Original Research Article

On the difference in cycling pattern on linear and higher-order effect designs

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ABSTRACT

This paper seeks to determine the nature and extent to which the influence of cycling affects the pattern of convergence on Linear, Interactive, and Quadratic order effect designs. Cycling, a phenomenal problem associated with the construction of $D-$optimal designs, impedes the rate of convergence whenever it occurs in a variance exchange process. The variance exchange algorithmic search method based on the philosophy of numerically searching the design space for optimum $D-$designs was adopted. Two and three-factor response functions are used in the investigation of even and odd-sized $N-$point designs. Generated data from designs of sizes 10 and 11 were employed in the investigation. Numerical illustrations were given to ascertain the pattern of convergence on each of the $m-$degree polynomial designs. The computations and graphs were conducted in R version 4.1.1 (2021). The results show that cycling patterns differ with respect to the degree of the response function whether it is of even or odd-sized design, or has two or three variables.

Keywords: ***C****ycling, Variance exchange process, D-optimality, Linear and Higher-order effects, Response function, Even and odd N-point designs.*

I. INTRODUCTION

Optimum experimental designs depend upon the model or models to be fitted to the data, on the values of the parameters of the models [1]. The first- and second-order model response polynomials in p factors considered in this study are respectively

$$E(y) = a_0 + \sum_{i=1}^{p} a_i x_i \tag{1}$$

$$E(y) = a_0 + \sum_{i=1}^{p} a_i x_i + \sum_{i=1}^{p-1} \sum_{j=i+1}^{p} a_{ij} x_i x_j + \sum_{i=1}^{p} a_{ii} x_i^2 \tag{2}$$

The theory of continuous designs considers minimization of the general measure of imprecision $\eta\{M(\xi)\}$ on the designs, eqns. (1) and (2). The function to be minimized depends on the measure ξ through the information matrix $M(\xi)$. A special case of this is the $D-$optimality and its relationship to $G-$optimality. For $D-$optimality, [1] and [2] hold that

$$\eta\{M(\xi)\} = \log\left|M^{-1}(\xi)\right| = -\log\left|M(\xi)\right|, \tag{3}$$

the generalized variance of the parameter estimates, or its logarithm $-\log\left|M(\xi)\right|$, is minimized. This imply the determinant of the information matrix $M(\xi)$ is maximized.

[3], [4], [5], and [1] hold it that continuous designs that are $D-$optimum are also $G-$optimum, that is to say they minimize the maximum over $\mathcal{X}$ of the variance

$$d(x,\ \xi) = f^T(x) M^{-1}(\xi) f(x) \qquad (4)$$

a function of both the design ξ and the point at which the prediction is made.

Variance exchange algorithm is one of several algorithms for finding continuous measures ξ that minimize $\eta\{M(\xi)\}$. However, while [6] established that optimal designs guaranteed from ultimate convergence by maximizing the determinant of information matrix $M(\xi)$ using the variance exchange process is usually not feasible, [7] attributed it to cycling.

Cycling, when it occurs in such an iterative process, halts the monotonically increasing sequence of determinants of the information matrices with attendant consequence on the rate of convergence; the determinant and variances of points at this point equals the determinant of the information matrix and variances of points one, two or a fraction of one or two places before it in the exchange process.

Cycling apart from inducing the rate of convergence, may show remarkable influence in the pattern of convergence according to different degrees of response functions involved whichever the number of variables p or the size N of the design.

In this article we will investigate the influence of cycling as it affects first and second order effect designs with respect to the convergent patterns.

1.1 Literature Review

Variance exchange algorithm, an iterative technique for getting a $D-$optimum exact design has enjoyed a rich literature by several authors. Among the prominent works referenced in this paper are those published by [8], the modification of Fedorov's Exchange Algorithm. In their algorithm, they switched each point in the design with the candidate point that maximizes the Fedorov's delta function, a procedure only twice as fast; Other works consulted include those of [9], who developed exchange and interchange procedures to search for optimal design; [10], who generalized the original Modified Fedorov Exchange Algorithm (MFEA). They did this by permitting only certain stages at each iteration, with these stages corresponding to its same number of points, in the designs with the smallest variance; the KL-EA of [11], which was another modification of the Fedorov's exchange algorithm. In the KL-EA, a point with minimum variance in the design and a point in the candidate set with maximum variance are exchanged such that the Fedorov's delta function is maximized; [12], who reviewed some algorithms for constructing discrete $D-$optimal designs. We referred to similar iterative search structure of the variance exchange process by [13], who proposed some exchange algorithms for Constructing Model-Robust Experimental Designs; the exchange algorithm by [14], which made some refinements on the Fedorov's exchange algorithm done by simultaneously adding or exchanging two or more points at each step. This reduced the number of steps needed to construct a D-optimal design; Other Authors referred to in this work are [15], A variable-neighborhood search algorithm for finding optimal run orders in the presence of serial correlation.

