Construction and Convergence of the C-S Combined Mean Method for Multiple Polynomial Zeros

Abstract

In this article, we have combined two well known third order methods one is Chebyshev and another is Super-Halley to form an iterative method of third for solving polynomial equations with multiple polynomial zeros. This constructed method is basically the mean of the methods Chebyshev and Super-Halley, so we name the method as C-S Combined Mean Method. We have proposed some local convergence theorems of this C-S Combined Mean Method to establish the computation of a polynomial with known multiple zeros. For the establishment of this local convergence theorem, the key role is performed by a function(Real valued) termed as the function of initial conditions. Function of initial conditions $\boldsymbol{I}$ is a mapping from the set $\boldsymbol{D}$ into the set $\boldsymbol{M}$, where $\boldsymbol{D}$ (subset of $\boldsymbol{M}$) is the domain of the C-S Combined mean iterative scheme. Here the initial conditions uses the information only at the initial point and are given in the form $\boldsymbol{I(w_0)}$ which belongs to $\boldsymbol{J}$, where $\boldsymbol{J}$ is an in interval on the positive real line which also contains $\mathbf{0}$ and $\boldsymbol{w_0}$ is the starting point. We have used the notion of gauge function which also plays very important role in establishing the convergence theorem. Here we have used two types of initial conditions over an arbitrary normed field and established local convergence theorems of the constructed C-S Combined mean method. The error estimations are also found in our convergence analysis. For simple zero, the method as well as the results hold good.

Keywords: Local convergence, Gauge function, Chebyshev method, Super-Halley method, Initial conditions, Polynomial zeros, Multiple zeros, Normed field

MSC Classification: 65H04 , 12Y05

1 Introduction

A very fundamental and a centuries-old topic in numerical analysis is to find the zeros of single variable polynomial equations, which has many applications in engineering and other applied disciplines. As Galois theorem states that for the polynomial equation that have a general solution are those of degree less or equal to four. Therefore for the computation of zeros of a polynomial of higher degree our focus goes to iterative methods.
In the literature of iterative method for solving non-linear equations, Chebyshev and Super-Halley method are among the efficient methods in solving non-linear equation along with Newton's and Halley method. Recently Osada ([9]), Neta ([10]), Chun and Neta ([11]), Ren and Argyros ([12]), Ivanov ([13]) and many others have studied iterative method for solving an equation of non-linear type having multiple roots.
Chebyshev method for multiple zeros ([1], [2], [3]) is defined as following:

$$C(w) = \begin{cases} w - \frac{p^2}{2}\frac{g(w)}{g'(w)}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right), & \text{if } g'(w) \neq 0, \\ w, & \text{otherwise.} \end{cases} \tag{1}$$

and Super-Halley method for multiple zeros [4] is defined as following:

$$S(w) = \begin{cases} w - \frac{g(w)}{2g'(w)}\left(p + \frac{1}{1-\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right), & \text{if } g'(w) \neq 0, \\ w, & \text{otherwise.} \end{cases} \tag{2}$$

Motivated by ongoing research in this field, we have combined the above two methods to construct the C-S Combined Mean Method.
In 2009 Proinov [[14]] established two forms of local convergence theorems about Newton's technique under two types of initial conditions. Recently, Proinov [[5], [6]] and later Ivanov [7] have introduced convergence theorems for the Picard iterative scheme given as below

$$w_{m+1} = T(w_m),\ m = 0, 1, 2, \ldots, \tag{3}$$

where $T : D \to M$ is the function of iteration defined in a metric space M and $D \subset M$. Later Proinov and Ivanov [[15]] and Ivanov [[8]] used the same two types of initial conditions to establish local convergence theorems about the Halleys method and the Chebyshevs method ((1)) for multiple polynomial zeros. In 2020 Ivanov [[7]] used the same two types of initial conditions to establish local convergence theorems of Super-Halley method (2) for multiple polynomial zeros.

Here, we investigate convergence of the C-S combined mean method for polynomial zeros which are multiple in nature with the help of the same initial conditions as in the Proinov [[5], [6], [14]] and Ivanov [7].
In this paper section 2 is devoted to the preliminaries, necessary in establishing our results. Construction of our C-S Combined mean method is presented in section 3. We have devoted Section 4 in establishing two types of local convergence analysis of the proposed C-S combined mean method.

2 Preliminaries

In this paper, J will be treated as an interval in the positive real line containing 0, that is J is of the form $[0, a]$, $[0, a)$ or $[0, \infty)$. $S_l(u)$ is the polynomial defined as

$$S_l(u) = 1 + u + u^2 + \ldots + u^{l-1}. \tag{4}$$

If $l = 0$, then here we take $S_l(u) = 0$. Here, we will use that 0^0 equal to 1.

Definition 1 ([5]) A function $\varphi : J \to R_+$ is called quasi-homogeneous of order $r \geq 0$ on J if there exists a non decreasing function $\Psi : J \to R_+$ such that

$$\varphi(u) = u^r \Psi(u) \; for \; all \; u \in \; J.$$

Following are some properties of the above function.

