
Original Research Article

On the Spanning Tree Packing Number of the Complement of Generalized Petersen Graph and Cocktail Party Graph

Abstract

For any graph G , the spanning tree packing number of $\sigma(G)$, is the maximum number of edge-disjoint spanning trees contained in G . In this study, we determined the maximum number of edge-disjoint spanning trees of the generalized Petersen graph and cocktail graph.

Keywords: spanning tree number, Petersen graph, edge-disjoint.

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1 Introduction

Connectivity is one of the most basic concepts of graph-theoretical subjects, both in a combinatorial sense and in an algorithm sense. As we know, the classical connectivity has two equivalent definitions, one is the cut version; the other is the path version. The generalized connectivity of a graph G is a natural generalization of the path version definition. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an S -Steiner tree connecting S (S-tree) is a subgraph $T = (V', E')$ of that of G that is a tree with $S \subseteq V'$. The generalized edge-connectivity is related to two important problems. For a given graph G and $S \subseteq V(G)$, the problem of finding a maximum set of edge-disjoint Steiner trees connecting S in G is called the Steiner tree packing problem. In this study, we determined the maximum number of edge-disjoint spanning trees of the generalized Petersen graph and cocktail graph.

2 Preliminary Notes

This section contains some of the fundamental concepts necessary for the understanding of the study.

Definition 2.1. [4] The complement \overline{G} of a graph G is the graph with vertex set $V(G)$ such that two vertices are adjacent in \overline{G} if and only if these vertices are not adjacent in G .

Definition 2.2. [4] An edge-connectivity $\lambda(G)$ of a graph G is the minimum number of edges in graph G whose deletion disconnects the graph.

Theorem 2.1. [2] *The edge connectivity of G satisfies $\lambda(G) \geq 2k$ if and only if for any set E_k of k edges of G , then the subgraph $H = G - E_k$ has an edge disjoint trees.*

Corollary 2.2. [2] *If $\lambda(G) \geq 2k$ then G has k edge-disjoint spanning trees. The lower bound is*

$$\left\lfloor \frac{\lambda(G)}{2} \right\rfloor \leq \sigma(G),$$

where the upper bound is

$$\sigma(G) \leq \left\lfloor \frac{|E(G)|}{|V(G) - 1|} \right\rfloor.$$

Remark 2.1. [6] For any complement of generalized Petersen graph $\overline{G(n, k)}$ with $n \geq 3$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $|E(\overline{G(n, k)})| = 2n(n-2)$ and $deg_{\overline{G(n, k)}}(v) = 2(n-2)$ for all $v \in V(\overline{G(n, k)})$.

Remark 2.2. [3] For any Cocktail Party graph CP_n with $n \geq 2$,

$$|E(CP_n)| = 2n(n-1), \text{ and } deg_{CP_n}(v) = 2(n-1) \text{ for all } v \in V(CP_n)$$

Remark 2.3. [4] For any regular graph G , $\delta(G) = deg_G(v)$.

Remark 2.4. [7] For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, $n = \lfloor n \rfloor$ if and only if $n \leq x \leq n + 1$.

Theorem 2.3. [12] *Let G be a graph with order p , minimum degree $\delta(G)$, and edge connectivity $\lambda(G)$. If $\lambda(G) \geq \frac{1}{2}p$, then $\lambda(G) = \delta(G)$.*

Remark 2.5. [5] For any connected cubic graph G for which $|V(G)| \geq 6$, $\sigma(G) = 1$.

Definition 2.3. [4] A vertex-connectivity $\kappa(G)$ of a graph G is the minimum number of nodes in a graph G whose deletion disconnects G .

Definition 2.4. [4] A graph H is subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a spanning subgraph of G .

Definition 2.5. [5] For any graph G , the spanning tree packing number (STP) denoted by $\sigma(G)$, is the maximum number of edge-disjoint spanning trees contained in G .

Definition 2.6. [6] Generalized Petersen graph is connected cubic graph consisting of an inner star polygon (n, k) and an outer regular polygon (n) with corresponding vertices in the inner and outer polygons connected by the edges where $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$.

