COMMON FIXED POINT THEOREMS FOR $(\mathfrak{F}, \phi)$ - CONTRACTIVE MAPPINGS ON C*-ALGEBRA VALUED $\mathfrak{B}$-METRIC SPACES

Original Research Article

Abstract

The this paper, we extends the new concept of common fixed point theorems in C^*-algebra valued $\mathfrak{b}$-metric space $(AVBMS)$ via $(\mathfrak{F}, \phi)$ - contractive mappings. Investigated are the common fixed points criteria for existence and uniqueness. Additionally, provide an illustrate an example.

Keywords: C^*-algebra valued, $\mathfrak{b}$-metric spaces, common fixed point.

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1 Introduction

In 2014, Ma et al. [?] proposed the idea of C^*-algebra valued metric space $(AVMS)$ and investigated certain fixed point theorems for self-mapping under various contractive circumstances. Furthermore, the concept of C^*-$(AVMS)$ is generalized to that of C^*-$(AVBMS)$, where $\mathfrak{B}$ is an element of C^*-algebra greater than 1 and the triangle inequality is altered into $D_b(\varsigma, \mathfrak{y}) \leq \tau(D_b(\varsigma, \mathfrak{z}) + D_b(\mathfrak{z}, \mathfrak{y}))$. Theorems for self-map with contractive condition are then established using various fixed point theorems [?]. Besides, Alsulami et al. [?] the classic Banach fixed point theorems have been examined for their fixed point outcomes can be used to produce C^*-$(AVMS)$, C^*-$(AVBMS)$ in fixed point theory. [?][?]. During this article, we indicate $\mathfrak{A}$ as an unitary C^*-algebra and $\mathfrak{A}_h = \{\mathfrak{a} \in \mathfrak{A} : \mathfrak{a} = \mathfrak{a}^*\}$. Especially, an aspect $\mathfrak{a} \in \mathfrak{A}$ is a positive factor, if $\mathfrak{a} = \mathfrak{a}^*$ and $\sigma(\mathfrak{a}) \subseteq \mathcal{R}^+$, where $\sigma(\mathfrak{a})$ is the spectrum of $\mathfrak{a}$.There is a natural partial order in $\mathfrak{A}_h$ placed by $\mathfrak{a} \leq \mathfrak{b}$ iff $\theta \leq (\mathfrak{b} - \mathfrak{a})$, where θ implies the zero factor in $\mathfrak{A}$. Then, let $\mathfrak{A}^+$ and $\mathfrak{A}'$ represent the set $\{\mathfrak{a} \in \mathfrak{A} : \theta \leq \mathfrak{a}\}$ and the set $\{\mathfrak{a} \in \mathfrak{A} : \mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}, \ \forall \ \mathfrak{b} \in \mathfrak{A}\}$, respectively and $|\mathfrak{a}| = (\mathfrak{a}^*\mathfrak{a})^{\frac{1}{2}}$.

Definition 1.1. [?] Let χ be a non-empty set and $\tau \in \mathfrak{A}$ such that $\tau \geq I$. Suppose that the mapping $D_b : \chi \times \chi \to \mathfrak{A}$ is held, the following constraints.

(i) $\theta \leq D_b(\varsigma, \mathfrak{y})$ and $D_b(\varsigma, \mathfrak{y}) = \theta$ iff $(\varsigma = \mathfrak{y})$;

(ii) $D_b(\varsigma, \mathfrak{y}) = D_b(\varsigma, \mathfrak{y})$;

(iii) $D_b(\varsigma, \mathfrak{y}) \leq \tau(D_b(\varsigma, \mathfrak{z}) + D_b(\mathfrak{z}, \mathfrak{y}))\ \forall\ \varsigma, \mathfrak{y}, \mathfrak{z} \in \chi$.

Then, D_b is called C^*-$(AVBM)$ on χ and $(\chi, \mathfrak{A}, D_b)$ is called C^*-$(AVBMS)$.

Definition 1.2. [?]. Let $(\chi, \mathfrak{A}, D_b)$ be C^*-$(AVBMS)$. Imagine that $\{\varsigma_n\}$ is an order in χ and $\varsigma \in \chi$. If at all $\epsilon > \theta$, $\exists\ \mathcal{N} \ni \forall\ n > \mathcal{N}$, $||D_b(\varsigma_n, \mathfrak{x})|| \leq \epsilon$ then $\{\varsigma_n\}$ is allegedly convergent with respect to $\mathfrak{A}$, and $\{\varsigma_n\}$ converges to ς, i.e., we use $\lim_{n\to\infty} \varsigma_n = \varsigma$. If at all $\epsilon > \theta$, $\exists\ \mathcal{N} \ni \forall\ n, m > \mathcal{N}$, $||D_b(\varsigma_n, \varsigma_m)|| \leq \epsilon$, then $\{\varsigma_n\}$ is referred to as a Cauchy sequence in χ. $(\chi, \mathfrak{A}, D_b)$ is referred to as a complete C^*-$(AVBMS)$ if each Cauchy sequence is convergent in χ.

Remark 1.1. [?] Let $\mathfrak{A}$ be a C^*-algebra and assume that ϕ is a linear functional on $\mathfrak{A}$. Define $\phi^*(\mathfrak{a}) = \phi(\mathfrak{a}^*)\ \forall\ \mathfrak{a} \in \mathfrak{A}$. Then, ϕ^* likewise has a linear function $\mathfrak{A}$. And the function ϕ is known as a self-adjoint if $\phi^* = \phi$.

A linear function ϕ on $\mathfrak{A}$ is called positive if $\phi(\mathfrak{a}^*\mathfrak{a}) \geq 0$ for all $\mathfrak{a} \in \mathfrak{A}$. We indicate the positivity of ϕ by $\phi \geq \theta$. For two self-adjoint linear function ϕ_1, ϕ_2, There are $(\phi_2 - \phi_1) \geq \theta$ when $\phi_2 \geq \phi_1$.