(P1) A function g is quasi-homogeneous function of degree $r = 0$ on J if and only if g is non-decreasing on J.
(P2) If f and g are quasi-homogeneous functions of degree $r \geq 0$ and $s \geq 0$ on J, then fg is quasi-homogeneous of degree $r + s$ on J.
(P3) If two functions f and g are quasi-homogeneous of degree $r \geq 0$ on J, then $f + g$ is also quasi-homogeneous of degree r on J.

We will use these properties in proving Lemmas and Theorems in the later section.

Definition 2 ([6]) A function $\varphi : J \to R_+$ is called gauge function of order $r \geq 1$ on J if it satisfies the following conditions:

(i) φ is quasi-homogeneous function of degree r on J.
(ii) $\varphi(u) \leq u$ for all $u \in J$.

A gauge function φ of order r on J is said to be a strict gauge function if the last inequality is strict whenever $u \in J \setminus \{0\}$.

Lemma 2.1 ([6]) If $\varphi : J \to R_+$ is a quasi-homogeneous function of degree $r \geq 1$ on an interval J and $R \in J \setminus \{0\}$ is a fixed point of φ, then φ is a gauge function of order r on $[0, R]$. Moreover, if $r > 0$, then φ is a strict gauge function on $[0, R)$.

Definition 3 ([5]) Let $T : D \subset M \to M$ be a map on an arbitrary set M. A function $I : D \to R_+$ is said to be a function of initial conditions of T (with gauge function φ on J) if there exists a function $\varphi : J \to J$ such that

$$I(T(w)) \leq \varphi(I(w)) \; with\; all\; w \in \; D \; with\; Tx \in \; D\; and\; I(w) \in \; J. \tag{5}$$

Definition 4 ([5]) Let $T : D \subset M \to M$ be a map on a arbitrary set M and $I : D \to R_+$ be a function of initial conditions of T with gauge function on J. Then, a point $w \in D$ is said to be an initial point of T if $I(w) \in J$ and all of the iterates $T^m w (m = 0, 1, 2, \ldots)$ are well defined and belong to D.

Definition 5 ([6]) Let $T : D \subset M \to M$ be an operator in a normed space $(M, \|.\|)$, and let $I : D \to R_+$ be a function of initial conditions of T with gauge function on J. Then T is said to be an iterated contraction with respect to I at a point $\zeta \in D$(with control function ϑ) if $I(\zeta) \in J$ and

$$\|Tx - \zeta\| \leq \vartheta(I(w))\|w - \zeta\| \;\; for\;\; all\; w \in \; D \; with\; I(w) \in \; J, \tag{6}$$

where $\vartheta : J \to [0, 1)$ is a non-decreasing function.
We will use the following two theorems of Ivanov ([7]) to establish our result.

Theorem 1 *([7]) Let $T : D \subset M \to M$ be an iteration function, $\zeta \in F$ and $I : F \to R_+$ defined by (12). Suppose $\phi : J \to R_+$ is a quasi-homogeneous function of degree $p \geq 0$ and for each $w \in F$ with $I(w) \in J$, the following two conditions are satisfied:*

(i) w belongs to the set D;
(ii) $\|Tx - \zeta\| \leq \phi(I(w))\|w - \zeta\|$.

Let also $w_0 \in F$ be an initial guess such that

$$I(w_0) \in J \; and\; \phi(I(w_0)) < 1, \tag{7}$$

then the following statements hold.

(i) Then the Picard iteration (3) is well defined and converges to ζ with order $r = p + 1$.
(ii) For all $m \geq 0$, we have the following error estimates:
$\|w_{m+1} - \zeta\| \leq \mu^{r^m}\|w_m - \zeta\|$ *and* $\|w_m - \zeta\| \leq \mu^{s_m(r)}\|w_0 - \zeta\|$,
where $\mu = \phi(I(w_0))$.
(iii) The Picard iteration (3) converges to ζ with Q-order $r = p+1$ and with the following error estimates:
$\|w_{m+1} - \zeta\| \leq (Rd)^{1-r}\|w_m - \zeta\|^r$ *for all* $m \geq 0$,
where R is the minimal solution of the equation $\phi(u) = 1$ in the interval $J \setminus \{0\}$.

Theorem 2 *([7]) Let $T : D \subset M \to M$ be an iteration function, $\zeta \in F$ and $I : D \subset F \to R_+$ defined by (32). Suppose $\vartheta : J \to R_+$ is a nonzero quasi-homogeneous function of degree $p \geq 0$ and for each $w \in F$ with $I(w) \in J$, the following two conditions are satisfied:*

(i) *w belongs to the set* D;
(ii) $\|Tx - \zeta\| \leq \vartheta(I(w))\|w - \zeta\|$.