2 METHODOLOGY

We adopted the search method based on the philosophy of numerically searching the design space for optimum designs. Here we start with an estimate ξ_N of the optimum design ξ^* for the problem, itself not $D-$optimal, and improve on it iteratively until $D-$optimality conditions are satisfied. The next step in the algorithm is to improve the starting design by switching the least variance point[1] in the design with the

[1]A point is an ordered list of numbers. It is used interchangeably with the term 'vector', and lowercase letters in roman boldface are used to denote them.

highest variance points from its complement and evaluate the effect of the change on the $D-$optimality criterion. We repeat this process until no further improvement on the value of the determinant in the entire iteration through the factor settings. The output of this procedure is a design whose determinant can no

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longer be improved by further exchange of variance points. The influence of cycling known to affect the determinants at this stage of the iterative process induces different patterns of convergence which may be noticeable for different degrees of response functions applicable. This influence can be illustrated with the following statements.

Statement 1.0

Suppose $\xi_N^{(k)}$ be an $N-$point design measure of linear or interactive order effects at the k^{th} iteration in an exchange process searching for exact $D-$optimum design. If cycling occurs at this point, then for a $2-$ and $3-$factor response functions,

$$\det M\left(\xi_N^{(k)}\right)=\det M\left(\xi_N^{(k-1)}\right)$$

or

$$\det M\left(\xi_N^{(k)}\right)=\det M\left(\xi_N^{(k-2)}\right)$$

where $\xi_N^{(k-1)}$ and $\xi_N^{(k-2)}$, are the measures of designs, one and two steps backward respectively in the sequence.

Statement 2.0

Suppose $\xi_N^{(k)}$ be an $N-$point design measure of quadratic order effect at the k^{th} iteration in an exchange process searching for exact $D-$optimum design. For a $2-$factor response function at the point of cycling,

$$\det M\left(\xi_N^{(k)}\right)=\det M\left(\xi_N^{(k-)}\right),$$

where $\xi_N^{(k-)}$, is a measure of design a fraction of one and two steps backward in the sequence.

Statement 3.0

Suppose $\xi_N^{(k)}$ be an initial $N-$ point design measure of quadratic order effect searching for exact $D-$optimum design. Then for a $3-$factor response function, $\xi_N^{(1)}$ is nonsingular for all $N \in \mathbb{N}$ [2]

3. RESULTS And DISCUSSION

In this study, $10-$, and $11-$point designs are used for each of the $m-$degree polynomials in $2-$ and $3-$factors.

3.1 Linear, Two-Factor Response Function

A function of two independent variables x_1, x_2 is written as

$$f(\mathbf{x})=(x_1,\ x_2) \tag{5}$$

The linear statistical model for the two $2-$factor response function given by eqn. (1) becomes

[2] $\mathbb{N}$ is a list of possible design points, called a candidate set.

$$E(y)=E\left[f(\mathbf{x})\right]=a_0+\sum_{i=1}^{2}a_i x_i\ ,$$

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and the experimental set is

$$\tilde{X} = \left\{ x_1, x_2 ;\, x_1 = -2,-1,0,1,2,\ x_2 = -1,-0.5,0.5,1 \right\},\ E(\mathbf{e}) = 0,\ \mathrm{Var}(e) = \sigma_e^2$$

Each point in the 10, and $11-$point designs will be represented by $\mathbf{x}_i$; where $\mathbf{x}_i = (1, x_1, x_2)$.

