicians, however, argue that $\sqrt[3]{-8}$ should remain undefined and it only should be allowed to write $\sqrt[3]{-8} = -\sqrt[3]{8}$ (which is, by the way, what we did in our general case).
  The more restrictive mathematicians see, e.g., the danger of getting contradictions if one would allow to write $\sqrt[3]{-8} = -2$. Such a contradiction cited quite often could be: $-2 = \sqrt[3]{-8} \neq \sqrt[6]{(-8)(-8)} = \sqrt[6]{64} = +2$. A solution to eliminate this kind of contradiction would be to interdict the replacement $\sqrt[n]{x}$ by $\sqrt[n \bullet k]{x \uparrow k}$ in cases when $x<0$. Anyway, we do not want to step deeper into this discussion whether (for n odd and $x<0$) $\sqrt[n]{x}$ should remain undefined or not. This point is not really important regarding the content of this paper.

3. Be $x \in \mathbf{N}$, $x \geq 2$: Criteria for $\sqrt[n]{x}$ being irrational ($n \geq 2$).

Theorem 1.

Be $x \in \mathbf{N}$ and $x = x_1^{i1} \bullet x_2^{i2} \bullet\bullet\bullet\ x_k^{ik} \bullet\bullet\bullet\ x_m^{im}$ be the prime factorization of x.
Then, $\sqrt[n]{x}$ is rational for $n=n_0 \Leftrightarrow \exists\ n_0 \geq 2$: $\forall\ k=1, \ldots, m\ \exists\ \nu \in \mathbf{N}$, $\nu = \nu(k)$ such that $n_0 \bullet \nu = i_k$.

Proof of Theorem 1.

Assumption: $\sqrt[n]{x}$ is rational $\Rightarrow \exists\ p, q$ with $\sqrt[n]{x} = p/q$, p and q being coprime.

$$\sqrt[n]{x} = p/q \Leftrightarrow x = p^n/q^n$$

Therefore, $$x \bullet q^n = p^n \quad (1)$$

Let the prime factorization of x be: $x = x_1^{i1} \bullet x_2^{i2} \bullet\bullet\bullet\ x_k^{ik} \bullet\bullet\bullet\ x_m^{im}$.

The prime factorization of q cannot contain any of the primes x_k, which are part of the prime factorization of x, because otherwise x_k would appear on the left side of eq. (1), but – because of p and q being coprime – x_k would not be part of the right side of eq. (1). Therefore, x_k has to be part of the prime factorization of p. Let us now denote with ν the exponent of x_k in the prime factorization of p. In order to obtain x_k having the same exponent on both sides of eq. (1) it has to be required that n fulfills the condition $n \bullet \nu = i_k$. Thus, n has to be a divisor (resulting in an integer) of i_k. And because this requirement has to be fulfilled for all $k=1,2, \ldots , m$ we realize that n has to be integer divisor of $i1$, $i2$, …, im. In addition, it becomes evident that, besides the primes x_i , p cannot contain a prime factor $p^* \neq x_i\ \forall\ i$, because it could not be part of q's prime factorization (both being coprime). Therefore, it also becomes evident: q's prime factorization does not contain neither any of the prime factors x_i $\forall i$, nor any other prime. Therefore, we see that $q=1$ must hold and this implies that $\sqrt[n]{x}$ is not only a rational but even a natural number.

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Based on the prime factors of $x = x_1^{i1} \bullet x_2^{i2} \bullet\bullet\bullet\ x_k^{ik} \bullet\bullet\bullet\ x_m^{im}$ we define $D_k(x)=\{n \in \mathbf{N} |\ n \geq 2, \exists\, \nu \in \mathbf{N}$ such that $n \bullet \nu = i_k\}$ to denote the set of potential divisors of i_k, i.e. this means that i_k is an integer multiple of ν. Examples: For $i_k=15$, $D_k=\{3,5,15\}$ and for $i_k=18$, $D_k=\{2,3,6,9,18\}$. *Remark:* D_k includes, e.g., all prime factors of i_k.

Short summary: Theorem 1 implies that $\sqrt[n]{x}$, $x \in \mathbf{N}$ is a natural number exactly for the elements of $D^*(x) := D_1(x) \cap D_2(x) \cap \ldots \cap D_k(x) \cap \ldots D_m(x)$, i.e. for $x \in \mathbf{N}$, we see that $\sqrt[n]{x} \in \mathbf{N}$ *iff.*: $n \in D^*(x)$. This, of course, implies that $\sqrt[n]{x}$ is irrational *iff.:* $D^*(x)=\emptyset$.