Definition 1.3. [?] If $\phi : \mathfrak{A} \to \mathcal{B}$ is a linear mapping in C^*-algebra, It is said to be positive if $\phi(\mathfrak{A}^+) \subseteq \phi(\mathcal{B}^+)$. When this occurs, $\phi(\mathfrak{A}_h) \subseteq \phi(\mathcal{B}_h)$, and the map of restriction $\phi : \mathfrak{A}_h \to \mathcal{B}_h$ is increasing. if $\mathcal{B} = \mathfrak{A}$ the positive linear map is thus referred to as positive functional, and it meets the following propositions **??** and **??**.

Proposition 1.1. *[?] Let $\mathfrak{A}$ be a C^*-algebra with* 1, *thereafter, a positive functional is bounded and* $\phi(1) = \| \phi \|$.

Proposition 1.2. *[?] Let $\mathfrak{A}$ be a C^*-algebra with* 1 *and let ϕ be a bounded linear functional on $\mathfrak{A}$* $\ni \phi(\mathfrak{a}) = \| \phi \| \| \mathfrak{a} \|$. *There exists positive component $\mathfrak{a} \in \mathfrak{A} \ni \phi$ is a positive functional.*

Definition 1.4. [?] Let the non-decreasing function $\mathfrak{F} : \mathfrak{A}^+ \to \mathfrak{A}^+$ be a positive linear map that complies with the following restrictions:

(i) $\mathfrak{F}$ is continuous;

(ii) $\mathfrak{F}(\mathfrak{a}) = \theta$ iff $\mathfrak{a} = \theta$;

(iii) $\lim_{n\to\infty} \mathfrak{F}^n(\mathfrak{a}) = \theta$.

Definition 1.5. [?] Suppose that $\mathfrak{A}$ and $\mathcal{B}$ are C^*-algebra. $\mathfrak{A}$ mapping $\mathfrak{F} : \mathfrak{A} \to \mathcal{B}$ is purported to be C^*-homomorphism if :

(i) $\mathfrak{F}(\mathfrak{a}\varsigma + \mathfrak{b}\mathfrak{y}) = \mathfrak{a}\mathfrak{F}(\varsigma) + \mathfrak{b}\mathfrak{F}(\mathfrak{y})\ \forall\ \mathfrak{a}, \mathfrak{b} \in C$ and $\varsigma, \mathfrak{y} \in \mathfrak{A}$;

(ii) $\mathfrak{F}(\varsigma, \mathfrak{y}) = \mathfrak{F}(\varsigma)\mathfrak{F}(\mathfrak{y})\ \forall\ \varsigma, \mathfrak{y} \in \mathfrak{A}$;

(iii) $\mathfrak{F}(\varsigma^*) = \mathfrak{F}(\varsigma)^*\ \forall\ \varsigma \in \mathfrak{A}$;

(iv) $\mathfrak{F}$ maps the unit in $\mathfrak{A}$ to the unit in $\mathcal{B}$.

Definition 1.6. [?] Let $\mathfrak{A}$ and $\mathcal{B}$ be a C^*-algebra space and let $\mathfrak{F} : \mathfrak{A} \to \mathcal{B}$ be a homomorphism, then $\mathfrak{F}$ is called an $*$-isomorphism if it is one-to-one $*$-isomorphism. We declare that C^*-algebra $\mathfrak{A}$ is $*$-isomorphism to a C^*-algebra $\mathcal{B}$ if $\exists$ $*$-isomorphism of $\mathfrak{A}$ onto B.

property 1.1. *[?] Let $\mathfrak{A}$ and B be C^*-algebra space and $\mathfrak{F} : \mathfrak{A} \to \mathcal{B}$ is a C^*-homomorphism $\forall\ \varsigma \in \mathfrak{A}$, there are $\sigma(\mathfrak{F}(\varsigma)) \subset \sigma(\varsigma$ and $\| \mathfrak{F}(\varsigma) \| \leq \| \mathfrak{F} \|$.*

Corollary 1.1. *[?] Every C^*-homomorphism is bounded.*

Corollary 1.2. *[?] Suppose that $\mathfrak{F}$ is C^*-isomorphism from $\mathfrak{A}$ to $\mathcal{B}$, then $\sigma(\mathfrak{F}(\varsigma)) \subset \sigma(\varsigma)$ and $\| \mathfrak{F}(\varsigma) \| \leq \| \varsigma \|\ \forall\ \varsigma \in \mathfrak{A}$.*

Lemma 1.3. *[?] Every $*$-homomorphism is positive.*

2 Main Results

Theorem 2.1. *Let* $(\chi, \mathfrak{A}, D_b)$ *is a complete* C^*-$(AVBMS)$. *Let* $L, M : \chi \to \chi$ *be a contractive mapping and*

$$\mathfrak{F}\left(D_b\left(L\varsigma, M\mathfrak{y}\right)\right) \leq \mathfrak{F}\left(N\left(\varsigma, \mathfrak{y}\right)\right) - \phi\left(D_b\left(\varsigma, \mathfrak{y}\right)\right) \tag{2.1}$$

$$N\left(\varsigma, \mathfrak{y}\right) \leq \begin{bmatrix} \alpha D_b\left(\varsigma, \mathfrak{y}\right) + \frac{\beta[1+D_b(\varsigma, L\varsigma)]D_b(\mathfrak{y}, M\mathfrak{y})}{1+D_b(\varsigma, \mathfrak{y})} \\ +\gamma\left[D_b\left(\varsigma, L\varsigma\right) + D_b\left(\mathfrak{y}, M\mathfrak{y}\right)\right] + \delta\left[D_b\left(\varsigma, M\mathfrak{y}\right) + D_b\left(\mathfrak{y}, L\varsigma\right)\right] \end{bmatrix}$$

For all $\varsigma, \mathfrak{y} \in \chi$, *where* $\tau \in \mathfrak{A}_+^{'}$, $\alpha+\beta+\gamma+\delta \geq 0$ *with* $\tau\alpha+\beta+\gamma(\tau+1)+\delta\left(\tau(\tau+1)\right) < 1$. $\mathfrak{F}$ *and* ϕ *are* $*$*-homomorphisms and the constraint* $\mathfrak{F}\left(\mathfrak{a}\right) \leq \phi(\mathfrak{a})$. *Then* L *and* M *have a unique common fixed point in* χ.

