Spanning Tree Packing of Lexicographic Product of Graphs Resulting from Path and Complete Graphs

Original Article

Abstract

For any graphs G of order n, the spanning tree packing number, denoted by σ, of a graph G is the maximum number of edge-disjoint spanning tree contained in G. In this study determine the spanning packing number of lexicographic product of graphs resulting from two path graphs

Keywords: connectivity, edge-disjoint spanning tree packing number, lexicographic product.

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1 Introduction

The spanning-tree packing number of a graph G, denoted by $\sigma(G)$, which has n vertices, represents the maximum count of edge-disjoint spanning trees present in G. This quantity has been utilized as a measure to assess the reliability of communication networks and has been extensively explored by various researchers. To explore this topic further, one can refer to the surveys conducted by Palmer [5] and Ozeki and Yamashita [4]. Determining the maximum number of edge-disjoint spanning trees in a given graph G can be accomplished in polynomial time, as described in [10].

Peng and Tay [11], conducted a study on the spanning-tree packing numbers of Cartesian products formed by combining different sets of complete graphs, cycles, and complete multipartite graphs. Subsequently, Ku, Wang, and Hung [12] derived the following outcome: for two connected graphs G and H, the spanning-tree packing number of their Cartesian product is greater than or equal to the sum of the spanning-tree packing numbers of G and H minus one. In [13], Li, H. et.al. obtained a sharp lower bound for the spanning-tree packing number of lexicographic product graphs.

In this paper, we determine the exact values of the spanning tree packing number of lexicographic product of graphs resulting from path graph P_n and complete graphs, K_{2n}

2 Preliminary Notes

This section contains some of the fundamental concepts necessary for the understanding of the study. Definitions that are not in this paper can be found on [6], [7], [8].

Definition 2.1. [2] A set of subgraphs of G are edge disjoint if no two of them have an edge in common.

Definition 2.2. [3] A bridge is an edge $e = uv$ in a connected graph whose removal reults in a disconnected graph.

Corollary 2.1. *[4] If $\lambda(G) \geq 2k$ then G has k edge-disjoint spanning trees. The lower bound is*

$$\left\lfloor \frac{\lambda(G)}{2} \right\rfloor \leq \sigma(G),$$

where the upper bound is

$$\sigma(G) \leq \left\lfloor \frac{|E(G)|}{|V(G)-1|} \right\rfloor.$$

Theorem 2.2. *[13] Let G and H be two connected nontrivial graphs, and let $\sigma(G) = k$, $\sigma(H) = l$, $|V(G)| = n_1(n_1 \geq 2)$, and $|V(G)| = n_2(n_2 \geq 2)$ the following are true:*

(i.) *If $kn_2 = ln_1$, then $\sigma(G[H]) \geq kn_2(= ln_1)$;*

(ii.) *If $ln_1 > kn_2$, then $\sigma(G[H]) \geq kn_2 - \left\lceil \frac{kn_2-1}{n_1} = l - 1 \right\rceil$; and*

(iii.) *If $ln_1 < kn_2$, then $\sigma(G[H]) \geq kn_2 \left\lceil \frac{kn_2}{n_1+1} + l \right\rceil$.*

Moreover, the bounds are sharp (i.e. there exist a graph such that the equality holds)

Definition 2.3. [3] An acyclic graph is a graph that has no cycles.

Definition 2.4. [3] A tree is a connected acyclic graph.

Definition 2.5. [3] A graph G is complete if every pair of distinct vertices are edjacent in G. A complete graph of n vertices is denoted by K_n. The graph K_1 is a trivial graph.

Definition 2.6. [3] A graph h is a spanning subgraph of G if H is subgraph of G such that $V(G) = V(G)$.

Definition 2.7. [3] A spanning tree of a graph G is a spanning subgraph of G that is a tree.

Definition 2.8. [13] For any graph G the spanning tree packing number (STP), denoted by $\sigma(G)$, is the maximum number of edge disjoint trees contained in G.

Definition 2.9. [3] The composition (lexicographic product) $G[H]$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = uvv' \in E(H)$.

3 Main Results

Proposition 3.1. *Let G be a connected nontrivial graph. If G contains a bridge, then $\sigma(G) = 1$.*

Proof: Suppose G has a bridge e_0 and suppose further $\sigma(G) = 1$. Then there exist at least two edge disjoint spanning tree, say T_1 and T_2. A contradiction since A and B are edge disjoint. Therefore, $\sigma(G) = 1$. □

Proposition 3.2. *Let G and H be nontrivial connected graph. Then $\sigma(G \cup H) = 0$.*

Proof: Let G and H be a nontrivial connected graphs. Suppose $\sigma(G \cup H) \neq= 0$. Then there exist at least a spanning tree, T_0, in G and H such that for all $v \in V(G)$, $v \in V(T_0)$. However, G and

H are disjoint in $G \cup H$, Thus, there can be no spanning subgraph connecting the vertices of G and H. This is a contradiction in the assumption that $\sigma(G \cup H) \neq 0$. Therefore, $\sigma(G \cup H) = 0$. □

Remark 3.1. For a path P_n where $n \geq 3$, $\sigma(P_n) = 1$.

