Original Research Article

On the Spanning Tree Packing of Lexicographic Product Resulting from Path and Complete Graphs

Abstract

For any graphs G of order n, the spanning tree packing number σ of a graph G is the maximum number of edge-disjoint spanning tree contianed in G. This paper aims to determine the spanning packing number of lexicographic product of graphs resulting from path and complete graphs

Keywords: connectivity, edge-disjoint spanning tree packing number, lexicographic product.

2020 Mathematics Subject Classification: 05C35

1 Introduction

The field of mathematics plays a vital role in several fields. One of the important areas in mathematics is graph theory which is used in structural models. This structural arrangements of various objects or technologies lead to new inventions and adjustments in the existing environment for enhancement in those fields. The origin of graph theory started with the problem of Konigsberg problen in 1735. This problems leads to the concept of Euclidean graph when Euler studied the problem of Konigbergs bridge and the constructed a structure to solve the problem called Euclerian graph. Any how, the term "graph" was introduced by Sylvester in 1878 where he draw an analogy between "Quantic invariants" and covariants of algebra and molecular diagrams.

As the time goes by many mathematicians studied the different areas in graph theory. One of the problem is the spanning tree packing number (STP) denoted by $\sigma(G)$ which is maximum number of edge disjoint spanning tree contained in G by the work of E.M. Palmer [2]. On the otherhand, in [5], presents a comprehensive exploration of spanning trees in finite simple graphs, which are graphs without loops or multiple edges. The aim is to provide a survey of spanning trees and investigate various problems related to them, which serve as extensions of the renowned Hamiltonian path problem. In this study we determine the spanning-tree packing number of lexicographic product of graphs resulting from two path graphs.

2 Preliminary Notes

This section contains some of the fundamental concepts necessary for the understanding of the study.

Definition 2.1. [2] A set of subgraphs of G are edge disjoint if no two of them have an edge in common.

Definition 2.2. [4] A bridge is an edge $e = uv$ in a connected graph whose removal reults in a disconnected graph.

Corollary 2.1. *[5] If $\lambda(G) \geq 2k$ then G has k edge-disjoint spanning trees. The lower bound is*

$$\left\lfloor \frac{\lambda(G)}{2} \right\rfloor \leq \sigma(G),$$

where the upper bound is

$$\sigma(G) \leq \left\lfloor \frac{|E(G)|}{|V(G) - 1|} \right\rfloor.$$

Theorem 2.2. *[3] Let G and H be two connected nontrivial graphs, and let $\sigma(G) = k$, $\sigma(H) = l$, $|V(G)| = n_1 (n_1 \geq 2)$, and $|V(G)| = n_2 (n_2 \geq 2)$ the following are true:*

(*i.*) *If $kn_2 = ln_1$, then $\sigma(G[H]) \geq kn_2 (= ln_1)$;*

(*ii.*) *If $ln_1 > kn_2$, then $\sigma(G[H]) \geq kn_2 - \left\lceil \frac{kn_2 - 1}{n_1} = l - 1 \right\rceil$; and*

(*iii.*) *If $ln_1 < kn_2$, then $\sigma(G[H]) \geq kn_2 \left\lceil \frac{kn_2}{n_1 + 1} + l \right\rceil$.*

Moreover, the bounds are sharp (i.e. there exist a graph such that the equality holds)

Definition 2.3. [4] An acyclic graph is a graph that has no cycles.

Definition 2.4. [4] A tree is a connected acyclic graph.

Definition 2.5. [4] A graph G is complete if every pair of distinct vertices are edjacent in G. A complete graph of n vertices is denoted by K_n. The graph K_1 is a trivial graph.

Definition 2.6. [4] A graph h is a spanning subgraph of G if H is subgraph of G such that $V(G) = V(G)$.

Definition 2.7. [4] A spanning tree of a graph G is a spanning subgraph of G that is a tree.