Proof. Let $\varsigma_0 \in \chi$ and define $\varsigma_n = L\varsigma_{n-1},\ \varsigma_{n+1} = M\varsigma_n$ we have

$$\begin{aligned} \mathfrak{F}\left(D_b\left(\varsigma_n, \varsigma_{n+1}\right)\right) &= \mathfrak{F}\left(D_b\left(L\varsigma_{n-1}, M\varsigma_n\right)\right) \\ &\leq \mathfrak{F}\left(N\left(\varsigma_{n-1}, \varsigma_n\right)\right) - \phi\left(D_b\left(\varsigma_{n-1}, \varsigma_n\right)\right) \\ &= \mathfrak{F}\begin{pmatrix} \alpha D_b\left(\varsigma_{n-1}, \varsigma_n\right) + \frac{\beta\left[1+D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right)\right]D_b(\varsigma_n, M\varsigma_n)}{1+D_b\left(\varsigma_{n-1}, \varsigma_n\right)} \\ +\gamma\left[D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right) + D_b\left(\varsigma_n,\ M\varsigma_n\right)\right] \\ +\delta\left[D_b\left(\varsigma_{n-1}, M\varsigma_n\right) + D_b\left(\varsigma_n, L\varsigma_{n-1}\right)\right] \end{pmatrix} - \phi\left(D_b\left(\varsigma_{n-1}, \varsigma_n\right)\right) \\ &= \mathfrak{F}(\alpha)\mathfrak{F}\left(D_b\left(\varsigma_{n-1},\ \varsigma_n\right)\right) + \frac{\mathfrak{F}(\beta)\mathfrak{F}\left[1+D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right)\right]D_b(\varsigma_n M\varsigma_n)}{1+D_b\left(\varsigma_{n-1}, \varsigma_n\right)} \\ &+\mathfrak{F}(\gamma)\mathfrak{F}\left[D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right) + D_b\left(\varsigma_n, M\varsigma_n\right)\right] \\ &+\mathfrak{F}(\delta)\mathfrak{F}\left[D_b\left(\varsigma_{n-1}, M\varsigma_n\right) + D_b\left(\varsigma_n, L\varsigma_{n-1}\right)\right] - \phi\left(D_b\left(\varsigma_{n-1}, \varsigma_n\right)\right). \end{aligned}$$

Therefore

$$\begin{aligned} \parallel \mathfrak{F}\left(D_b\left(\varsigma_n, \varsigma_{n+1}\right)\right) \parallel &= \parallel \mathfrak{F}\left(D_b\left(L\varsigma_{n-1}, M\varsigma_n\right)\right) \parallel \\ &\leq \parallel \mathfrak{F}(\alpha) \parallel \parallel \mathfrak{F}\left(D_b\left(\varsigma_{n-1}, \varsigma_n\right)\right) \parallel + \parallel \mathfrak{F}(\beta) \parallel \parallel \frac{\mathfrak{F}\left[1+D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right)\right]D_b(\varsigma_n, M\varsigma_n)}{1+D_b\left(\varsigma_{n-1}, \varsigma_n\right)} \parallel \\ &+ \parallel \mathfrak{F}(\gamma) \parallel \parallel \mathfrak{F}\left[D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right) + D_b\left(\varsigma_n, M\varsigma_n\right)\right] \parallel \\ &+ \parallel \mathfrak{F}(\delta) \parallel \parallel \mathfrak{F}\left[D_b\left(\varsigma_{n-1}, M\varsigma_n\right) + D_b\left(\varsigma_n, L\varsigma_{n-1}\right)\right] \parallel \\ &- \parallel \phi\left(D_b\left(\varsigma_{n-1}, \varsigma_n\right)\right) \parallel \to 0 \text{ as } n \to +\infty. \end{aligned}$$

Consider the fact that ϕ and $\mathfrak{F}$ are strongly monotone functions. There are

$$\begin{aligned} D_b\left(\varsigma_n, \varsigma_{n+1}\right) &= D_b(L\varsigma_{n-1}, M\varsigma_n) \\ &\leq \alpha D_b\left(\varsigma_{n-1}, \varsigma_n\right) + \frac{\beta\left[1+D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right)\right]D_b(\varsigma_n, M\varsigma_n)}{1+D_b\left(\varsigma_{n-1}, \varsigma_n\right)} \\ &+\gamma\left[D_b\left(\varsigma_{n-1}, L\varsigma_{n-1}\right) + D_b\left(\varsigma_n, M\varsigma_n\right)\right] + \delta[D_b\left(\varsigma_{n-1}, M\varsigma_n\right) + D_b(\varsigma_n, L\varsigma_{n-1})] \\ &= \alpha D_b\left(\varsigma_{n-1}, \varsigma_n\right) + \frac{\beta\left[1+D_b\left(\varsigma_{n-1}, \varsigma_n\right)\right]D_b\left(\varsigma_n, \varsigma_{n+1}\right)}{1+D_b\left(\varsigma_{n-1}, \varsigma_n\right)} \\ &+\gamma\left[D_b\left(\varsigma_{n-1}, \varsigma_n\right) + D_b\left(\varsigma_n, \varsigma_{n+1}\right)\right] + \delta[D_b\left(\varsigma_{n-1}, \varsigma_{n+1}\right) + D_b(\varsigma_n, \varsigma_n)] \\ &= (\alpha+\gamma)D_b\left(\varsigma_{n-1},\ \varsigma_n\right) + (\beta+\gamma)D_b\left(\varsigma_n, \varsigma_{n+1}\right) + \delta\left(D_b\left(\varsigma_{n-1}, \varsigma_{n+1}\right)\right) \\ &\leq (\alpha+\gamma)D_b\left(\varsigma_{n-1},\ \varsigma_n\right) + (\beta+\gamma)D_b\left(\varsigma_n, \varsigma_{n+1}\right) \\ &+\tau\delta\left(D_b\left(\varsigma_{n-1}, \varsigma_n\right) + D_b\left(\varsigma_n, \varsigma_{n+1}\right)\right) \\ &= (\alpha+\gamma+\tau\delta\ )D_b\left(\varsigma_{n-1},\ \varsigma_n\right) + (\beta+\gamma+\tau\delta)D_b\left(\varsigma_n, \varsigma_{n+1}\right). \end{aligned}$$