Let also, $w_0 \in F$ *be an initial guess such that*

$$I(w_0) \in J \text{ and } \vartheta(I(w_0)) \leq \psi(I(w_0)), \tag{8}$$

where ψ *is defined by*

$$\psi(u) = 1 - u(1 + \vartheta(u)).$$

Then the Picard iteration (3) is well defined and converges to ζ *with the following error estimates:*

$$\|w_{m+1} - \zeta\| \leq \theta\mu^{r^m}\|w_m - \zeta\| \text{ and } \|w_m - \zeta\| \leq \theta^m\mu^{s_m(r)}\|w_0 - \zeta\| \text{ for all } m \geq 0, \tag{9}$$

where $\mu = \frac{\vartheta(I(w_0))}{\psi(I(w_0))}$ *and* $\theta = \psi(I(w_0))$. *In addition, if the second inequality in (8) is strict, then the order of convergence of Picard iteration (3) is at least* $r = p + 1$

3 Recurrence relation for the method

Here, we have derived a relation of the C-S Combination Method combining the two third order iterative method namely Chebyshev and Super-Halley method. For $g(w) \neq 0$, we define the C-S Combined Mean method as follows

$$\begin{aligned} T(w) &= \frac{1}{2}C(w) + \frac{1}{2}S(w) \\ &= w - \frac{p^2}{4}\frac{g(w)}{g'(w)}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right) + \frac{g(w)}{4g'(w)}\left(p + \frac{1}{1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right) \\ &= w - \frac{1}{2}\frac{g(w)}{g'(w)}\left[\frac{p^2}{2}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right) + \frac{1}{2}\left(p + \frac{1}{1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right)\right] \end{aligned}$$

Thus our C-S combined mean method has the following form

$$T(w) = \begin{cases} w - \frac{1}{2}\frac{g(w)}{g'(w)}\left[\frac{p^2}{2}\left(\frac{3-p}{p} + \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right) + \frac{1}{2}\left(p + \frac{1}{1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}}\right)\right], & \text{if } g'(w) \neq 0, \\ w, & \text{otherwise.} \end{cases} \tag{10}$$

The domain of the C-S Combined Mean iteration function T (10) is the set D, Which is defined below:

$$D = \{w \in F : g'(w) \neq 0 \Rightarrow 1 - \frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)} \neq 0\}. \tag{11}$$

4 Local Convergence of Combined Mean Method

Let $g \in F[w]$ be a polynomial which having degree $q(\geq 2)$, such that all the zeros of g are in F, and also let $\zeta \in F$ be a zero of the polynomial g such that

multiplicity of ζ is p.
Here $(F, \|.\|)$ denotes a field having a norm and $F[w]$ is the ring of polynomial on the field F.
Here, we examine the convergence of C-S Combined mean method (10) with the help of function of initial conditions I, which is a map from D from R_+ and is defined as follows:

$$I(w) = I_g(w) = \frac{\|(w-\zeta)\|}{d}, \tag{12}$$

here d represents the distance from ζ to the closest zero of g other than ζ; if ζ is a only zero of g then we set $I(w) = 0$.

Lemma 4.1 Let $g \in F[w]$ be a $q(\geq 2)$ degree polynomial having all zeros in F, where F is a field. If $\zeta_1, \ldots, \zeta_s$, are the all zeros of g, multiplicity of the zeros being $p_1, \ldots, p_s$, respectively. Then

(i) If $w \in F$ be such that for those w, $g(w) \neq 0$, then for any one of $i = 1, \ldots, s$, we have the following
$\frac{g'(w)}{g(w)} = \frac{p_i+\gamma_i}{w-\zeta_i}$, where $\gamma_i = (w-\zeta_i)\sum_{j\neq i}\frac{p_j}{w-\zeta_j}$.

(ii) If $w \in F$ is not of g and g', then for any $i = 1, \ldots, s$, we have
$\frac{g''(w)}{g'(w)} = \frac{(p_i+\gamma_i)^2-(p_i+\delta_i)}{(w-\zeta_i)(p_i+\gamma_i)}$, where $\delta_i = (w-\zeta_i)^2\sum_{j\neq i}\frac{p_j}{(w-\zeta_j)^2}$.

Proof

$$(i)\ From\ \frac{g'(w)}{g(w)} = \sum_{j=1}^{s}\frac{p_j}{w-\zeta_j},\ we\ have$$

$$\begin{aligned}
\frac{g'(w)}{g(w)} &= \sum_{j=1}^{s}\frac{p_j}{w-\zeta_j} \\
&= \frac{p_i}{w-\zeta_i} + \sum_{j\neq i}\frac{p_j}{w-\zeta_j} \\
&= \frac{p_i+\gamma_i}{w-\zeta_i},\ where\ \gamma_i = (w-\zeta_i)\sum_{j\neq i}\frac{p_j}{w-\zeta_j}.
\end{aligned}$$

Which proves the first part of the lemma.
(ii)Using the above identity and the following two identities

$$\frac{g''(w)}{g'(w)} = \frac{g'(w)}{g(w)} - \frac{g(w)}{g'(w)}\sum_{j=1}^{s}\frac{p_j}{(w-\zeta_j)^2}\ and\ \sum_{j=1}^{s}\frac{p_j}{(w-\zeta_j)^2} = \frac{p_i+\delta_i}{(w-\zeta_i)^2},$$

we get

$$\frac{g''(w)}{g'(w)} = \frac{(p_i+\gamma_i)^2-(p_i+\delta_i)}{(w-\zeta_i)(p_i+\gamma_i)},\ where\ \delta_i = (w-\zeta_i)^2\sum_{j\neq i}\frac{p_j}{(w-\zeta_j)^2}.$$

□

Lemma 4.2 Let $w, \zeta \in F$ and $\zeta_1, \dots, \zeta_s \in F$ be the list of all zeros of g which are other than ζ, then for any of $i = 1, \dots, s$, the inequality listed below is accurate.

$$\|w - \zeta_j\| \geq (1 - I(w))d, \tag{13}$$

where $I : F \to R_+$ is defined by (12).

