Some Fixed Point Theorems in $e-$ Complete E-Metric Space

Original Research Article

Abstract

In this study, we present fixed-point results for various contractions in $\mathtt{E}-$ Metric Space along with several corollaries. Finding a fixed point result in $\mathtt{E}-$ Metric Space can also be done via implicit relations. In the investigation of our primary result stands, all the theorems will be in non-solid cone and we cannot assume interior point is non-empty.

Keywords: $\mathtt{E}-$*Metric Space; e−Complete; Non-Solid Cones; Semi-interior points.*

2010 Mathematics Subject Classification: 47H09; 47H10; 54H25.

1 Introduction

The notion of Cone metric space introduced by Huang and Zhang [4] replacing the real numbers by an ordered Banach space, also they discovered fixed point theorems in cone metric space using contraction mapping. In 2008, Rezapour and Hamlbarani [12] redefined fixed point result without assumptions of normality of cone in Banach space. Al-Rawashdeh et.al. [1] introduced the concept of $\mathtt{E}-$metric space in 2012. TVS-valued cone Banach space were defined by Mehmood et al.[9] in 2014.

Typically, the Banach space $\mathtt{E}$ was taken into account with a defined order in relation to the positive solid cone $\mathtt{E}^+$ of $\mathtt{E}$, i.e., by assuming that interior of $\mathtt{E}^+$ is not empty. The non-solid cones were just briefly mentioned in a few results [6, 7]. When non-solid cones are involved, [7] took into account the quasi interior points of P rather than interior points.

In 2017, Polyrakis [2] introduced the notions of semi-interior points and see some other article related to cone and $\mathtt{E}-$metric space [3, 5, 7]. Mehmood et al. [8] generalized the Banach,Kannan and Chatterjea contractions in the setting of $\mathtt{E}-$metric space via non-solid cones.

In this article, we develope Ciric type and some other different kinds of contraction in the setting of $\mathtt{E}-$metric space. Also finally define some implicit relation in the setting of $\mathtt{E}-$metric space.

A normed space $\mathtt{E}$ with $\|\cdot\|$, its ordered by the positive cone $\mathtt{E}^+$, then $u, v \in \mathtt{E}$, *such that* $u \preceq v$ iff $v - u \in \mathtt{E}^+$.

Definition 1.1. [1] An ordered space $\mathtt{E}$ is a vector space over $\mathbb{R}$ with a partial order relation $\preceq$ such that

(OS_1) $\forall u, v \ \ \& \ \ z \in \mathtt{E}, if \ \ u \preceq v \implies u + z \preceq v + z,$

(OS_2) $\forall \upsilon \in \mathbb{R}^+ \ \ \& \ \ u \in \mathtt{E}, u \succeq 0_{\mathtt{E}}, \upsilon u \succeq 0_{\mathtt{E}}.$

Likewise, if $\mathtt{E}$ contains a norm, it is referred to as a normed ordered space.

Definition 1.2. A normed ordered space $\mathtt{X}$'s positive cone $\mathtt{E}^+$ is known as:

1. Normal, $\exists \ \ N > 0 \ni$
$$0 \leq u \leq v \implies \|u\| \leq N\|v\|$$
$\forall u, v \in \mathtt{X}$.
2. Solid, then $\mathtt{E}^+$ has non-empty interior,

3 Reflexive, if and only if $\mathtt{E}^+ \cap U$ is weakly compact, where U is the unit ball in $\mathtt{X}$,

4 Strongly reflexive, if and only if $\mathtt{E}^+ \cap U$ is compact.

In the above definition, reflexive and strongly reflexive cones can be defined in [7].

Definition 1.3. [1] Given that $\mathtt{E}$ is an ordered space over real scalars, let $\mathtt{X}$ be a nonempty set. For all u, v and z in $\mathtt{X}$, an ordered $\mathtt{E}$-metric on $\mathtt{X}$, denoted by the $\mathtt{E}-$ valued function $p^{\mathtt{E}} : \mathtt{X} \times \mathtt{X} \to \mathtt{E}$, we have

(e_1) $p^{\mathtt{E}}(u, v) \geq 0_{\mathtt{E}}$ and $p^{\mathtt{E}}(u, v) = 0_{\mathtt{E}}$ if and only if $u = v$,

(e_2) $p^{\mathtt{E}}(u, v) = p^{\mathtt{E}}(v, u)$,

(e_3) $p^{\mathtt{E}}(u, v) \preceq p^{\mathtt{E}}(u, z) + p^{\mathtt{E}}(z, v)$.

Then the tuple $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ is identified as $\mathtt{E}$-metric space.

The definition that follows presumes that int $\left(\mathtt{E}^+\right)$ is not empty.