This indicates that

$$D_b\left(\varsigma_n, \varsigma_{n+1}\right) \leq \frac{\alpha+\gamma+\tau\delta}{\beta+\gamma+\tau\delta}D_b\left(\varsigma_n, \varsigma_{n-1}\right) \tag{2.2}$$

$$D_b\left(\varsigma_n, \varsigma_{n+1}\right) \leq hD_b\left(\varsigma_n, \varsigma_{n-1}\right)$$

where,

$$h = \frac{\alpha+\gamma+\tau\delta}{\beta+\gamma+\tau\delta} < 1.$$

As a result,

$$\| D_b(\varsigma_{n-1},\varsigma_n) \| \, \| D_b(\varsigma_n,\varsigma_{n+1}) \| \leq \| h \| \, \| D_b(\varsigma_n,\varsigma_{n-1}) \| \to 0, \ as\ n,m \to +\infty.$$

Let $n > m$

$$D_b(\varsigma_n,\varsigma_m) \leq \tau D_b(\varsigma_n,\varsigma_{n-1}) + \tau^2 D_b(\varsigma_{n-1},\varsigma_{n-2}) + \ldots + \tau^{n-m} D_b(\varsigma_{m-1},\varsigma_m)$$

When using the theorem's constraint,

$$\begin{aligned}
\mathfrak{F}(D_b(\varsigma_n,\varsigma_m)) \quad & \leq \mathfrak{F}(\tau D_b(\varsigma_n,\varsigma_{n-1})) + \mathfrak{F}\left(\tau^2 D_b(\varsigma_{n-1},\varsigma_{n-2})\right) + \ldots + \mathfrak{F}\left(\tau^{n-m} D_b(\varsigma_{m-1},\varsigma_m)\right) \\
& \leq \mathfrak{F}(\tau)\mathfrak{F}(D_b(\varsigma_n,\varsigma_{n-1})) + \mathfrak{F}\left(\tau^2\right)\mathfrak{F}(D_b(\varsigma_{n-1},\varsigma_{n-2})) + \ldots \\
& + \mathfrak{F}\left(\tau^{n-m}\right)\mathfrak{F}(D_b(\varsigma_{m-1},\varsigma_m)) \\
& \leq \mathfrak{F}(\tau N(\varsigma_n,\varsigma_{n-1})) - \phi(\tau D_b(\varsigma_n,\varsigma_{n-1})) + \mathfrak{F}\left(\tau^2 N(\varsigma_{n-1},\varsigma_{n-2})\right) \\
& - \phi\left(\tau^2 D_b(\varsigma_{n-1},\varsigma_{n-2})\right) + \ldots + \mathfrak{F}\left(\tau^{n-m} N(\varsigma_{m-1},\varsigma_m)\right) \\
& - \phi\left(\tau^{n-m} D_b(\varsigma_{m-1},\varsigma_m)\right) \\
& = \mathfrak{F}\begin{pmatrix} \alpha D_b(\varsigma_n,\varsigma_{n-1}) + \frac{\beta\left[1+D_b(\varsigma_{n-1},\varsigma_n)\right] D_b(\varsigma_n,\varsigma_{n+1})}{1+D_b(\varsigma_{n-1},\varsigma_n)} \\ +\gamma\tau\left[D_b(\varsigma_{n-1},\varsigma_n) + D_b(\varsigma_n,\varsigma_{n+1})\right] \\ +\delta\tau\left[D_b(\varsigma_{n-1},\varsigma_{n+1}) + D_b(\varsigma_n,\varsigma_n)\right] \end{pmatrix} - \phi(\tau D_b(\varsigma_n,\varsigma_{n-1})) + \ldots \\
& + \mathfrak{F}\begin{pmatrix} \alpha\tau^{n-m} D_b(\varsigma_{m-1},\varsigma_m) + \frac{\beta\tau^{n-m}\left[1+D_b(\varsigma_m,\varsigma_{m-1})\right] D_b(\varsigma_{m-1},\varsigma_{m-2})}{1+D_b(\varsigma_m,\varsigma_{m-1})} \\ +\gamma\tau^{n-m}\left[D_b(\varsigma_m,\varsigma_{m-1}) + D_b(\varsigma_{m-1},\varsigma_{m-2})\right] \\ +\delta\tau^{n-m}\left[D_b(\varsigma_m,\varsigma_{m-2}) + D_b(\varsigma_{m-1},\varsigma_{m-1})\right] \end{pmatrix} \\
& - \phi\left(\tau^{n-m} D_b(\varsigma_{m-1},\varsigma_m)\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathfrak{F}(D_b(\varsigma_n,\varsigma_m)) \quad = & \begin{pmatrix} \mathfrak{F}(\alpha)\mathfrak{F}(\tau)\mathfrak{F}(D_b(\varsigma_n,\varsigma_{n-1})) \\ +\mathfrak{F}(\beta)\mathfrak{F}(\tau)\mathfrak{F}\left(\frac{\left[1+D_b(\varsigma_{n-1},\varsigma_n)\right] D_b(\varsigma_n,\varsigma_{n+1})}{1+D_b(\varsigma_{n-1},\varsigma_n)}\right) \\ +\mathfrak{F}(\gamma)\mathfrak{F}(\tau)\mathfrak{F}\left[D_b(\varsigma_{n-1},\varsigma_n) + D_b(\varsigma_n,\varsigma_{n+1})\right] \\ +\mathfrak{F}(\delta)\mathfrak{F}(\tau)\mathfrak{F}\left[D_b(\varsigma_{n-1},\varsigma_{n+1}) + D_b(\varsigma_n,\varsigma_n)\right] \end{pmatrix} - \phi(\tau D_b(\varsigma_n,\varsigma_{n-1})) + \ldots \\
+ & \begin{pmatrix} \mathfrak{F}(\alpha)\mathfrak{F}\left(\tau^{n-m}\right)\mathfrak{F}(D_b(\varsigma_{m-1},\varsigma_m)) \\ +\mathfrak{F}(\beta)\mathfrak{F}\left(\tau^{n-m}\right)\mathfrak{F}\left(\frac{\left[1+D_b(\varsigma_m,\varsigma_{m-1})\right] D_b(\varsigma_{m-1},\varsigma_{m-2})}{1+D_b(\varsigma_m,\varsigma_{m-1})}\right) \\ +\mathfrak{F}(\gamma)\mathfrak{F}\left(\tau^{n-m}\right)\mathfrak{F}\left[D_b(\varsigma_m,\varsigma_{m-1}) + D_b(\varsigma_{m-1},\varsigma_{m-2})\right] \\ +\mathfrak{F}(\delta)\mathfrak{F}\left(\tau^{n-m}\right)\mathfrak{F}\left[D_b(\varsigma_m,\varsigma_{m-2}) + D_b(\varsigma_{m-1},\varsigma_{m-1})\right] \end{pmatrix} - \phi\left(\tau^{n-m} D_b(\varsigma_{m-1},\varsigma_m)\right).
\end{aligned}$$