Proof According to the definition of d we have $d \leq \|\zeta - \zeta_j\|$ for all $j = 1, \dots, s$. So using above and triangle inequality we have the following

$$\|w - \zeta_j\| = \|\zeta - \zeta_j + w - \zeta\| \geq \|\zeta - \zeta_j\| - \|w - \zeta\| \geq (1 - I(w))d.$$

□

4.1 First Kind of Local Convergence theorem

Here, $F[w]$ is the ring of polynomials over the field F. Let g be a polynomial of degree $q(\geq 2)$, which is in $F[w]$. In this section of the article we will establish the convergence of the C-S Combined mean method (10) using the function of initial condition $I : D \to R_+$ which is defined in (12).
Next, we define the functions ϕ_{sn}, ϕ_{sd} and ϕ_c.

$$\phi_{sn}(u) = (q - p)\left(\frac{2(q - p)u}{1 - u} + q\right)u^2, \tag{14}$$

$$\phi_{sd}(u) = \frac{2(p - qu)((2p - q)u^2 - 2pu + p)}{1 - u}, \tag{15}$$

$$\phi_c(u) = \frac{2(q - p)^3 u + p(q - p)(3q - 2p)}{2(p - qu)^3} u^2. \tag{16}$$

where $q \geq p$ and $p \geq 1$.
Clearly, ϕ_{sn} is positive in the clo-open interval $[0, 1)$. Easily we can show that ϕ_{sn} quasi-homogeneous on the clo-open interval $[0, 1)$ of degree 2. The second function ϕ_{sd} is decreasing as well as positive on the clo-open interval $[0, \tau)$, where τ is defined by

$$\tau = \begin{cases} \frac{p}{q}, & \text{if } q \geq 2p, \\ \frac{p}{p + \sqrt{p(q-p)}}, & \text{if } q < 2p. \end{cases} \tag{17}$$

Easily we can show that on the clo-open interval $[0, \frac{p}{q})$ the function ϕ_c is a second degree quasi-homogeneous function So, we can now define the function $\phi_s : [0, \tau) \to R_+$ defined by

$$\phi_s(u) = \frac{\phi_{sn}}{\phi_{sd}} = \frac{(q - p)(q + (q - 2p)u)u^2}{2(p - qu)((2p - q)u^2 - 2pu + p)}. \tag{18}$$

From the properties (P1) and (P2), we can easily say that on the interval $[0, \tau)$, ϕ_s is quasi-homogeneous of degree 2
Now, we define a function $\phi : [0, \tau) \to R_+$ defined by

$$\phi(u) = \frac{\phi_s(u)}{2} + \frac{\phi_c(u)}{2}. \tag{19}$$

As $\phi_s(u)$ and $\phi_c(u)$ are both second degree quasi homogeneous function, so by property (P3), ϕ is also quasi-homogeneous function of the same degree 2 in the clo-open interval $[0, \tau)$.

Lemma 4.3 Suppose that $g(w) \in F[w]$ be a $q(\geq 2)$ degree polynomial which splits over F, and let $\zeta \in F$ be a multiple zero of $g(w)$, multiplicity being p. Let $w \in F$ satisfies the following

$$I(w) < \tau, \tag{20}$$

where the function I is defined by (12) and τ is defined by (17). Then the following two statements (i) and (ii) are true.

(i) w is in D, the domain of the C-S Combined mean method and is defined in (11).
(ii) $\|Tx - \zeta\| \leq \phi(I(w))\|w - \zeta\|$, where ϕ defined in (19).

Proof Let $w \in F$ satisfy the inequality (20). If any of $p = q$ or $w = \zeta$ or both are true, then $Tx = \zeta$ and therefore both the statements of the lemma holds. So we assume that $p \neq q$ and $w \neq \zeta$. Let $\zeta_1, \ldots, \zeta_s$ be the list of all distinct zeros of g with multiplicities $p_1, \ldots, p_s$, respectively. Let $\zeta = \zeta_i, p = p_i, \gamma = \gamma_i$ and $\delta = \delta_i$ for some $1 \leq i \leq s$, where γ_i and δ_i defined in Lemma (4.1).
To prove the first part of the lemma we have to show that $g'(w) \neq 0$ implies $1 - \frac{g(w)}{g'(w)} \frac{g''(w)}{g'(w)} \neq 0$.