The initial and complement extended matrices for linear order effect, $2-$factor, $10-$, and $11-$point designs are given respectively as follows.

$$X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -0.5 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 0.5 \\ 1 & 0 & -1 \\ 1 & 0 & -0.5 \\ 1 & 1 & 0.5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{pmatrix},\ X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & 0.5 \\ 1 & -1 & -0.5 \\ 1 & -1 & 1 \\ 1 & 0 & 0.5 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & -0.5 \\ 1 & 2 & -0.5 \\ 1 & 2 & 0.5 \end{pmatrix}; \qquad X_{11}^{(1)} = \begin{pmatrix} 1 & -2 & 0.5 \\ 1 & -2 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -0.5 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & -0.5 \\ 1 & 1 & 0.5 \\ 1 & 2 & -1 \\ 1 & 2 & -0.5 \\ 1 & 2 & 0.5 \end{pmatrix},\ X_{11}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 \\ 1 & -2 & -0.5 \\ 1 & -1 & 0.5 \\ 1 & -1 & 1 \\ 1 & 0 & -0.5 \\ 1 & 0 & 0.5 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

3.2 Linear, Three-Factor Response Function

The linear statistical model, eqn. (1), for the $3-$factor response function is

$$E(y) = E\left[f(\mathbf{x}) \right] = a_0 + \sum_{i=1}^{3} a_i x_i\ ,$$

and the experimental set is

$$\tilde{X} = \left\{ x_1, x_2, x_3 : x_1 = -2,-1,0,1,2;\ x_2 = -1,0,1; x_3 = 1,1 \right\},\ E(\mathbf{e}) = 0 \text{ and } \mathrm{Var}(e) = \sigma_e^2$$
$$\mathbf{x}_i = (1, x_1, x_2, x_3)$$

The initial and complement extended matrices for linear order effect, $3-$factor, $10-$, and $11-$point designs are given respectively as follows.

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$$X_{10}^{(1)}=\begin{pmatrix}1&-2&-1&1\\1&-2&0&1\\1&-2&1&-1\\1&-1&-1&-1\\1&-1&0&1\\1&1&-1&1\\1&1&0&1\\1&1&1&1\\1&2&-1&1\\1&2&1&-1\end{pmatrix},\ X_{10}^{(1)c}=\begin{pmatrix}1&-2&-1&-1\\1&-2&0&-1\\1&-2&1&1\\1&-1&-1&1\\1&-1&0&-1\\1&-1&1&-1\\1&-1&1&1\\1&1&-1&-1\\1&1&0&-1\\1&1&1&-1\\1&2&-1&-1\\1&2&0&-1\\1&2&0&1\\1&2&1&1\end{pmatrix};\quad X_{11}^{(1)}=\begin{pmatrix}1&-2&-1&-1\\1&-2&0&1\\1&-2&1&-1\\1&-1&-1&1\\1&-1&0&-1\\1&-1&1&1\\1&1&0&-1\\1&1&1&-1\\1&2&-1&-1\\1&2&0&1\\1&2&1&-1\end{pmatrix},\ X_{11}^{(1)c}=\begin{pmatrix}1&-2&-1&1\\1&-2&0&-1\\1&-2&1&1\\1&-1&-1&-1\\1&-1&0&1\\1&-1&1&-1\\1&1&-1&-1\\1&1&-1&1\\1&1&0&1\\1&1&1&1\\1&2&-1&1\\1&2&0&-1\\1&2&1&1\end{pmatrix}$$

The pattern of convergence by determinants for the linear order effect designs is summarized in Table 1 below

Table 1: Pattern of convergence for the linear order designs

$\xi_N^{(k)}$	$\left\|M\left(\xi_{10}^{(k)}\right)\right\|$		$\left\|M\left(\xi_{11}^{(k)}\right)\right\|$	
	2-factor	3- factor	2-factor	3- factor
1	1390.0	12480	1545.25	14840
2	1739.0	15484	2071.25	25568
3	2006.0	25340	2605.25	35968
4	2251.5	28476	2854.00	39368
5	2349.0	32000	2926.00	41760
6	2352.0	32000	3024.00	44160
7	2349.0		3024.00	44928
				44928