Consider the fact that ϕ and $\mathfrak{F}$ are strongly monotone functions. There are

$$\begin{aligned}
D_b(\varsigma_n,\varsigma_m) \quad & \leq \alpha\tau D_b(\varsigma_n,\varsigma_{n-1}) + \frac{\beta\left[1+D_b(\varsigma_{n-1},\varsigma_n)\right] D_b(\varsigma_n,\varsigma_{n+1})}{1+D_b(\varsigma_{n-1},\varsigma_n)} \\
& + \gamma\tau\left[D_b(\varsigma_{n-1},\varsigma_n) + D_b(\varsigma_n,\varsigma_{n+1})\right] + \delta\tau\left[D_b(\varsigma_{n-1},\varsigma_{n+1}) + D_b(\varsigma_n,\varsigma_n)\right] + \ldots \\
& + \alpha\tau^{n-m} D_b(\varsigma_{m-1},\varsigma_m) + \frac{\beta\tau^{n-m}\left[1+D_b(\varsigma_m,\varsigma_{m-1})\right] D_b(\varsigma_{m-1},\varsigma_{m-2})}{1+D_b(\varsigma_m,\varsigma_{m-1})} \\
& + \gamma\tau^{n-m}\left[D_b(\varsigma_m,\varsigma_{m-1}) + D_b(\varsigma_{m-1},\varsigma_{m-2})\right] \\
& + \delta\tau^{n-m}\left[D_b(\varsigma_m,\varsigma_{m-2}) + D_b(\varsigma_{m-1},\varsigma_{m-1})\right].
\end{aligned}$$

So,

$$D_b(\varsigma_n,\varsigma_m) \leq \begin{pmatrix} \|\alpha\|\|\tau\|\|D_b(\varsigma_n,\varsigma_{n-1})\| \\ +\|\beta\|\|\tau\|\|\frac{[1+D_b(\varsigma_{n-1},\varsigma_n)]D_b(\varsigma_n,\varsigma_{n+1})}{1+D_b(\varsigma_{n-1},\varsigma_n)}\| \\ +\|\gamma\|\|\tau\|\|D_b(\varsigma_{n-1},\varsigma_n)+D_b(\varsigma_n,\varsigma_{n+1})\| \\ +\|\delta\|\|\tau\|\|D_b(\varsigma_{n-1},\varsigma_{n+1})+D_b(\varsigma_n,\varsigma_n)\|+\ \dots \\ +\|\alpha\|\|\tau^{\ n-m}\|\|D_b(\varsigma_{m-1},\varsigma_m)\| \\ +\|\beta\|\|\tau^{n-m}\|\|\frac{[1+D_b(\varsigma_m,\varsigma_{m-1})]D_b(\varsigma_{m-1},\varsigma_{m-2})}{1+D_b(\varsigma_m,\varsigma_{m-1})}\| \\ +\|\gamma\|\|\tau^{n-m}\|\|D_b(\varsigma_m,\varsigma_{m-1})+D_b(\varsigma_{m-1},\varsigma_{m-2})\| \\ +\|\delta\|\|\tau^{n-m}\|\|D_b(\varsigma_m,\varsigma_{m-2})+D_b(\varsigma_{m-1},\varsigma_{m-1})\| \end{pmatrix} \to 0,\ as\ n.m\to+\infty.$$

Then $\{\varsigma_n\}$ is Cauchy sequence. Since $(\chi,\mathfrak{A},D_b)$ is a complete C^*-$(AVBMS)$ $\exists\ \mathfrak{u}\in\chi\ni\varsigma_n\to\mathfrak{u}$ as $n\to\infty$. Now that

$$\begin{aligned}
D_b(\mathfrak{u},M\mathfrak{u}) &\leq \tau[D_b(\mathfrak{u},\varsigma_{n+1})+D_b(\varsigma_{n+1},M\mathfrak{u})] \\
&= \tau[D_b(\varsigma_{n+1},M\mathfrak{u})+D_b(\mathfrak{u},\varsigma_{n+1})] \\
&= \tau[D_b(L\varsigma_n,M\mathfrak{u})+D_b(\mathfrak{u},\varsigma_{n+1})] \\
\mathfrak{F}(D_b(\mathfrak{u},M\mathfrak{u})) &= \tau[\mathfrak{F}(D_b(L\varsigma_n,M\mathfrak{u}))+\mathfrak{F}(D_b(\mathfrak{u},\varsigma_{n+1}))] \\
&\leq \tau[\mathfrak{F}(N(\varsigma_n,\mathfrak{u}))-\phi D_b(\varsigma_n,\mathfrak{u})]+\tau[\mathfrak{F}D_b(\mathfrak{u},\varsigma_{n+1})] \\
\|\mathfrak{F}(D_b(L\varsigma_n,M\mathfrak{u}))\| &\leq \|\tau\|\|\mathfrak{F}(D_b(\mathfrak{u},\varsigma_{n+1}))\|+\|\tau\|\|\mathfrak{F}\alpha\|\|D_b(\varsigma_n,\mathfrak{u})\| \\
&+\|\tau\|\|\mathfrak{F}\beta\|\|\frac{[1+D_b(\varsigma_n,L\varsigma_n)]D_b(\mathfrak{u},M\mathfrak{u})}{1+D_b(\varsigma_n,\mathfrak{u})}\| \\
&+\|\tau\|\|\mathfrak{F}\gamma\|\|D_b(\varsigma_n,\varsigma_{n+1})+D_b(\mathfrak{u},M\mathfrak{u})\| \\
&+\|\tau\|\|\mathfrak{F}\delta\|\|D_b(\varsigma_n,M\mathfrak{u})+D_b(\mathfrak{u},L\varsigma_n)\|-\|\tau\|\|\phi D_b(\varsigma_n,\mathfrak{u})\|.
\end{aligned}$$