From Lemma (4.2) and equation (20), we get

$$\|w - \zeta_j\| \geq (1 - I(w))d > 0 \text{ ,as } \tau < 1 \tag{21}$$

for each $j \neq i$. Above assures $g(w) \neq 0$. Then, Lemma (4.1) gives the following

$$\frac{g'(w)}{g(w)} = \frac{p + \gamma}{w - \zeta}, \; where \; \gamma = (w - \zeta) \sum_{j \neq i} \frac{p_j}{w - \zeta_j}. \tag{22}$$

Using the triangle inequality and equation (21), we have the following

$$\|\gamma\| \leq \|w - \zeta\| \sum_{j \neq i} \frac{p_j}{\|w - \zeta_j\|} \leq \frac{\|w - \zeta\|}{(1 - I(w))d} \sum_{j \neq i} p_j = \frac{(q - p)I(w)}{1 - I(w)}. \tag{23}$$

Using the triangle inequality, equation(23) and as $I(w) < \tau \leq \frac{p}{q}$, we get the following

$$\|p + \gamma\| \geq p - \|\gamma\| \geq p - \frac{(q - p)I(w)}{1 - I(w)} = \frac{p - qI(w)}{1 - I(w)} > 0. \tag{24}$$

Hence, $p + \gamma \neq 0$. This implies $g'(w) \neq 0$.
Then, from the Lemma (4.1), we have the following

$$\frac{g''(w)}{g'(w)} = \frac{(p + \gamma)^2 - (p + \delta)}{(w - \zeta)(p + \gamma)}, \; where \; \delta = (w - \zeta)^2 \sum_{j \neq i} \frac{p_j}{(w - \zeta_j)^2}. \tag{25}$$

Now, from (22) and (25), we have

$$1 - \frac{g(w)}{g'(w)} \frac{g''(w)}{g'(w)} = 1 - \frac{(w-\zeta)}{p+\gamma} \frac{(p+\gamma)^2 - (p+\delta)}{(w-\zeta)(p+\gamma)} = \frac{p+\delta}{(p+\gamma)^2}. \tag{26}$$

Therefore, by triangle inequality, (21) and $I(w) < \tau$, we have the following estimate

$$\|\delta\| \leq \frac{(q-p)I(w)^2}{(1-I(w))^2} \; and \; \|p+\delta\| \geq p - \|\delta\| \geq \frac{\phi_{sd}(I(w))}{2(p-nE(w))(1-I(w))} > 0. \tag{27}$$

From above, we conclude that

$$\left\| 1 - \frac{g(w)}{g'(w)} \frac{g''(w)}{g'(w)} \right\| > 0.$$

Therefore we can say that $w \in D$. Which proves (i). From the construction of the C-S Combined mean method, we have the following

$$\begin{aligned}
Tx - \zeta &= w - \zeta - \frac{p^2}{4} \frac{(w-\zeta)}{(p+\gamma)} \left[\frac{3-p}{p} + \frac{(p+\gamma)^2 - (p+\delta)}{(p+\gamma)^2} \right] \\
&\quad - \frac{(w-\zeta)}{4(p+\gamma)} \left[p + \frac{(p+\gamma)^2}{p+\delta} \right] \\
&= \frac{(w-\zeta)}{2} \left[1 - \frac{p}{2} \frac{3(p+\gamma)^2 - p(p+\delta)}{(p+\gamma)^3} \right] \\
&\quad + \frac{(w-\zeta)}{2} \left[1 - \frac{2p^2 + p\delta + 2p\gamma + \gamma^2}{2(p+\gamma)(p+\delta)} \right] \\
&= \frac{(w-\zeta)}{2} \left[\frac{2\gamma^3 + 3p\gamma^2 + p^2\delta}{2(p+\gamma)^3} \right] + \frac{(w-\zeta)}{2} \left[\frac{2\gamma\delta + p\delta - \gamma^2}{2(p+\gamma)(p+\delta)} \right] \\
&= \kappa(w-\zeta),
\end{aligned}$$

where

$$\kappa = \frac{1}{2} \left(\left[\frac{2\gamma^3 + 3p\gamma^2 + p^2\delta}{2(p+\gamma)^3} \right] + \left[\frac{2\gamma\delta + p\delta - \gamma^2}{2(p+\gamma)(p+\delta)} \right] \right). \tag{28}$$

Using (23), (24) and (27), we now estimate $\|\kappa\|$ and is as follows.

$$\begin{aligned}
\|\kappa\| &\leq \frac{1}{2} \left(\left\| \left[\frac{2\gamma^3 + 3p\gamma^2 + p^2\delta}{2(p+\gamma)^3} \right] \right\| + \left\| \left[\frac{2\gamma\delta + p\delta - \gamma^2}{2(p+\gamma)(p+\delta)} \right] \right\| \right) \\
&\leq \frac{1}{2} \left(\left[\frac{2\|\gamma\|^3 + 3p\|\gamma\|^2 + p^2\|\delta\|}{2\|(p+\gamma)\|^3} \right] + \left[\frac{2\|\gamma\|\|\delta\| + p\|\delta\| + \|\gamma\|^2}{2\|(p+\gamma)\|\|(p+\delta)\|} \right] \right) \\
&\leq \frac{2\left(\frac{(q-p)I(w)}{1-I(w)}\right)^3 + 3p\left(\frac{(q-p)I(w)}{1-I(w)}\right)^2 + p^2\frac{(q-p)I(w)^2}{(1-I(w))^2}}{4\left(\frac{p-qI(w)}{1-I(w)}\right)^3} \\
&\quad + \frac{2\frac{(q-p)I(w)}{1-I(w)}\frac{(q-p)I(w)^2}{(1-I(w))^2} + p\frac{(q-p)I(w)^2}{(1-I(w))^2} + \left(\frac{(q-p)I(w)}{1-I(w)}\right)^2}{4\frac{p-qI(w)}{1-I(w)}\frac{\phi_{sd}(I(w))}{2(p-qI(w))(1-I(w))}} \\
&= \frac{1}{2} \left[\frac{\phi_{sn}(I(w))}{\phi_{sd}(I(w))} + \phi_c(I(w)) \right] \\
&= \frac{\phi_c(I(w))}{2} + \frac{\phi_c(I(w))}{2} = \phi(I(w)).
\end{aligned}$$