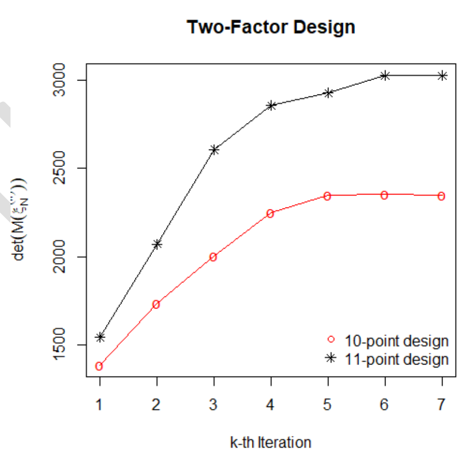

Fig 1 (a) Pattern of convergence for linear effect, two-factor designs

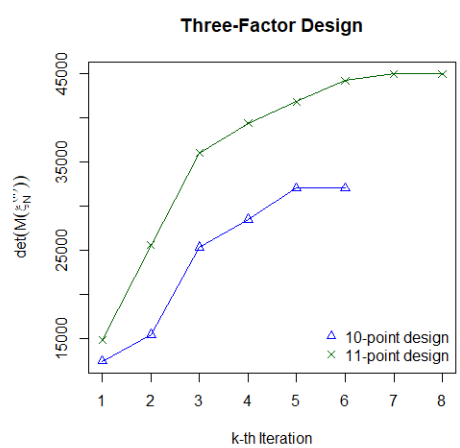

Fig 1 (b) Pattern of convergence for linear effect, three-factor designs

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3.3 Mixed, Two-factor Response Function

The mixed[3] statistical model, eqn. (2), for the two $2-$factor response function is

$$E\left[f(\mathbf{x})\right] = a_0 + \sum_{i=1}^{2} a_i x_i + \sum_{j=i+1}^{2} a_{ij} x_i x_j$$

and the experimental set is

$$\tilde{X} = \{x_1, x_2;\, x_1 = -2, -1, 0, 1, 2,\ x_2 = -1, -0.5, 0.5, 1\},\ E(\mathbf{e}) = 0,\ \mathrm{Var}(e) = \sigma_e^2$$
$$\mathbf{x}_i = (1, x_1, x_2, x_1 x_2)$$

The initial and complement extended matrices for interactive order effect, $2-$factor, $10-$, and $11-$point designs are given respectively as follows.

$$X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -0.5 & 1 \\ 1 & -2 & 0.5 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & -0.5 & 0.5 \\ 1 & 0 & -0.5 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 2 & 0.5 & 1 \end{pmatrix}, X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 1 & -2 & 1 & -2 \\ 1 & -1 & 0.5 & -0.5 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0.5 & 0 \\ 1 & 1 & -0.5 & -0.5 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 2 & -0.5 & -1 \\ 1 & 2 & 1 & 2 \end{pmatrix}; X_{11}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 \\ 1 & -2 & -0.5 & 1 \\ 1 & -1 & -0.5 & 0.5 \\ 1 & -1 & 0.5 & -0.5 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0.5 & 0 \\ 1 & 1 & -0.5 & -0.5 \\ 1 & 1 & 0.5 & 0.5 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \end{pmatrix}, X_{11}^{(1)^c} = \begin{pmatrix} 1 & -2 & 0.5 & -1 \\ 1 & -2 & 1 & -2 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & -0.5 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 2 & -0.5 & -1 \\ 1 & 2 & 0.5 & 1 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

3.4 Mixed, Three-Factor Response Function

The mixed statistical model, eqn. (2), for the two $3-$factor response function is

$$E\left[f(\mathbf{x})\right] = a_0 + \sum_{i=1}^{3} a_i x_i + \sum_{i=1}^{2} \sum_{j=i+1}^{3} a_{ij} x_i x_j$$

and the experimental set is

$$\tilde{X} = \{x_1, x_2, x_3 : x_1 = -2, -1, 0, 1, 2;\ x_2 = -1, 0, 1; x_3 = 1, 1\},\ E(\mathbf{e}) = 0 \text{ and } \mathrm{Var}(e) = \sigma_e^2$$
$$\mathbf{x}_i = (1, x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3)$$

The initial and complement extended matrices for interactive order effect, $3-$factor, $10-$, and $11-$point designs are given respectively as follows.

