On the 2-Rainbow Domination Number Over the γ-set of Some Classes of Graphs

Original Article

Abstract

A 2-rainbow domination function (2-RDF) is a function $f : V(G) \to 2^{[2]}$ on a vertex set $V(G)$ of a graph G, such that for any vertex v with $f(x) = \emptyset$, we have $\bigcup_{u \in N(v)} f(u) = [2]$. The weight of f is $\|f\| = \sum_{v \in V(G)} |f(v)|$. The minimum value of $\|f\|$ is the 2-rainbow domination number denoted by $\gamma_{r2}(G)$. In this study, we determine the 2-rainbow domination number of some special classes of graphs over its corresponding γ-set, specifically, path, cycle, fan, and wheel graphs.

Keywords: domination, rainbow dominating function, rainbow domination number.
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1 Introduction

One of the enthralling branches in graph theory is the domination in graphs that is presently effectively in excess of several investigations that have been referred to and presented since this has published in [1]. Several authors have defined and analyzed various varieties of domination. One of such is the concept of rainbow domination by Bresar et al [9]. It determines the k-rainbow domination number of a graph G denoted by $\gamma_{rk}(G)$, which is the minimum weight of a k-rainbow dominating function. The k-rainbow dominating function (kRDF) is a mapping from $V(G)$ to the power set of $\{1, 2, ..., k\}$. Prior to the release of the concept of k-rainbow domination, Bresar and Sumenjak in [6], introduced the 2-rainbow domination in graphs for which exact values of 2-rainbow domination number of several classes of graphs can be found, and it is shown that for the generalized Petersen graphs $GP(n, k)$ this number is between $\lceil \frac{4n}{5} \rceil$ and n. Further study on the 2-rainbow domination in the Petersen graph is also investigated in [8]. In this study we investigate the concept of 2-rainbow domination over a γ-set of some classes of graphs.

2 Preliminary Notes

This section contains some of the fundamental concepts necessary for the understanding of the study. Related definitions that are not in this study can be found in [2], [3], [4], [5], [6], and [9].

Definition 2.1. [1] If S and T are any two nonempty subsets of $V(G)$, we say S dominates T if $T \subseteq N[S]$. If S dominates $V(G)$ then S is a dominating set od G. The domination number of G. denotes by $\gamma(G)$, is the minimum cardinality of dominating set of G. A set whose cardinality is the domination number of a graph G is called γ-set of G.

Definition 2.2. [9] Let $[k] = \{1, 2, 3, ..., k\}$, where the elements of $[k]$ are called colors. Given a graph G and a positive integer k, a k-rainbow domination function (or k-RDF) is function $f : V(G) \to 2^{[k]}$, such that for any vertex v with $f(x) = \emptyset$, we have

$\bigcup_{u\in N(v)} f(u) = [k]$. Let $\|f\| = \sum_{v\in V(G)} |f(v)|$; we refer to $\|f\|$ as the weight of f. The k-rainbow domination number, $\gamma_{rf}(G)$, of graph G, is the minimum value of $\|f\|$.

3 Main Results

This section deals about the 2-rainbow dominating number of special classes of graphs over γ-set of some classes of graphs, specifically, path, cycle, fan, and wheel.

Theorem 3.1. If P_n is a path graph of order $n \geq 3$, then

$$\gamma_{r2}(P_n) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \bmod 3 \\ \frac{2n-2}{3}, & \text{if } n \equiv 1 \bmod 3 \\ \frac{2n-1}{3}, & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

Proof: Let $P_n = \{v_1, v_2, \ldots, v_n\}$ with $n \geq 3$ and let $\emptyset \neq S \subseteq v(P_n)$ be a γ-set of P_n. Consider the following cases:

Case 1: $n \equiv 0 (\text{mod } 3)$

Let $V(G) = S \cup S_1 \cup S_2$ such that $S = \{v_2, v_5, \ldots, v_{n-2}\}$, $S_1 = \{v_1, v_4, \ldots, v_{n-3}\}$ and $S_2 = \{v_3, v_6, \ldots, v_n\}$. Clearly, $|S_1| = |S_2| = \frac{n}{3}$. Define a function $f : V(P_n) \to 2^{[2]}$ by

$$f(v) = \begin{cases} \emptyset, & \text{if } v \in S \\ \{1\}, & \text{if } v \in S_1 \\ \{2\}, & \text{if } v \in S_2 \end{cases}$$