Which proves (ii). □

Theorem 3 *Let $g \in F[w]$ be a polynomial of degree $q \geq 2$ that splits over F, and let $\zeta \in F$ be a zero of g such that the multiplicity of ζ is p. Let $w_0 \in F$ satisfies the following initial condition*

$$I(w_0) < \tau \text{ and } \phi(I(w_0)) < 1, \tag{29}$$

where $I : F \to R_+$ is defined in (12) and ϕ is defined in (19). Then the following three statements are true.

(i) Iterative sequence (10) of the C-S Combined mean method is defined and converges to ζ having order of convergence 3.

(ii) Error estimates are as follows

$$\|w_{m+1}-\zeta\| \leq \mu^{3^m} \|w_m-\zeta\| \text{ and } \|w_m-\zeta\| \leq \mu^{(3^m-1)/2}\|w_0-\zeta\|, \text{for all} m \geq 0, \tag{30}$$

where $\mu = \phi(I(w_0))$.

(iii) A posteriori error estimate given below

$$\|w_{m+1} - \zeta\| < \frac{1}{(Ud)^2}\|w_m - \zeta\|^3, \text{for all} m \geq 0, \tag{31}$$

where $U \in (0, \tau)$ is the unique solution of $\phi(u) = 1$ in $(0, \tau)$.

Proof Lemma (4.3) and theorem (1) gives the proof. □

4.2 Second Kind of Local Convergence theorem

Let $g \in F[w]$ be a polynomial which having degree $q(\geq 2)$, such that all the zeros of g are in F, and also let $\zeta \in F$ be a zero of the polynomial g, multiplicity of ζ being p.
Here $(F, \|.\|)$ denotes a field having a norm and $F[w]$ is the ring of polynomial on the field F.
Here, we examine the convergence of C-S Combined mean method (10) with the help of function of initial conditions I, which is a map from D from R_+ and is defined as follows:

$$I(w) = I_g(w) = \frac{\|(w - \zeta)\|}{\rho(w)}, \tag{32}$$

here $\rho(w)$ represents the distance from the zero w to the closest zero of g other than ζ; if ζ is a only zero of g then we set $I(w) = 0$.
Now, we define two real functions ϑ_s and ϑ_c, for $q > p \geq 1$, by

$$\vartheta_s(u) = \frac{(q-p)(q+2(q-p)u)u^2}{2(p-(q-p)u)(p-(q-p)u^2)} \tag{33}$$

and

$$\vartheta_c(u) = \frac{2(q-p)^3u^3 + p(q-p)(3q-2p)u^2}{2(p-(q-p)u)^3}. \tag{34}$$

Clearly, the functions ϑ_s and ϑ_c are quasi-homogeneous functions of degree 2 on $[0, \tau_1]$, where τ_1 is defined by

$$\tau_1 = \begin{cases} \frac{p}{q-p}, & \text{if } q \geq 2p, \\ \sqrt{\frac{p}{q-p}}, & \text{if } q < 2p. \end{cases} \tag{35}$$

Now, we can define a function $\vartheta : [0, \tau_1) \to R_+$ defined by

$$\vartheta(u) = \frac{\vartheta_s(u)}{2} + \frac{\vartheta_c(u)}{2}. \tag{36}$$

As both the functions $\vartheta_s(u)$ and $\vartheta_c(u)$ are quasi-homogeneous, therefore by property (P3) we can say that ϑ is quasi-homogeneous of degree 2 in the interval $[0, \tau_1)$.

Lemma 4.4 Let $g \in F[w]$ be a polynomial of degree $q(\geq 2)$ which splits over F, and let $\zeta \in F$ be a zero of g with multiplicity p. Let $w \in F$ be such that

$$I(w) < \tau_1, \tag{37}$$

where the function I is defined by (32).Then:

(i) w is in D, the domain of the C-S Combined mean method and is defined in (11).
(ii) $\|Tx - \zeta\| \leq \vartheta(I(w))\|w - \zeta\|$, where ϑ is defined in (36).