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[3] The term 'mixed' refers to 'interactive' portion of variables in a second response polynomial. In this paper, the two will be used interchangeably.

$$X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & -1 & 2 & 2 & 1 \\ 1 & -2 & 0 & -1 & 0 & 2 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 2 & -1 & 1 & -2 & 2 & -1 \\ 1 & 2 & 1 & -1 & 2 & -2 & -1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}, \quad X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & 1 & 2 & -2 & -1 \\ 1 & -2 & 0 & 1 & 0 & -2 & 0 \\ 1 & -2 & 1 & -1 & -2 & 2 & -1 \\ 1 & -2 & 1 & 1 & -2 & -2 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 1 & -2 & 2 & -1 \\ 1 & 2 & 0 & -1 & 0 & -2 & 0 \\ 1 & 2 & 0 & 1 & 0 & 2 & 0 \end{pmatrix}$$

$$X_{11}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & -1 & 2 & 2 & 1 \\ 1 & -2 & 0 & 1 & 0 & -2 & 0 \\ 1 & -2 & 1 & -1 & -2 & 2 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 \\ 1 & 2 & -1 & 1 & -2 & 2 & -1 \\ 1 & 2 & 1 & -1 & 2 & -2 & -1 \end{pmatrix}, \quad X_{11}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & 1 & 2 & -2 & -1 \\ 1 & -2 & 0 & -1 & 0 & 2 & 0 \\ 1 & -2 & 1 & 1 & -2 & -2 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -1 & -2 & -2 & 1 \\ 1 & 2 & 0 & -1 & 0 & -2 & 0 \\ 1 & 2 & 0 & 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}$$

The pattern of convergence by determinants for the interactive order effect designs is summarized in Table 2

Table 2: Pattern of convergence for mixed effect designs

$\xi_N^{(k)}$	$\left\|M\left(\xi_{10}^{(k)}\right)\right\|$		$\left\|M\left(\xi_{11}^{(k)}\right)\right\|$	
	2-factor	3-factor	2-factor	3- factor
1	10395.00	24717312	10435.50	47553280
2	22834.69	89879040	22698.00	150100992
3	32832.00	159045632	42623.13	294461440
4	50544.00	242581504	49428.00	363802624
5	75688.87	301989888	57011.62	542703616
6	50544.00	340787200	63036.00	542703616
7		340787200	68688.00	
8			68688.00	

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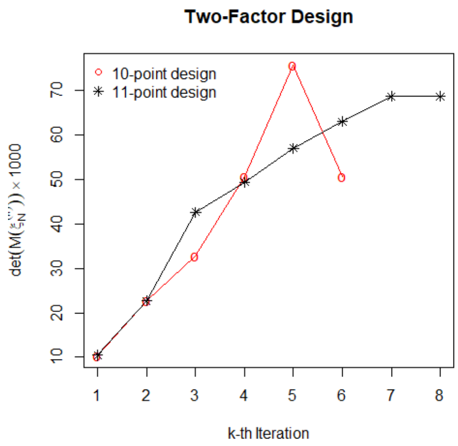

Fig 2 (a) Pattern of convergence for mixed effect, two-factor designs

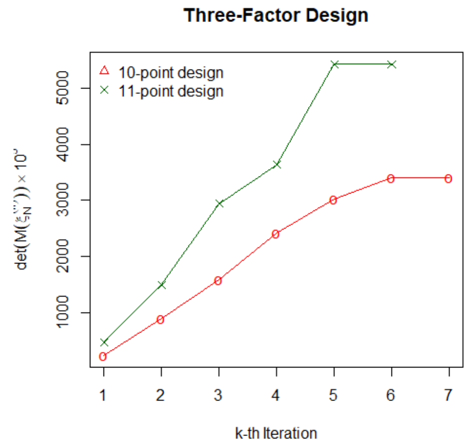

Fig 2 (b) Pattern of convergence for mixed effect, three-factor designs

3.5 Quadratic, Two-Factor Response Function

The quadratic statistical model, eqn. (2), for the two $2-$factor response function is

$$E\left[f(\mathbf{x})\right] = a_0 + \sum_{i=1}^{2} a_i x_i + \sum_{j=i+1}^{2} a_{ij} x_i x_j + \sum_{i=1}^{2} a_{ii} x_i^2$$

and the experimental set is

$$\tilde{X} = \left\{x_1, x_2;\, x_1 = -2,-1,0,1,2,\ x_2 = -1,-0.5,0.5,1\right\},\ E(\mathbf{e}) = 0,\ \mathrm{Var}(e) = \sigma_e^2$$

$$\mathbf{x}_i = \left(1, x_1, x_2, x_1 x_2, x_1^2, x_2^2\right)$$

The initial and complement quadratic order effect extended design matrices for $2-$factor, $10-$, and $11-$point designs are given respectively as follows.