Examples:

I. Be $x \in \mathbf{N}$ and the prime factorization of x is $x = p_0 \bullet p_1^{i1} \bullet p_2^{i2} \bullet\bullet\bullet\ p_k^{ik} \bullet\bullet\bullet\ p_m^{im}$. Then, $\forall n \geq 2$ $\sqrt[n]{x} \notin \mathbf{Q}$. Thus, if the prime factorization of x contains (at least) one prime with exponent of 1 then $\sqrt[n]{x}$ cannot be rational.

II, Be $x \in \mathbf{N}$ and the prime factorization of x is $x = p_0^{2\mu+1} \bullet p_1^{i1} \bullet p_2^{i2} \bullet\bullet\bullet\ p_k^{ik} \bullet\bullet\bullet\ p_m^{im}$, $\mu \in \mathbf{N}$. Then, $\forall n$ being even, $n \geq 2$: $\sqrt[n]{x} \notin \mathbf{Q}$. Thus, if the prime factorization of x contains (at least) one prime with an odd exponent, then $\sqrt[n]{x}$ cannot be rational for all n being even.

4. Be $x \in$ **Q**$^+$, $x \neq 1$: Criteria for $\sqrt[n]{x}$ being irrational ($n \geq 2$).

Theorem 2.

Be $x \in$ **Q**$^+$, $x \neq 1$, where **Q**$^+ := \{y \in Q \mid y > 0\}$ and $x = p/q$, p and q are coprime.

Let $p = p_1^{i1} \bullet p_2^{i2} \bullet\bullet\bullet\ p_k^{ik} \bullet\bullet\bullet\ p_m^{im}$ and $q = q_1^{j1} \bullet q_2^{j2} \bullet\bullet\bullet\ q_k^{jk} \bullet\bullet\bullet\ q_l^{jl}$

be the prime factorization of p and q, p_k and q_j denoting mutually different primes $\forall k \in \{1 \ldots m\}$, (in case of p) and $\forall k \in \{1 \ldots l\}$, (in case of q).

$\sqrt[n]{x}$ is rational for some $n \in$ **N** $\Leftrightarrow$ $\exists$ $n_0 \geq 2$: $\forall$ $k = 1, \ldots, m$ $\exists \nu_1 \in$ **N,** $\nu_1 = \nu_1(k)$ (in case of p) and $\forall$ $k = 1, \ldots, l$ (in case of q) $\exists \nu_2 \in$ **N,** $\nu_2 = \nu_2(k)$ such that $n_0 \bullet \nu_1 = i_k$ and $n_0 \bullet \nu_2 = j_k$.

Proof of Theorem 2.

Let $x \in$ **Q**$^+$**.** $\exists$ $p,q \in$ **N**, p,q coprime and $x = p/q$.

Assumption: $\sqrt[n]{x}$ is rational. $\Rightarrow \exists$ $r,s \in$**N**, r,s being coprime and

$$\sqrt[n]{x} = \sqrt[n]{p/q} = \frac{r}{s} \quad \Leftrightarrow \quad p/q = r^n / s^n \quad \Leftrightarrow \quad p \bullet s^n = q \bullet r^n. \qquad (2)$$

Eq. (2) means:

$$p_1^{i1} \bullet p_2^{i2} \bullet\bullet\bullet\ p_k^{ik} \bullet\bullet\bullet\ p_m^{im} \bullet s^n = q_1^{j1} \bullet q_2^{j2} \bullet\bullet\bullet\ q_k^{jk} \bullet\bullet\bullet\ q_l^{il} \bullet r^n \qquad (3)$$

By way of example, let us now consider p_k^{ik}: so prime p_k is part of the left side of eq. (3) $\Rightarrow p_k$ has also to appear on the right side of eq. (3). As $\forall q_i$: $q_i \neq p_k$ $\forall k$ (p and q are coprime): p_k has to be a prime factor of r^n, and, therefore p_k has also to be a prime factor of r. Let us assume that the prime factorization of r contains p_k exactly with exponent ν. Then r^n contains p_k exactly with factor $p_k^{\nu \bullet n}$. To satisfy eq. (3), the exponent of p_k on the left side of eq. (3), i.e. i_k . has to be identical with the exponent of p_k on the right side of eq. (3), i.e. with $\nu \bullet n$. Thus, we found exactly the condition stated in Theorem 2, i.e. $\nu \bullet n = i_k$. For all primes being part of the prime factorization of p and q (with frequency i_k and j_k resp.), values of ν and n (i.e. a constant $n = n_0$) have to exist which satisfy the condition $\nu \bullet n_0 = i_k$ and $\nu \bullet n_0 = j_k$, respectively. ■

Assume the prime factorization of p and q as introduced in Theorem 2.
Let

- $D_k(p)$ denote the set of potential divisors of exponent i_k, $k \in \{1 \ldots m\}$ and
- $D_k(q)$ denote the set of potential divisors of exponent j_k, $k \in \{1 \ldots l\}$.
- $D^*(p) := D_1(p) \cap D_2(p) \cap \ldots \cap D_k(p) \cap \ldots D_m(p)$, and
- $D^*(q) := D_1(q) \cap D_2(q) \cap \ldots \cap D_k(q) \cap \ldots D_l(q)$.