Definition 2.8. [3] For any graph G the spanning tree packing number (STP), denoted by $\sigma(G)$, is the maximum number of edge disjoint trees contained in G.

Definition 2.9. [4] The composition (lexicogrpahic product) $G[H]$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ in which (u, v) is adjacent to (u', v') if and only if either $uu' \in E(G)$ or $u = uvv' \in E(H)$.

3 Main Results

In this section, we establish some results of spanning tree packing number of lexicographic product of two paths.

Proposition 3.1. *Let G be a connected nontrivial graph. If G contains a bridge, then $\sigma(G) = 1$.*

Proof: Suppose G has a bridge e_0 and suppose further $\sigma(G) = 1$. Then there exist at least two edge disjoint spanning tree, say T_1 and T_2. A contradiction since A and B are edge disjoint. Therefore, $\sigma(G) = 1$. □

Proposition 3.2. *Let G and H be nontrivial connected graph. Then $\sigma(G \cup H) = 0$.*

Proof: Let G and H be a nontrivial connected graphs. Suppose $\sigma(G \cup H) \neq= 0$. Then there exist at least a spanning tree, T_0, in G and H such that for all $v \in V(G)$, $v \in V(T_0)$. However, G and H are disjoint in $G \cup H$, Thus, there can be no spanning subgraph connecting the vertices of G and H. This is a contradiction in the assumption that $\sigma(G \cup H) \neq 0$. Therefore, $\sigma(G \cup H) = 0$. □

Remark 3.1. For a path P_n where $n \geq 3$, $\sigma(P_n) = 1$.

Proposition 3.3. *Let P_n and P_m be two paths. Then $\sigma(P_n[P_m]) = n$, where $m = n$.*

Proof: Let P_n and P_m be the two paths for $m, n \geq 3$. Then by Corollary 2.1,

$$\begin{aligned}
\sigma(G) &\leq \left\lfloor \frac{|E(G)|}{|V(G) - 1|} \right\rfloor \\
\sigma(P_n[P_m]) &\leq \left\lfloor \frac{|E(P_n[P_m])|}{|V(P_n[P_m]) - 1|} \right\rfloor \\
&\leq \left\lfloor \frac{|E(P_m)||(P_n)| + |E(P_n)||V(P_m)|^2)|}{|V(P_n[P_m]) - 1|} \right\rfloor \\
&\leq \left\lfloor \frac{(m-1)n + (n-1)m^2)}{mn - 1} \right\rfloor.
\end{aligned}$$

Since $m = n$ by assumption, we have

$$\begin{aligned}
\sigma(P_n[P_m]) &\leq \left\lfloor \frac{(n-1)n + (n-1)n^2}{n^2 - 1} \right\rfloor \\
&= \left\lfloor \frac{(n^2 + n)(n-1)}{(n^2 - 1)} \right\rfloor \\
&= \left\lfloor \frac{(n(n+1))(n-1)}{(n^2 - 1)} \right\rfloor \\
&= \left\lfloor \frac{n(n^2 - 1)}{(n^2 - 1)} \right\rfloor \\
&= \lfloor n \rfloor \\
&= n.
\end{aligned}$$

Thus, $\sigma(P_n[P_m]) \leq n$.

By Theorem 2.2

$$\sigma(P_n[P_m]) \geq kn = lm.$$

Since $\sigma(P_n) = 1$, by Remark 3.1. Thus, $\sigma(P_n[P_m]) \geq n$.