Using the property of ϕ, we have

$$\begin{aligned}
\|\mathfrak{F}(D_b(L\varsigma_n,M\mathfrak{u}))\| &\leq \|\tau\|\|\mathfrak{F}(D_b(\mathfrak{u},\varsigma_{n+1}))\|+\|\tau\|\|\mathfrak{F}\alpha\|\|D_b(\varsigma_n,\mathfrak{u})\| \\
&+\|\tau\|\|\mathfrak{F}\beta\|\|\frac{[1+D_b(\varsigma_n,L\varsigma_n)]D_b(\mathfrak{u},M\mathfrak{u})}{1+D_b(\varsigma_n,\mathfrak{u})}\| \\
&+\|\tau\|\|\mathfrak{F}\gamma\|\|D_b(\varsigma_n,\varsigma_{n+1})+D_b(\mathfrak{u},M\mathfrak{u})\| \\
&+\|\tau\|\|\mathfrak{F}\delta\|\|D_b(\varsigma_n,M\mathfrak{u})+D_b(\mathfrak{u},L\varsigma_n)\|
\end{aligned}$$

where $\mathfrak{F}$ is strongly monotone, then

$$\begin{aligned}
\|(D_b(L\varsigma_n,M\mathfrak{u}))\| &\leq \|\tau\|\|(D_b(\mathfrak{u},\varsigma_{n+1}))\|+\|\tau\|\|\alpha\|\|D_b(\varsigma_n,\mathfrak{u})\| \\
&+\|\tau\|\|\beta\|\|\frac{[1+D_b(\varsigma_n,L\varsigma_n)]D_b(\mathfrak{u},M\mathfrak{u})}{1+D_b(\varsigma_n,\mathfrak{u})}\| \\
&+\|\tau\|\|\gamma\|\|D_b(\varsigma_n,\varsigma_{n+1})+D_b(\mathfrak{u},M\mathfrak{u})\| \\
&+\|\tau\|\|\delta\|\|D_b(\varsigma_n,M\mathfrak{u})+D_b(\mathfrak{u},L\varsigma_n)\| \\
&= \|\tau\|\|(D_b(\mathfrak{u},\varsigma_{n+1}))\| \\
&+\|\tau\|\begin{bmatrix} \|\alpha\|\|D_b(\varsigma_n,\mathfrak{u})\| \\ +\|\beta\|\|\frac{[1+D_b(\varsigma_n,\varsigma_{n+1})]D_b(\mathfrak{u},M\mathfrak{u})}{1+D_b(\varsigma_n,\mathfrak{u})}\| \\ +\|\gamma\|\|D_b(\varsigma_n,\varsigma_{n+1})+D_b(\mathfrak{u},M\mathfrak{u})\| \\ +\|\delta\|\|D_b(\varsigma_n,M\mathfrak{u})+D_b(\mathfrak{u},\varsigma_{n+1})\| \end{bmatrix}
\end{aligned}$$

as $\varsigma_n\to\mathfrak{u}$ and $\varsigma_{n+1}\to\mathfrak{u}$ as $n\to\infty$, we obtain

$$\|1-\tau\beta-\tau\gamma-\tau\delta\|\|D_b(\mathfrak{u},M\mathfrak{u})\|\leq\begin{pmatrix} \|\tau\|\|\alpha\|\|D_b(\varsigma_n,\mathfrak{u})\| \\ +\|\tau\|\|1+\delta\|\|D_b(\mathfrak{u},\varsigma_{n+1})\| \end{pmatrix}\to 0\ as\ n\to\infty.$$

Hence $\|D_b(M\mathfrak{u},\mathfrak{u})\|=0$ since $\|1-\tau\beta-\tau\gamma-\tau\delta\|>0$. As a result, $M\mathfrak{u}=\mathfrak{u}$ which is $\mathfrak{u}$is a fixed point of M. Similarly we can demonstrate this $L\mathfrak{u}=\mathfrak{u}$. Hence $L\mathfrak{u}=M\mathfrak{u}=\mathfrak{u}$. This show that $\mathfrak{u}$ is common fixed point of L and M.

Let $\mathfrak{v}$ be a different fixed point common to L and M. (i.e) $L\mathfrak{v} = M\mathfrak{v} = \mathfrak{v} \ni \mathfrak{u} \neq \mathfrak{v}$ we have $D_b(\mathfrak{u}, \mathfrak{v}) = D_b(L\mathfrak{u}, M\mathfrak{v})$ then,

$$\begin{aligned} \mathfrak{F}(D_b(\mathfrak{u}, \mathfrak{v})) &= \mathfrak{F}(D_b(L\mathfrak{u}, M\mathfrak{v})) \leq \mathfrak{F}(N(\mathfrak{u}, \mathfrak{v})) - \phi(D_b(\mathfrak{u}, \mathfrak{v})) \\ \| \mathfrak{F}(D_b(L\mathfrak{u}, M\mathfrak{v})) \| &\leq \| \mathfrak{F}\alpha \| \| D_b(\mathfrak{u}, \mathfrak{v}) \| + \| \mathfrak{F}\beta \| \| \tfrac{[1+D_b(\mathfrak{u}, L\mathfrak{u})]D_b(\mathfrak{v}, M\mathfrak{v})}{1+D_b(\mathfrak{u}, \mathfrak{v})} \| \\ &+ \| \mathfrak{F}\gamma \| \| D_b(\mathfrak{u}, L\mathfrak{u}) + D_b(\mathfrak{v}, M\mathfrak{v}) \| \\ &+ \| \mathfrak{F}\delta \| \| D_b(\mathfrak{u}, M\mathfrak{v}) + D_b(\mathfrak{v}, L\mathfrak{u}) \| - \| \phi D_b(\mathfrak{u}, \mathfrak{v}) \| \end{aligned}$$