Analogously to Section 3 we want to present a similar short summary:

Theorem 2 implies that $\sqrt[n]{x}$, $x \in$ **Q**$^+$**,** $x \neq 1$ is a rational number exactly for the elements of $D^*(p) \cap D^*(q)$. This, of course, implies that $\sqrt[n]{x} = \sqrt[n]{p/q}$ is irrational *iff*: $n \notin D^*(p) \cap D^*(q)$.

Remark: $D^*(p)$ denotes the set of common divisors of all exponents i_k appearing in the prime factorization of p. (Analogously, for $D^*(q)$).

5. Be $x \in$ **R**$^+$ **\ Q** : Criteria for $\sqrt[n]{x}$ being irrational ($n \geq 2$).

In this section, we prove that for all $n \geq 2$ the n^{th} root of any positive irrational number x, i.e. $\sqrt[n]{x}$, is irrational. Let **R**$^+$ denote **R**$^+$:= {$y \in$ **R** | y>0}.

Theorem 3. For all $x \in$ **R**$^+$ **\ Q** and $n \in$ **N**, $n \geq 2$: $\sqrt[n]{x}$ is an irrational number.

Proof of Theorem 3.

Be $x \in$ **R**$^+$ **\ Q**, $n \in$ **N**, $n \geq 2$. Assume: $\exists$ $y=p/q \in$ **Q**, $p,q \in$ **N**\{0} such that $y = \sqrt[n]{x} = p/q \Rightarrow p^n \in$ **Z** and $q^n \in$ **Z**, i.e. $x=p^n/q^n \in$ **Q**, which contradicts $x \notin$ **Q**. ∎

The proof of this theorem is amazingly short and straightforward.

6. Summary and Conclusions.

Let us now come back to our claims formulated in Section 1 namely that, applying the results of this paper, it will be extremely easy to answer the questions posed in the first section, namely whether the n^{th} root of x_i, $i \in$ {0,1,2} is still a rational number or not. Intentionally, we have given the numbers x_i not as conventional decimal numbers (which anyway would contain a very large number of decimal digits) but already using their prime factorization.

Based on the prime factorization and directly based on the results of Section 3 the reader will notice that $\sqrt[n]{x_0}$, $n \geq 2$ will be a rational number *iff.* $n \in$ {2,5,10} and an irrational number otherwise. The evident reason for this result is that the prime 101 possesses an exponent of 10 and only for the 3 values of n mentioned, $\exists$ $\nu \in$ **N** such that $n \bullet \nu = 10$ (please note the assumption $n \geq 2$).

$\sqrt[n]{x_1}$ and $\sqrt[n]{x_2}$ are irrational $\forall$ $n \geq 2$, because the corresponding prime factorization contains a prime, the exponent of which is 1 and $n \bullet \nu = 1$ does not possess any solution for $n \geq 2$, $\nu \in$ **N**.

This paper has elaborated criteria which represent both necessary and sufficient conditions for $\sqrt[n]{x}$, $n \in$ **N**, $n \geq 2$, $x \in$ **R** being still rational numbers. Indeed, our criteria which are often based on the properties of prime factorization of integers, are surprisingly simple and can be derived in an astonishingly straightforward manner. In summary, we found a complete answer to the question "For which $n \in$ **N** and $x \in$ **R** is $\sqrt[n]{x}$ an irrational number?"

Our results comprise:

- $\sqrt[n]{x}$, for $x \in$ **N**, $x \geq 2$, $n \geq 2$ is still a natural number if all exponents appearing in the prime factorization of x satisfy some simple condition (see Section 3);
- $\sqrt[n]{x} = \sqrt[n]{p/q}$, for $x \in$ **Q**$^+$, $x \neq 1$, $n \geq 2$ is still a rational number if all exponents appearing in the prime factorization of, both, p and q satisfy some simple condition (see Section 4);
- $\sqrt[n]{x}$, for $x \in$ **R**$^+$ \ **Q**, $n \geq 2$ is always an irrational number (see Section 5).

Our search for criteria to be as simple as possible was motivated primarily by our desire to present non-trivial mathematical facts to people (e.g., to students) in a straightforward manner. At the same time, we tried to cover as many cases as possible – even an uncountably infinite number of different roots. The desire to elaborate simple proofs was already the main motivation of a recent publication of this author [7]. Hints for the construction of proofs being as simple as possible can be found, e.g., in [8].

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Declarations

- **Ethical approval**
  not applicable
- **Availability of data and materials**
  not applicable

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