Proof Let $w \in F$ satisfy the inequality (20). If any of $p = q$ or $w = \zeta$ or both are true, then $Tx = \zeta$ and therefore both the statements of the lemma holds. So we assume that $p \neq q$ and $w \neq \zeta$. Let $\zeta_1, \ldots, \zeta_s$ be the list of all distinct zeros of g with multiplicities $p_1, \ldots, p_s$, respectively. Let $\zeta = \zeta_i, p = p_i, \gamma = \gamma_i$ and $\delta = \delta_i$ for some $1 \leq i \leq s$, where γ_i and δ_i defined in Lemma (4.1).
To prove the first part of the lemma we have to show that $g(w) \neq 0$ and $g'(w) \neq 0$ implies $1 - \frac{g(w)}{g'(w)} \frac{g''(w)}{g'(w)} \neq 0$. Clearly we can write the following

$$\|w - \zeta_j\| \geq \rho(w) > 0 \tag{38}$$

for each $j \neq i$. This assures that $g(w) \neq 0$. Then, Lemma (4.1) gives the following

$$\frac{g'(w)}{g(w)} = \frac{p + \gamma}{w - \zeta}, \; where \, \gamma = (w - \zeta) \sum_{j \neq i} \frac{p_j}{w - \zeta_j}. \tag{39}$$

Using the triangle inequality and (38), we have the following:

$$\|\gamma\| \leq \|w - \zeta\| \sum_{j \neq i} \frac{p_j}{\|w - \zeta_j\|} \leq \frac{\|w - \zeta\|}{\rho(w)} \sum_{j \neq i} p_j = (q - p)I(w). \tag{40}$$

Using the triangle inequality, equation (40) and $I(w) < \tau_1$, we have the following:

$$\|p + \gamma\| \geq p - \|\gamma\| \geq p - (q - p)I(w) > 0. \tag{41}$$

Hence, $p+\gamma \neq 0$. This implies $g'(w) \neq 0$.
Then from the Lemma (4.1), we have the following

$$\frac{g''(w)}{g'(w)} = \frac{(p+\gamma)^2-(p+\delta)}{(w-\zeta)(p+\gamma)}, \; where \; \delta = (w-\zeta)^2 \sum_{j\neq i} \frac{p_j}{(w-\zeta_j)^2}. \tag{42}$$

Now from (39) and (42), we have

$$1-\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)} = 1-\frac{(w-\zeta)}{p+\gamma}\frac{(p+\gamma)^2-(p+\delta)}{(w-\zeta)(p+\gamma)} = \frac{p+\delta}{(p+\gamma)^2}. \tag{43}$$

Therefore, by triangle inequality, (38) and $I(w) < \tau_1$, we have the following estimate

$$\|\delta\| \leq (q-p)I(w)^2 \; and \; \|p+\delta\| \geq p-\|\delta\| \geq p-(q-p)I(w)^2 \geq 0. \tag{44}$$

From above, we conclude that

$$\left\|1-\frac{g(w)}{g'(w)}\frac{g''(w)}{g'(w)}\right\| > 0.$$

Therefore we can say that $w \in D$. Which proves (i) From the construction of the C-S Combined mean method, we have the following

$$\begin{aligned}
Tx-\zeta &= w-\zeta-\frac{p^2}{4}\frac{(w-\zeta)}{(p+\gamma)}\left[\frac{3-p}{p}+\frac{(p+\gamma)^2-(p+\delta)}{(p+\gamma)^2}\right] \\
&- \frac{(w-\zeta)}{4(p+\gamma)}\left[p+\frac{(p+\gamma)^2}{p+\delta}\right] \\
&= \frac{(w-\zeta)}{2}\left[1-\frac{p}{2}\frac{3(p+\gamma)^2-p(p+\delta)}{(p+\gamma)^3}\right] \\
&+ \frac{(w-\zeta)}{2}\left[1-\frac{2p^2+p\delta+2p\gamma+\gamma^2}{2(p+\gamma)(p+\delta)}\right] \\
&= \frac{(w-\zeta)}{2}\left[\frac{2\gamma^3+3p\gamma^2+p^2\delta}{2(p+\gamma)^3}\right]+\frac{(w-\zeta)}{2}\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right] \\
&= \kappa(w-\zeta),
\end{aligned}$$

where

$$\kappa = \frac{1}{2}\left(\left[\frac{2\gamma^3+3p\gamma^2+p^2\delta}{2(p+\gamma)^3}\right]+\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right]\right). \tag{45}$$

We now use the estimates (40), (41) and (44) to estimate $\|\kappa\|$ and is given as following

$$\begin{aligned}
\|\kappa\| &\leq \frac{1}{2}\left(\left\|\left[\frac{2\gamma^3+3p\gamma^2+p^2\delta}{2(p+\gamma)^3}\right]\right\|+\left\|\left[\frac{2\gamma\delta+p\delta-\gamma^2}{2(p+\gamma)(p+\delta)}\right]\right\|\right) \\
&\leq \frac{1}{2}\left(\left[\frac{2\|\gamma\|^3+3p\|\gamma\|^2+p^2\|\delta\|}{2\|(p+\gamma)\|^3}\right]+\left[\frac{2\|\gamma\|\|\delta\|+p\|\delta\|+\|\gamma\|^2}{2\|(p+\gamma)\|\|(p+\delta)\|}\right]\right) \\
&\leq \frac{2\left((q-p)I(w)\right)^3+3p\left((q-p)I(w)\right)^2+p^2(q-p)I(w)^2}{4\left(p-(q-p)I(w)\right)^3} \\
&+ \frac{2(q-p)I(w)(q-p)I(w)^2+p(q-p)I(w)^2+\left((q-p)I(w)\right)^2}{4(p-(q-p)I(w))(p-(q-p)I(w)^2)} \\
&= \frac{2(q-p)^3I(w)^3+p(q-p)(3q-2p)I(w)^2}{2(p-(q-p)I(w))^3}
\end{aligned}$$