$$X_{10}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 & 4 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & -1 & -0.5 & 0.5 & 1 & 0.25 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -0.5 & 0 & 0 & 0.25 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -0.5 & -0.5 & 1 & 0.25 \\ 1 & 1 & 0.5 & 0.5 & 1 & 0.25 \\ 1 & 2 & -1 & -2 & 4 & 1 \\ 1 & 2 & 0.5 & 1 & 4 & 0.25 \end{pmatrix},\quad X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -0.5 & 2 & 4 & 0.25 \\ 1 & -2 & 0.5 & -1 & 4 & 0.25 \\ 1 & -2 & 1 & -2 & 4 & 1 \\ 1 & -1 & 0.5 & -0.5 & 1 & 0.25 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -0.5 & -1 & 1 & 0.25 \\ 1 & 2 & 1 & 2 & 4 & 1 \end{pmatrix}$$

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$$X_{11}^{(1)} = \begin{pmatrix} 1 & -2 & -1 & 2 & 4 & 1 \\ 1 & -2 & -0.5 & 1 & 4 & 0.25 \\ 1 & -1 & -0.5 & 0.5 & 1 & 0.25 \\ 1 & -1 & 0.5 & -0.5 & 1 & 0.25 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & 0.5 & 0 & 0 & 0.25 \\ 1 & 1 & -0.5 & -0.5 & 1 & 0.25 \\ 1 & 1 & 0.5 & 0.5 & 1 & 0.25 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 & 4 & 1 \end{pmatrix}, \quad X_{11}^{(1)c} = \begin{pmatrix} 1 & -2 & 0.5 & -1 & 4 & 0.25 \\ 1 & -2 & 1 & -2 & 4 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & -0.5 & 0 & 0 & 0.25 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 2 & -0.5 & -1 & 4 & 0.25 \\ 1 & 2 & 0.5 & 1 & 4 & 0.25 \\ 1 & 2 & 1 & 2 & 4 & 0.25 \end{pmatrix}$$

3.6 Quadratic, Three-Factor Response Function

The quadratic statistical model, eqn. (2), for the $3-$factor response function is

$$E\left[f(\mathbf{x}) \right] = a_0 + \sum_{i=1}^{3} a_i x_i + \sum_{i=1}^{3} a_{ii} x_i^2 + \sum_{i=1}^{2} \sum_{j=i+1}^{3} a_{ij} x_i x_j$$

and the experimental set is

$$\tilde{X} = \left\{ x_1, x_2, x_3 : x_1 = -2, -1, 0, 1, 2;\ x_2 = -1, 0, 1; x_3 = 1, 1 \right\},\ E(\mathbf{e}) = 0 \text{ and } \operatorname{Var}(e) = \sigma_e^2$$

$$\mathbf{x}_i = (1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2, x_1 x_2, x_1 x_3, x_2 x_3)$$

The initial and complement quadratic order effect extended design matrices for $3-$factor, $10-$, and $11-$point designs are given respectively as follows.

$$X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & -1 & 4 & 1 & 1 & 2 & 2 & 1 \\ 1 & -2 & 0 & 1 & 4 & 0 & 1 & 0 & -2 & 0 \\ 1 & -2 & 1 & 1 & 4 & 1 & 1 & -2 & -2 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 2 & 0 & 1 & 4 & 0 & 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & -1 & 4 & 1 & 1 & 2 & -2 & -1 \end{pmatrix}, \quad X_{10}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & 1 & 4 & 1 & 1 & 2 & -2 & -1 \\ 1 & -2 & 0 & -1 & 4 & 0 & 1 & 0 & 2 & 0 \\ 1 & -2 & 1 & -1 & 4 & 1 & 1 & -2 & 2 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -1 & 4 & 1 & 1 & -2 & -2 & 1 \\ 1 & 2 & -1 & 1 & 4 & 1 & 1 & -2 & 2 & -1 \\ 1 & 2 & 0 & -1 & 4 & 0 & 1 & 0 & -2 & 0 \\ 1 & 2 & 1 & 1 & 4 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}$$