Then f is a 2-RDF of G with

$$\|f\| = \sum_{v\in V(P_n)} |f(v)| = |S| + |S_1| + |S_2| = 0 + \frac{n}{3} + \frac{n}{3} + \frac{2n}{3}.$$

Now, to show that $\|f\|$ is the minimum weight of P_n, suppose $\|f\|$ is not the minimum weight of P_n. Then there exists a function $g : V(P_n) \to 2^{[2]}$ such that $\|g\| < \|f\|$. Since S is a γ-set of P_n, then $g(v) = \emptyset$ for each $v \in S$. Now, since $\|g\| < \|f\|$, then there exist $u, w \in V(G) \setminus S$ such that $g(u) = \{1\} = g(w)$, a contadiction. Thus, $\|f\| = \frac{2n}{3}$ is the minimum weight of P_n. Hence,

$$\gamma_{r2}(P_n) = \frac{2n}{3}.$$

Case 2: $n \equiv 1 \pmod 3$

With the same assumption as case 1. Let $S = \{v_1, v_4, \ldots, v_n\}$, $S_1 = \{v_2, v_5, \ldots, v_{n-2}\}$ and $S_2 = \{v_3, v_6, \ldots, v_{n-1}\}$. Clearly, $|S_1| = \frac{n-1}{3}$ and $|S_2| = \frac{n-1}{3}$. Define a function $f : V(P_n) \to 2^{[2]}$ by

$$f(v) = \begin{cases} \emptyset, & \text{if } v \in S \\ \{1\}, & \text{if } v \in S_1 \\ \{2\}, & \text{if } v \in S_2 \end{cases}$$

Then f is a 2-RDF of G with

$$\|f\| = \sum_{v \in V(P_n)} |f(v)| = |S| + |S_1| + |S_2| = 0 + \frac{n-1}{3} + \frac{n-1}{3} + \frac{2n-2}{3}.$$

To show that $\|f\|$ is the minimum weight of P_n. On the contary, suppose $\|f\|$ is the minimum weight of P_n. Then there exists a function $g : V(P_n) \to 2^{[2]}$ such that $\|g\| < \|f\|$. Since S is a γ-set of P_n, then $g(v) = \emptyset$ for all $v \in S$. Now, since $\|g\| < \|f\|$, then there exist $u, w \in V(G) \setminus S$ such that $g(u) = \{1\} = g(w)$, a contadiction. Thus, $\|f\| = \frac{2n-2}{3}$ is the minimum weight of P_n. Hence,

$$\gamma_{r2}(P_n) = \frac{2n-2}{3}.$$

Case 3: $n \equiv 2 \pmod 3$

Choose $S = \{v_1, v_2, \ldots, v_{n-1}\}$, $S_1 = \{v_2, v_5, \ldots, v_n\}$ and $S_2 = \{v_3, v_6, \ldots, v_{n-2}\}$. Clearly, $|S_1| = \frac{n+1}{3}$ and $|S_2| = \frac{n-2}{3}$. Define a function $f : V(P_n) \to 2^{[2]}$ by

$$f(v) = \begin{cases} \emptyset, & \text{if } v \in S \end{cases}$$

$\{1\}$, if $v \in S_1$

$\{2\}$, if $v \in S_2$

Then f is a 2-RDF of G with

$$\|f\| = \sum_{v\in V(P_n)} |f(v)| = |S| + |S_1| + |S_2| = 0 + \frac{n+1}{3} + \frac{n-2}{3} + \frac{2n-1}{3}.$$

To show that $\|f\|$ is the minimum weight of P_n. On the contary, suppose $\|f\|$ is the minimum weight of P_n. Then there exists a function $g : V(P_n) \to 2^{[2]}$ such that $\|g\| < \|f\|$. Since S is a γ-set of P_n, then $g(v) = \emptyset$ for all $v \in S$. Now, since $\|g\| < \|f\|$, then there exist $u, w \in V(G) \setminus S$ such that $g(u) = \{1\} = g(w)$, a contadiction. Thus, $\|f\| = \frac{2n-1}{3}$ is the minimum weight of P_n. Hence,