Definition 2.7. [1] Ladder rung graph is graph of order n denoted by nP_2 the graph union of n copies of path graph P_2 .

Definition 2.8. [3] Cocktail party graph (CP_n) is a graph consisting of two rows of paired nodes in which all nodes except the paired ones are connected with straight lines. It is the graph complement of the ladder rung graph nP_2 .

3 Main Results

This section contains the main results of the study.

Remark 3.1. For any Generalized Petersen graph $G(n, k)$ for $n \geq 3$, $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $\sigma(G(n, k)) = 1$.

Theorem 3.1. For a generalized Petersen graph $G(n, k)$, $\sigma\overline{G(n, k)} = n - 2$ for all $n \geq 3$.

Proof: We consider the following cases:

Case 1: $n - 2 \leq \sigma\overline{G(n, k)}$.

By Corollary 2.2,

$$\left\lfloor \frac{\lambda(G)}{2} \right\rfloor \leq \sigma(G)$$

Now, by Remark 2.1, $\deg_{\overline{G(n, k)}}(v) = 2(n - 2)$, for all $v \in V(\overline{G(n, k)})$. Since $\overline{G(n, k)}$ is also a regular graph, it follows that $\delta(\overline{G(n, k)}) = \Delta(\overline{G(n, k)}) = \deg_{\overline{G(n, k)}}(v) = 2(n - 2)$, $\lambda(\overline{G(n, k)}) = \delta(\overline{G(n, k)})$. By Theorem 2.3

$$\begin{aligned} \delta(\overline{G(n, k)}) &\geq \frac{|V(\overline{G(n, k)})|}{2} \\ 2(n - 2) &\geq \frac{2n}{2} \\ 2n - 4 &\geq n \\ n &\geq 4. \end{aligned}$$

This is true since $|V(\overline{G(n, k)})| \geq 6$. Hence $\lambda\overline{G(n, k)} = 2(n - 2)$. Now we have

$$\sigma\overline{G(n, k)} \geq \left\lfloor \frac{\lambda(\overline{G(n, k)})}{2} \right\rfloor = \left\lfloor \frac{2(n - 2)}{2} \right\rfloor = n - 2.$$

Case 2: $\sigma\overline{G(n, k)} \leq n - 2$

On the other hand, by 2.2, and Remark 2.1, we get

$$\begin{aligned} \sigma\overline{G(n, k)} &\leq \left\lfloor \frac{|E(\overline{G(n, k)})|}{|V(\overline{G(n, k)})| - 1} \right\rfloor \\ &= \left\lfloor \frac{2n(n - 2)}{2n - 1} \right\rfloor \end{aligned}$$

Now, to show that $\left\lfloor \frac{2n(n - 2)}{2n - 1} \right\rfloor = n - 2$ for $n \geq 3$ we use Remark 2.4, i.e., we need to show that

$n - 2 \leq \left\lfloor \frac{2n(n - 2)}{2n - 1} \right\rfloor \leq (n - 2) + 1$. Consider the following claims:

Claim 1: $n - 2 \leq \frac{2n(n - 2)}{2n - 1}$

Verifying algebraically,

$$\begin{aligned} (n - 2)(2n - 1) &\leq 2n(n - 2) \\ 2n^2 - 2 - 4n + 2 &\leq 2n^2 - 4n \\ -n + 2 &\leq 0 \\ -n &\leq -2. \end{aligned}$$

This shows that Claim 1 is true since $n \geq 3$ by assumption.

Claim 2: $\frac{2n(n - 2)}{2n - 1} \leq (n - 2) + 1$

Verifying algebraically,

$$\begin{aligned}
 2n(n-2) &\leq n-1(2n-1) \\
 2n^2 - 4n &\leq 2n^2 - 2n - n + 1 \\
 -4n &\leq -3n + 1 \\
 -n &\leq -1
 \end{aligned}$$

This shows that claim 2 is true since $n \geq 3$ by assumption. Combining by claim 1 and claim 2, we get

$$\sigma(\overline{G(n,k)}) \leq \left\lfloor \frac{2(n-2)}{2} \right\rfloor = n-2.$$

By case 1 and case2, this implies that

$$n-2 \leq \sigma(\overline{G(n,k)}) \leq n-2.$$

Therefore, $\sigma(\overline{G(n,k)}) = n-2$. □

The following are consequence of Remark 3.1, and Theorem 3.1.