Hence, $n \leq \sigma(P_n[P_m]) \leq n$. Thus, $\sigma(P_n[P_m]) = n$. □

Proposition 3.4. *Let K_{2n} and K_{2m} be two complete graphs. Then*

$$\sigma(K_{2n}[K_{2m}]) = n\lfloor \frac{n}{2} \rfloor, \text{ where } n = m .$$

Proof:

By Corollary 2.1,

$$\begin{aligned}\sigma(G) &\leq \left\lfloor \frac{|E(G)|}{|V(G)-1|} \right\rfloor \\ \sigma(K_{2n}[K_{2m}]) &\leq \left\lfloor \frac{|E(K_{2n}[K_{2m}])|}{|V(K_{2n}[K_{2m}])-1|} \right\rfloor \\ &\leq \left\lfloor \frac{|E(K_{2m}||V(K_{2n})| + |E(K_{2n})||V(K_{2m})|^2}{|V(K_{2n}[K_{2m}])-1|} \right\rfloor\end{aligned}$$

Since $m = n$ by assumption, we have

$$\begin{aligned}\sigma(K_{2n}[K_{2m}]) &\leq \left\lfloor \frac{\frac{n(n-1)n}{2} + \frac{n(n-1)n^2}{2}}{n^2-1} \right\rfloor \\ &\leq \left\lfloor \frac{\frac{n^3-n^2+n^4-n^3}{2}}{n^2-1} \right\rfloor \\ &\leq \left\lfloor \frac{n^4-n^2}{2(n^2-1)} \right\rfloor \\ &\leq \left\lfloor \frac{n^2(n^2-1)}{2(n^2-1)} \right\rfloor \\ &\leq \left\lfloor \frac{n^2}{2} \right\rfloor.\end{aligned}$$

Thus, $\sigma(K_{2n}[K_{2m}]) \leq \left\lfloor \frac{n^2}{2} \right\rfloor$. By Theorem 2.2

$$\sigma(K_{2n}[K_{2m}]) \geq kn = lm.$$

Since $\sigma(K_{2n}[K_{2m}]) = n\left\lfloor \frac{n}{2} \right\rfloor$, by Remark 3.1. Thus, $\sigma(K_{2n}[K_{2m}]) \geq \left\lfloor \frac{n^2}{2} \right\rfloor$. Hence, $\left\lfloor \frac{n^2}{2} \right\rfloor \leq \sigma(K_{2n}[K_{2m}]) \leq n\left\lfloor \frac{n}{2} \right\rfloor$. Thus, $\sigma(K_{2n}[K_{2m}]) = n\left\lfloor \frac{n}{2} \right\rfloor$. □

4 Conclusion

In this article, we established some results of spanning tree packing number of lexicographic product of two paths.

References

[1] Barden, B., Libeskind-Hadas, R., Davis, J., Williams, W. (1999). One Edge-disjoint Spanning Trees in Hypercubes. Inform. Proc. Lett. 70(1), 1316.

[2] Bondy J.A., MurtyU.S.R. (2008). Graph THoery, GTM 244, Springer.

[3] Hengzhe L., Xueliang L., Mao Y. and Yue J. (2014). Note on the spanning-tree packing number of lexicographic product graphs. Elsevier B.V.

[4] Mao Y. and Xueliang L. (2016). Generalized Connectivity of Graphs. Springer. .

[5] Ozeki K. Yamashita T. (2011). Spanning tress: A survey, Graphs Combin. 27(1), 1-26

[6] Palmer E. (2001). On the spanning tree packing number of a graph: a survey, Discrete Math. 230, 13-21.

[7] Tan KSR, Cabahug Jr. IS. Safe Sets in Some Graph Families; 2022; 18(9): 1-7

[8] Mangubat DP, Cabahug Jr. IS. On the restrained cost eective sets of some special classes of graphs; 2022; 18(8): 22-24.

[9] Dinorog MG. (2022). Rings Domination Number of Some Mycielski Graphs. Asian Research Journal of Mathematics. 2022 Dec 3;18(12):16-26.Available:https://doi.org/10.9734/arjom/2022/v18i12621

[10] Ruaya KKB, Eballe RG, & Cabahug Jr. IS. Another Look of Rings Domination in Ladder Graph. Asian Research Journal of Mathematics. 2022;18(12):27-33