$\gamma_{r2}(P_n) = \frac{2n-1}{3}$. □

Corollary 3.2. If C_n is a cycle graph of order $n \geq 3$, then

$$\gamma_{r2}(C_n) = \gamma_{r2}(P_n) = \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \bmod 3 \\ \frac{2n-2}{3}, & \text{if } n \equiv 1 \bmod 3 \\ \frac{2n-1}{3}, & \text{if } n \equiv 2 \bmod 3 \end{cases}$$

Proof:

This immediately follows from the Theorem 3.1 by constructing the same S, S_1 and S_2 for all cases of n. □

Theorem 3.3. Let F_n be a fan grpah of order $n \geq 3$. Then $\gamma_{r2}(F_n) = n$.

Proof: Let $F_n = \{v_1, v_2, \ldots, v_n\} \cup \{v_0\}$ where $\deg(v_0) = n$ and let $\emptyset \neq S \subseteq V(G)$ be a γ-set of F_n. Choose $S = \{v_0\}$ and $S_1 = \{v_1, v_2, \ldots, v_n\}$. Define the function $f : V(F_n) \to 2^{[2]}$ by

$$f(v_i) = \begin{cases} \emptyset, & \text{if } i = 0 \\ \{1\}, & \text{if } i \text{ is odd} \\ \{2\}, & \text{if } i \text{ is even} \end{cases}$$

Then f is a 2-RDf of G with

$$\|f\| = \sum_{v \in V(F_n)} |f(v_i)| = |S| + |S_1| + |S_2| = 0 + \frac{n}{2} + \frac{n}{2} = n.$$

Clearly, $\|f\|$ is the minimum weight of F_n. Thus, $\gamma_{r2}(F_n) = n$. □

Theorem 3.4. If W_n is a wheel graph with $n \geq 3$, then

$$\gamma_{r2}(W_n) = \begin{cases} n, & \text{if } n \equiv 0 \bmod 2 \\ n+1, & \text{if } n \equiv 1 \bmod 2 \end{cases}$$

Proof:

Let $W_n = \{v_1, v_2, ..., v_n\} \cup \{v_0\}$ iwth $\deg(v_0) = n$ and let $\emptyset \neq S \subseteq V(G)$ be a γ-set of W_n. Consider the following cases:

Case 1: $n \equiv 0 \pmod 2$

Choose $S = \{v_0\}$ and $S_1 = \{v_1, v_2, ..., v_n\}$. Define a function $f : V(W_n) \to 2^{[2]}$ by

$$f(v_i) = \begin{cases} \emptyset, & \text{if } i = 0 \\ \{1\}, & \text{if } i \text{ is odd} \\ \{2\}, & \text{if } i \text{ is even} \end{cases}$$

Then f is a 2-RDf of G with

$$\|f\| = \sum_{v \in V(W_n)} |f(v_i)| = |S| + |S_1| + |S_2| = 0 + \frac{n}{2} + \frac{n}{2} = n.$$

Clearly, $\|f\|$ is the minimum weight of W_n. Thus, $\gamma_{r2}(W_n) = n$. □

Case 2: $n \equiv 1 \pmod 2$

Choose $S = \{v_0\}$ and $S_1 = \{v_1, v_2, ..., v_n\}$. Define a function $f : V(W_n) \to 2^{[2]}$ by

$$f(v_i) = \begin{cases} \emptyset, & \text{if } i = 0 \\ \{1\}, & \text{if } i \neq n \text{ and } i \text{ is odd} \end{cases}$$

$\{2\}$, if i is even

$\{1, 2\}$, if $i = n$.

Then f is a 2-RDf of G with

$$\|f\| = \sum_{v \in V(W_n)} |f(v_i)| = |S| + |S_1| + |S_2| = 0 + \frac{n}{2} + \frac{n}{2} + 1 = n + 1.$$

Clearly, $\|f\|$ is the minimum weight of W_n. Thus, $\gamma_{r2}(W_n) = n + 1$. □

4 Conclusion

In this article, we focuses on the concept of 2-rainbow dominating number within specific classes of graphs, namely path, cycle, fan, and wheel graphs, over its γ-set for which exact values are generated. Further research and analysis in this area can contribute to the development of graph theory and its applications in various fields.

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Competing Interests

The authors declare that they have no competing interests.

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