Proposition 3.3. *Let P_n and P_m be two paths. Then $\sigma(P_n[P_m]) = n$, where $m = n$.*

Proof: Let P_n and P_m be the two paths for $m, n \geq 3$. Then by Corollary 2.1,

$$\begin{aligned}
\sigma(G) &\leq \left\lfloor \frac{|E(G)|}{|V(G) - 1|} \right\rfloor \\
\sigma(P_n[P_m]) &\leq \left\lfloor \frac{|E(P_n[P_m])|}{|V(P_n[P_m]) - 1|} \right\rfloor \\
&\leq \left\lfloor \frac{|E(P_m)||(P_n)| + |E(P_n)||V(P_m)|^2)|}{|V(P_n[P_m]) - 1|} \right\rfloor \\
&\leq \left\lfloor \frac{(m-1)n + (n-1)m^2)}{mn - 1} \right\rfloor.
\end{aligned}$$

Since $m = n$ by assumption, we have

$$\begin{aligned}
\sigma(P_n[P_m]) &\leq \left\lfloor \frac{(n-1)n + (n-1)n^2}{n^2 - 1} \right\rfloor \\
&= \left\lfloor \frac{(n^2 + n)(n-1)}{(n^2 - 1)} \right\rfloor \\
&= \left\lfloor \frac{(n(n+1))(n-1)}{(n^2 - 1)} \right\rfloor \\
&= \left\lfloor \frac{n(n^2 - 1)}{(n^2 - 1)} \right\rfloor \\
&= \lfloor n \rfloor \\
&= n.
\end{aligned}$$

Thus, $\sigma(P_n[P_m]) \leq n$.

By Theorem 2.2

$$\sigma(P_n[P_m]) \geq kn = lm.$$

Since $\sigma(P_n) = 1$, by Remark 3.1. Thus, $\sigma(P_n[P_m]) \geq n$.

Hence, $n \leq \sigma(P_n[P_m]) \leq n$. Thus, $\sigma(P_n[P_m]) = n$. □

Proposition 3.4. *Let K_{2n} and K_{2m} be two complete graphs. Then*

$$\sigma(K_{2n}[K_{2m}]) = n\left\lfloor \frac{n}{2} \right\rfloor, \text{ where } n = m \ .$$

Proof:

By Corollary 2.1,

$$\sigma(G) \leq \left\lfloor \frac{|E(G)|}{|V(G) - 1|} \right\rfloor$$

$$\sigma(K_{2n}[K_{2m}]) \leq \left\lfloor \frac{|E(K_{2n}[K_{2m}])|}{|V(K_{2n}[K_{2m}]) - 1|} \right\rfloor$$

$$\leq \left\lfloor \frac{|E(K_{2m}||V(K_{2n})| + |E(K_{2n})||V(K_{2m})|^2}{|V(K_{2n}[K_{2m}]) - 1|} \right\rfloor$$

Since $m = n$ by assumption, we have

$$\sigma(K_{2n}[K_{2m}]) \leq \left\lfloor \frac{\frac{n(n-1)n}{2} + \frac{n(n-1)n^2}{2}}{n^2 - 1} \right\rfloor$$

$$\leq \left\lfloor \frac{\frac{n^3 - n^2 + n^4 - n^3}{2}}{n^2 - 1} \right\rfloor$$

$$\leq \left\lfloor \frac{n^4 - n^2}{2(n^2 - 1)} \right\rfloor$$

$$\leq \left\lfloor \frac{n^2(n^2 - 1)}{2(n^2 - 1)} \right\rfloor$$

$$\leq \left\lfloor \frac{n^2}{2} \right\rfloor.$$

Thus, $\sigma(K_{2n}[K_{2m}]) \leq \left\lfloor \frac{n^2}{2} \right\rfloor$. By Theorem 2.2

$$\sigma(K_{2n}[K_{2m}]) \geq kn = lm.$$

Since $\sigma(K_{2n}[K_{2m}]) = n\lfloor \frac{n}{2} \rfloor$, by Remark 3.1. Thus, $\sigma(K_{2n}[K_{2m}]) \geq \lfloor \frac{n^2}{2} \rfloor$. Hence, $\lfloor \frac{n^2}{2} \rfloor \leq \sigma(K_{2n}[K_{2m}]) \leq n\lfloor \frac{n}{2} \rfloor$. Thus, $\sigma(K_{2n}[K_{2m}]) = n\lfloor \frac{n}{2} \rfloor$. □

4 Conclusion

In this paper, we have successfully determined the precise values of the spanning tree packing number for the lexicographic product of graphs formed by combining a path graph P_n and complete graphs, K_{2n}. These findings may contribute to the understanding of spanning tree packing in these specific graph structures and provide valuable insights into their combinatorial properties.

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Competing Interests

The authors declare that they have no competing interests.

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