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$$X_{11}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & -1 & 4 & 1 & 1 & 2 & 2 & 1 \\ 1 & -2 & 0 & 1 & 4 & 0 & 1 & 0 & -2 & 0 \\ 1 & -2 & 1 & -1 & 4 & 1 & 1 & -2 & 2 & -1 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 2 & -1 & 1 & 4 & 1 & 1 & -2 & 2 & -1 \\ 1 & 2 & 1 & -1 & 4 & 1 & 1 & 2 & -2 & -1 \end{pmatrix}, \quad X_{11}^{(1)c} = \begin{pmatrix} 1 & -2 & -1 & 1 & 4 & 1 & 1 & 2 & -2 & -1 \\ 1 & -2 & 0 & -1 & 4 & 0 & 1 & 0 & 2 & 0 \\ 1 & -2 & 1 & 1 & 4 & 1 & 1 & -2 & -2 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & -1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -1 & 4 & 1 & 1 & -2 & -2 & 1 \\ 1 & 2 & 0 & -1 & 4 & 0 & 1 & 0 & -2 & 0 \\ 1 & 2 & 0 & 1 & 4 & 0 & 1 & 0 & 2 & 0 \\ 1 & 2 & 1 & 1 & 4 & 1 & 1 & 2 & 2 & 1 \end{pmatrix}$$

The pattern of convergence by determinants for the quadratic order effect designs is summarized in Table 3.

Table 3: Pattern of convergence for quadratic order designs

$\xi_N^{(k)}$	$\left\|M\left(\xi_{10}^{(k)}\right)\right\|$		$\left\|M\left(\xi_{11}^{(k)}\right)\right\|$	
	2-factor	3- factor	2-factor	3- factor
1	53230.5	0	254229.8	0
2	441414.6		643248.0	
3	910818.0		1472981.0	
4	1329552.0		1827636.0	
5	1521792.0		2033151.0	
6	1475712.0		1992587.0	

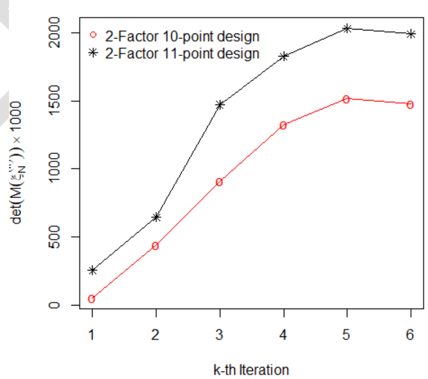

Fig 3 Pattern of convergence for quadratic effect, two-factor designs

3.7 Discussion

The Authors considered appraising the pattern of convergence due to cycling in linear, mixed and quadratic response polynomials. Generated data using two and three variables for each of the $m-$degree

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response functions were employed. Computations and graphs were conducted in R version 4.1.1 (R Core Team (2021)). The results of the analysis are summarized in Tables and graphs. The summaries of the pattern of convergence for degree 1 polynomials are presented in Table 1 and Figures 1(a) and 1(b); Tables 2 and Figures 2(a) & 2(b) present summaries for mixed polynomials while Table 3 gives the summaries for the quadratic portions in degree 2 respectively. The results show that

(i) Determinants increase monotonically to certain points and are seen to either becoming steady or reverting to specific earlier values for both the linear and mixed models except for the quadratic model.
(ii) Determinants for quadratic model for the $2-$factor situation reverts at certain points in the process to non-specific earlier values while those for the starting designs for the $3-$factors are all zero.

These results agree reasonably well with [6], when they stated respectively that a variance exchange process may have good starting designs but may not guarantee designs having maximum determinants, and [16] and [17], that starting designs may sometimes be singular.

4 CONCLUSION AND FUTURE SCOPE

This paper critically analyzed the convergent patterns of one- and two-degree response polynomials in a variance exchange process and comes to the conclusion that

(i) The first degree and the mixed polynomials follow a regular pattern of determinants cycling about specific values giving a different picture of irregular patterns with the second order degree, cycling about non-specific values.
(ii) The initial designs for all quadratic components of the second order polynomials have singular information matrices and cannot guarantee a variance exchange process.
(iii) The effect of cycling on the pattern of convergence for two- and three-degree polynomials therefore differs with respect to the number of variables of the designs involved.

The singular information matrices prevented the exchange process in the quadratic order effect for the two factor designs and therefore, emphasis on the research direction should be focused in future on a way to address such singular information matrices for the exchange process to be possible.

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