Using the property of ϕ, we get

$$\begin{aligned} \| \mathfrak{F}(D_b(L\mathfrak{u}, M\mathfrak{v})) \| &\leq \| \mathfrak{F}\alpha \| \| D_b(\mathfrak{u}, \mathfrak{v}) \| + \| \mathfrak{F}\beta \| \| \tfrac{[1+D_b(\mathfrak{u}, L\mathfrak{u})]D_b(\mathfrak{v}, M\mathfrak{v})}{1+D_b(\mathfrak{u}, \mathfrak{v})} \| \\ &+ \| \mathfrak{F}\gamma \| \| D_b(\mathfrak{u}, L\mathfrak{u}) + D_b(\mathfrak{v}, M\mathfrak{v}) \| \\ &+ \| \mathfrak{F}\delta \| \| D_b(\mathfrak{u}, M\mathfrak{v}) + D_b(\mathfrak{v}, L\mathfrak{u}) \| \end{aligned}$$

Where $\mathfrak{F}$ is strongly monotone, then

$$\begin{aligned} \| (D_b(L\mathfrak{u}, M\mathfrak{v})) \| &\leq \| \alpha \| \| D_b(\mathfrak{u}, \mathfrak{v}) \| + \| \beta \| \| \tfrac{[1+D_b(\mathfrak{u}, L\mathfrak{u})]D_b(\mathfrak{v}, M\mathfrak{v})}{1+D_b(\mathfrak{u}, \mathfrak{v})} \| \\ &+ \| \gamma \| \| D_b(\mathfrak{u}, L\mathfrak{u}) + D_b(\mathfrak{v}, M\mathfrak{v}) \| \\ &+ \| \delta \| \| D_b(\mathfrak{u}, M\mathfrak{v}) + D_b(\mathfrak{v}, L\mathfrak{u}) \| \\ &= \| \alpha + 2\delta \| \| D_b(\mathfrak{u}, \mathfrak{v}) \| \\ &\leq \| \tau\alpha + \beta + (\tau+1)\gamma + \tau(\tau+1)\delta \| \| D_b(\mathfrak{u}, \mathfrak{v}) \| \\ &< \| D_b(\mathfrak{u}, \mathfrak{v}) \| . \end{aligned}$$

Which is a contradiction. Hence $\| D_b(\mathfrak{u}, \mathfrak{v}) \| = 0$ and $\mathfrak{u} = \mathfrak{v}$. Thus $\mathfrak{u}$ is a unique common fixed point of L and M. □

Corollary 2.2. *Let $(\chi, \mathfrak{A}, D_b)$ is a complete C^*-(AVbMS). Let $M : \chi \to \chi$ be a contractive mapping and*

$$\mathfrak{F}(D_b(M^n\varsigma, M^n\mathfrak{y})) \leq \mathfrak{F}(L(\varsigma, \mathfrak{y})) - \phi(D_b(\varsigma, \mathfrak{y})) \tag{2.3}$$

$$L(\varsigma, \mathfrak{y}) \leq \alpha D_b(\varsigma, \mathfrak{y}) + \gamma[D_b(\varsigma, M^n\mathfrak{y}) + D_b(\mathfrak{y}, M^n\mathfrak{y})] + \delta[D_b(\varsigma, M^n\mathfrak{y}) + D_b(\mathfrak{y}, M^n\varsigma]$$

for all $\varsigma, \mathfrak{y} \in \chi$, where $\tau \in \mathfrak{A}'_+$, $\alpha + \gamma + \delta \geq 0$ with $\tau\alpha + \gamma(\tau+1) + \delta(\tau(\tau+1)) < 1$. $\mathfrak{F}$ and ϕ are $*$-homomorphisms and the constraint $\mathfrak{F}(\mathfrak{a}) \leq \phi(\mathfrak{a})$. Then M have a unique fixed point in X.

Corollary 2.3. *Let $(\chi, \mathfrak{A}, D_b)$ is a complete C^*-(AVbMS). Let $L : \chi \to \chi$ be a contractive mapping and*

$$\mathfrak{F}(D_b(L^n\varsigma, L^n\mathfrak{y})) \leq \mathfrak{F}(N(\varsigma, \mathfrak{y})) - \phi(D_b(\varsigma, \mathfrak{y})) \tag{2.4}$$

$$\begin{aligned} N(\varsigma, \mathfrak{y}) &\leq \alpha D_b(\varsigma, \mathfrak{y}) + \tfrac{\beta[1+D_b(\varsigma, L^n\varsigma)]D_b(\mathfrak{y}, L^n\mathfrak{y})}{1+D_b(\varsigma, \mathfrak{y})} \\ &+\gamma[D_b(\varsigma, L^n\varsigma) + D_b(\mathfrak{y}, L^n\mathfrak{y})] + \delta[D_b(\varsigma, L^n\mathfrak{y}) + D_b(\mathfrak{y}, L^n\varsigma] \end{aligned}$$

for all $\varsigma, \mathfrak{y} \in \chi$, where $\tau \in \mathfrak{A}'_+$, $\alpha + \beta + \gamma + \delta \geq 0$ with $\tau\alpha + \beta + \gamma(\tau+1) + \delta(\tau(\tau+1)) < 1$. $\mathfrak{F}$ and ϕ are $*$-homomorphisms and the constraint $\mathfrak{F}(\mathfrak{a}) \leq \phi(\mathfrak{a})$. Then L have a unique fixed point in χ.