$$+ \frac{(q-p)(q+2(q-p)I(w))I(w)^2}{2(p-(q-p)I(w))(p-(q-p)I(w)^2)}$$
$$= \frac{\vartheta_c(I(w))}{2} + \frac{\vartheta_s(I(w))}{2} = \vartheta(I(w)).$$

Proof of the lemma is therefore completed. □

Next, we state the convergence theorem second type.

Theorem 4 *Let $g \in F[w]$ be a polynomial of degree $q \geq 2$ which splits over F, and let $\zeta \in F$ be a zero of g such that the multiplicity of ζ is p. Let $w_0 \in F$ satisfies the following initial conditions*

$$I(w_0) < \tau_1 \text{ and } \vartheta(I(w_0)) \leq \psi(I(w_0)), \tag{46}$$

where the function I is defined in (32) and the function ψ is defined below as

$$\psi(u) = 1 - u(1 + \vartheta(u)). \tag{47}$$

Then, the C-S Combined mean method is defined and converges to ζ having the following error estimates

$$\|w_{m+1} - \zeta\| \leq \theta\mu^{3^m} \|w_m - \zeta\| \text{ and } \|w_{m+1} - \zeta\| \leq \theta^m \mu^{(3^m-1)/2} \|w_0 - \zeta\| \text{ forall } m \geq 0, \tag{48}$$

where $\theta = \psi(I(w_0))$ and $\mu = \frac{\vartheta(I(w_0))}{\psi(I(w_0))}$.

Proof Lemma (4.4) and Theorem (2) guarantees the proof. □

5 Conclusion

In the first part of this study, we combine the Chebyshev and Super-Halley methods to create a method for solving nonlinear equations. Secondly, we have showed the local convergence of the proposed method. This concept can be used in combining other methods in order to get more accurate results and rapid convergence.

Acknowledgments. We acknowledge the anonymous reviewers' thorough examination of our article as well as their many insightful remarks and suggestions.

Declarations

- Funding : No external funding has been received for this research.
- Conflict of interest/Competing interests : The authors declare no conflict of interest/competing interests.

References

[1] Chebychev, P.L.: Computation roots of an equation. In: Complete Works of P.L.Chebishev, Vol. 5, pp. 725. USSR Academy of Sciences, Moscow (in Russian) http://books.e-heritage.ru/book/10079542 (1838).

[2] J. F. Traub, Iterative Methods for the Solution of Equations, 2nd edn. Chelsea Publishing Company, New York (1982).

[3] N. Obreshkov, On the numerical solution of equations (in Bulgarian). Annuaire Univ. Sofia Fac. Sci. Phys. Math. 56, 7383 (1963).

[4] J. M. Gutirrez, M. A. Hernndez, An acceleration of Newtons method: Super-Halley method. Appl. Math. Comput. 2001, 117, 223239.

[5] P. D. Proinov, New general convergence theory for iterative processes and its applications toNewtonKantorovich type theorems. J. Complex. 2010, 26, 342.

[6] P. D. Proinov, General convergence theorems for iterative processes and applications to the Weierstrass root-finding method. arXiv:1503.05243 (2015).

[7] S. I. Ivanov, General Local Convergence Theorems about the Picard Iteration in Arbitrary Normed Fields with Applications to Super-Halley Method for Multiple Polynomial Zeros. Mathematics 2020, 8, 11, doi:10.3390/math8091599.

[8] S.I. Ivanov, On the convergence of Chebyshevs method for multiple polynomial zeros, Results Math. 69 (1) (2016) 93103, http://dx.doi.org/10.1007/s00025-015-0490-y.

[9] N. Osada, Asymptotic error constants of cubically convergent zero finding methods. J. Comput. Appl. Math. 196, 347357 (2006).

[10] B. Neta, New third order nonlinear solvers for multiple roots. Appl. Math. Comput. 202, 162170 (2008).

[11] C. Chun, B. Neta, A third-order modification of Newtons method for multiple roots. Appl. Math. Comput. 211, 474479 (2009).

[12] H. Ren, I. K. Argyros, Convergence radius of the modified Newton methodfor multiple zeros under Holder continuous derivative. Appl. Math. Comput.217, 612621 (2010)

[13] Stoil I. Ivanov, Unified Convergence Analysis of ChebyshevHalley Methods for Multiple Polynomial Zeros. Mathematics 10, no. 1 (2022): 135.

[14] P.D. Proinov, General local convergence theory for a class of iterative processes and its applications to Newtons method. J.Complex. 2009, 25, 3862.

[15] P.D. Proinov, S.I. Ivanov, On the convergence of Halleys method for multiple polynomial zeros. Mediterr. J. Math. 2015, 12, 555572.