Remark 3.2. The absolute difference between $\sigma(\overline{G(n,k)})$ and $\sigma(G(n,k))$ is increasing.

The next results shows the STP od Cocktail party graph denoted by CP_n .

Theorem 3.2. For a Cocktail party graph denoted by (CP_n) with $n \geq 2$,

$$\sigma(CP_n) = n-1.$$

Proof:

We consider the following cases:

Case 1: $n-1 \leq \sigma(CP_n)$

By Corollary 2.2,

$$\frac{\lambda(G)}{2} \leq \sigma(G).$$

Now, by Remark 2.2, $\deg_{CP_n}(v) = 2(n-1)$ for all $v \in V(CP_n)$. Since (CP_n) is also a regular graph, it follows that $\delta(CP_n) = \Delta(CP_n) = \deg_{CP_n} = 2(n-1)$. Now to show that $\sigma(CP_n) \geq \frac{|V(CP_n)|}{2}$ so that $\lambda(CP_n) = \sigma(CP_n)$. By Theorem 2.3

$$\begin{aligned}
 2(n-1) &\geq \frac{2n}{2} \\
 2n-2 &\geq n \\
 n-2 &\geq 0 \\
 n &\geq 2.
 \end{aligned}$$

Hence, $\lambda(CP_n) = \deg_{CP_n}(v) = 2(n-1)$. Now, by Corollary 2.2, we have

$$\begin{aligned}
 \sigma(G(n,k)) &\geq \left\lfloor \frac{2(n-1)}{2} \right\rfloor \\
 \sigma(G(n,k)) &= n-1.
 \end{aligned}$$

Case 2: $\sigma(CP_n) \leq n-1$

On the other hand by Corollary 2.2 and Remark 2.2

$$\begin{aligned}\sigma(\overline{CP_n}) &\leq \left\lfloor \frac{|E(\overline{CP_n})|}{|V(\overline{CP_n})| - 1} \right\rfloor \\ &= \left\lfloor \frac{2n(n-2)}{2n-1} \right\rfloor.\end{aligned}$$

To show that $\left\lfloor \frac{2n(n-2)}{2n-1} \right\rfloor = n-1$ for $n \geq 2$, we used Remark 2.5, i.e., we need to show that $n-1 \leq \frac{2n(n-2)}{2n-1} \leq (n-1) + 1$. Consider the following Claims below:

Claim 1: $n-1 \leq \frac{2n(n-2)}{2n-1}$.
Verifying algebraically,

$$\begin{aligned}(n-1)(2n-1) &\leq 2n(n-1) \\ 2n^2 - n - 2n + 1 &\leq 2n^2 - 2n \\ -n + 1 &\leq 0 \\ -n &\leq -1.\end{aligned}$$

Claim 2: $\frac{2n(n-2)}{2n-1} \leq (n-1) + 1$.
Verifying algebraically,

$$\begin{aligned}2n(n-1) &\leq n(2n-1) \\ 2n^2 - 2n &\leq 2n^2 - 2n \\ -2n &\leq -n \\ -n &\leq 0\end{aligned}$$

This shows that Claim 2 is true since $n \geq 2$ by assumption. Thus

$$\sigma(CP_n) \leq \left\lfloor \frac{2n(n-1)}{2n-1} \right\rfloor = n-1.$$

By case 1 and case 2 this implies that,

$$n-1 \leq \sigma(CP_n) \leq n-1.$$

Therefore,

$$\sigma(CP_n) = n-1$$

□

Remark 3.3. The spanning tree packing number of $\overline{CP_n}$ is 0.

Remark 3.4. The difference between $\sigma(CP_n)$ and $\sigma(\overline{CP_n})$ is increasing.

4 Conclusion

In this article, we determined the maximum number of edge-disjoint spanning trees of the generalized Petersen graph and cocktail graph.

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