Example 2.4. *Let $\chi = [0, 1]$ and $\mathfrak{A} = \mathbb{C}$ with a norm $\| \varsigma \| = |\varsigma|$ be a real C^*-algebra. We define $\mathbb{C}^+ = (\varsigma, \mathfrak{y}) \in \mathbb{C} : \varsigma = Re(\xi) \geq 0,\ \mathfrak{y} = Im(\xi) \geq 0\}$. The partial order $\leq$ with respect to the C^*-algebra $\mathbb{C}$. $Re(\varsigma_1) \leq Re(\varsigma_2)$ and $Im(\mathfrak{y}_1) \leq Im(\mathfrak{y}_2)\ \forall (\varsigma_1, \mathfrak{y}_1), (\varsigma_2, \mathfrak{y}_2) \in \mathbb{C}$. Let $D_b : \chi \times \chi \to \mathbb{C}$ suppose that $D_b(\varsigma, \mathfrak{y}) = 2(|\varsigma - \mathfrak{y}|,\ |\varsigma - \mathfrak{y}|)$ for $\varsigma, \mathfrak{y} \in \chi$. Then, $(\chi, \mathfrak{A}, D_b)$ is a C^*-algebra valued $\mathfrak{b}$-metric space where $\tau = 1$ in theorem* **??**. *Let $\mathfrak{F}, \phi : \mathbb{C}^+ \to \mathbb{C}^+$ be the mappings defined as follows: For $\mathcal{T} = (\varsigma, \mathfrak{y}) \in \mathbb{C}^+$*

$$\psi(\mathcal{T}) = \begin{cases} (\varsigma, \mathfrak{y}), & if\ \varsigma \leq 1\ and\ \mathfrak{y} \leq 1, \\ (\varsigma^2, \mathfrak{y}), & if\ \varsigma > 1\ and\ \mathfrak{y} \leq 1, \\ (\varsigma, \mathfrak{y}^2), & if\ \varsigma \leq 1\ and\ \mathfrak{y} > 1, \\ (\varsigma^2, \mathfrak{y}^2), & if\ \varsigma > 1\ and\ \mathfrak{y} > 1. \end{cases}$$

and for $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2) \in \mathbb{C}^+$ *with* $\mathcal{P} = \min\{\mathcal{G}_1, \mathcal{G}_2\}$,

$$\phi(\mathcal{G}) = \begin{cases} \left(\frac{\mathcal{P}^2}{2}, \frac{\mathcal{P}^2}{2}\right), & if\ \mathcal{P} \leq 1 \\ \left(\frac{1}{2}, \frac{1}{2}\right), & if\ \mathcal{P} > 2 \end{cases}$$

Then, $\mathfrak{F}$ *and* ϕ *are satisfying in definition* **??** *and* **??**. *Let* $L, M : \chi \to \chi$ *be defined as follows:*

$$L(\varsigma) = \begin{cases} 0, & if\ 0 \leq \varsigma \leq \frac{1}{2} \\ \frac{1}{4}, & if \frac{1}{2} < \varsigma \leq 1 \end{cases} \quad ; \ M(\varsigma) = \frac{1}{32}\ ,\ for\ \varsigma \in \chi$$

Then, L and M are satisfying in theorem **??**. *Let* $\alpha = \frac{1}{8}$, $\beta = 0$, $\gamma = \frac{1}{16}$ *and* $\delta = \frac{1}{16}$. *It demonstrates that:*

$$\mathfrak{F}(D_b(L\varsigma, M\mathfrak{y})) \leq \mathfrak{F}(N(\varsigma, \mathfrak{y})) - \phi(D_b(\varsigma, \mathfrak{y}))\ \forall\ \varsigma, \mathfrak{y} \in \chi\ with\ \mathfrak{y} \leq \varsigma.$$

Hence, Theorem **??** *is satisfied. Then demonstrate that 0 is a unique common fixed point of L and M.*

3 Conclusions

In Theorem 2.1 we have formulated a contractive conditions to modify and extend the concept of common fixed point theorem for C^*-algebra valued b-metric space via $(\phi, \mathfrak{F})$-contractive mapping. The existence and uniqueness of the result is presented in this article. We have also given some example which satisfies the contractive condition of our main result. Our result may be the vision for other authors to extend and improve several results in such spaces and applications to other related areas.

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References

[1] Alsulami, H.H., Agarwal, R.P., Karapinar, E., Khojasteh, F., A short note on C^*-valued contraction mappings, *J. Inequalities, Appl.* 2016, 50.

[2] Kamran, T. Postolache, M. Ghiura, A. Batul, S. Ali, R. The Banach contraction principle in C^*-algebra-valued b-metric spaces with application, *Fixed Point Theory Application*, 2016, 10.

[3] Ma, Z., Jiang, L., Sun, H. C^*-algebra-valued metric spaces and related fixed point theorems, *Fixed Point Theory Application*, 2014, 206.

[4] Ma, Z., Jiang, L., C^*-algebra-valued b-metric spaces and related fixed point theorems, *Fixed Point Theory Appl.*, 2015, 1, 1-12.

[5] Murphy, G.J. C^*-algebras and operator theory, *Academic Press: London, UK*, 1990.

[6] Ozer, O., Omran, S. On the generalized C^*-valued metric spaces related with Banach fixed point theory, *Int. J. Adv. Appl.sci.*, 2017, 4, 35–37.

[7] Rahmah Mustafa, Saleh Omran and Quang Ngoc Nguyen. Fixed Point Theory Using $\mathfrak{F}$ Contractive Mapping in C^*-Algebra Valued B-Metric Space, *Mathematics*, 2021, 9, 92.

[8] Schroer, B. Lectures on algebraic quantum field theory and operator algebra, *Acta Math. Sin.*, 2001, 23, 2205-2212.

[9] Sakai, S. C^*-Algebras and W^*-Algebras, *Springer: Berlin, Heidelberg*, 1998; Print ISBN 978-3-540-63633-5, Online ISBN 978-3-642-61993-9.

[10] Zhu, K. An Introduction to Operator Algebras, *CRC Press: Boca Raton, FL, USA*, 1961.