Definition 1.4. [8] Consider the $\mathtt{E}-$ metric space of an normed ordered space $\mathtt{E}$ and (u_n) in $\mathtt{X}$ is called convergent to $u \in \mathtt{X}$ *then,* $\forall \ c \in$ int $\left(\mathtt{E}^+\right), \exists$ a natural number $\mathbb{N}$,

$$p^{\mathtt{E}}\left(u_n, u\right) \ll c$$

$\forall n \geq \mathbb{N}$,

$$\lim_{n \to \infty} u_n = u,$$

or $u_n \to u$. The sequence (u_n) is called Cauchy, $\forall c \in$ int $\left(\mathtt{E}^+\right), \exists$ a natural number N_1 such that

$$p^{\mathtt{E}}\left(u_n, u_m\right) \ll c$$

$\forall n, m \leq N_1$

Definition 1.5. [2] Let $\mathtt{E}$ be a normed space with orders determined by the positive cone $\mathtt{E}^+$. We will represent the zero of $\mathtt{E}$ by the symbol $0_{\mathtt{E}}$. Let $U = \{u \in \mathtt{E} \ni \|u\| \leq 1\}$ represent the closed unit ball of $\mathtt{E}$, and by U^+ we refer to the positive portion of the unit ball formed by the set

$$U^+ = U \cap \mathtt{E}^+.$$

Definition 1.6. [2] A point $u_0 \in \mathtt{E}^+$is a semi-interior point of $\mathtt{E}^+$ if $\exists \rho > 0$ such that

$$u_0 - \rho U^+ \subseteq \mathtt{E}^+$$

Clearly any interior point of $\mathtt{E}^+$is semi-interior point. The Collection of all semi-interior points of $\mathtt{E}^+$ as indicated by $\left(\mathtt{E}^+\right)^{\ominus}$,

$$\forall u, v \in \mathtt{E}^+, u \lll v \iff v - u \in \left(\mathtt{E}^+\right)^{\ominus}$$

.

Example 1.1. *[2] Let* $\mathtt{X_n}$ *be the space of* $\mathbb{R}^2$ *ordered by the pointwise ordering and with norm* $\|u_n\|_n$ *having unit ball. The polygon* D_n *of* $\mathbb{R}^2$ *with vertices* $(1,0),(0,1),(-n,n),(-1,0),(0,-1),(n,-n)$. *The norm is given by*

$$\|(u,v)\|_n = \begin{cases} |u|+|v|, & \text{if } uv \geq 0 \\ \max\{|u|,|v|\} - \frac{n-1}{n}\min\{|u|,|v|\}, & \text{if } uv < 0. \end{cases}$$

Suppose that $\mathtt{E}$ *is the space of all the sequences* $(u_n) \ni u_n = (u_1^n, u_2^n) \in \mathtt{u_n}$ *with* $\|u_n\|_n \leq \gamma(u) \forall n \in \mathbb{N}$ *and* $\gamma > 0$, *depends on* u. *Assume that* $\mathtt{E}$ *is ordered by cone* $\mathtt{E}^+ = \{u = (u_n) \in \mathtt{X} : u_n \in \mathbb{R}_+^2 \text{ for any } n\}$ *and equipped with the norm*

$$\|u\|_\infty = \sup_{n\in\mathbb{N}} \|u_n\|_n .$$

Also, suppose that $\mathtt{X} = \mathtt{E}^+ - \mathtt{E}^+$ *is the subspace of* $\mathtt{E}$ *generated by* $\mathtt{E}^+$ *and ordered by* $\mathtt{E}^+$. *Let* $1 = (v_n)$, *where* $v_n = (1,1)$ *for any* $n \in \mathbb{N}$. *Then 1 is not an interior point of* $\mathtt{E}^+$. *For any* $m \in \mathbb{N}$ *and* $v = (v_n) \in \mathtt{X} \ni$,

$$v = (v_n) = \begin{cases} \zeta_n = (-2,2), & \text{for } n = m \\ v_n = (0,0), & \text{for } n \neq m \end{cases}$$

It is easy to show $\|v\|_\infty = \frac{2}{m}$ *and* $1 + v \notin \mathtt{E}^+$, *even though any* $u \in \mathtt{E}^+$ *is not an interior point of* $\mathtt{E}^+$, *therefore* $\mathtt{E}^+$ *has an empty interioc. The positive part of closed unit ball is*

$$U^+ = \{u \in \mathtt{E}^+ : \|u\|_\infty \leq 1\}.$$

For any $u = (u_n) \in U^+$ *it is easy to see that* $\|u_n\|_n = u_n^1 + u_n^2 \leq 1$,

$$1 - U^+ \subseteq \mathtt{E}^+$$

and 1 is a semi-interior point of $\mathtt{E}^+$. *Also the space* $\mathtt{X}$ *is not complete using Proposition 2.4 of [2].*

Example 1.2. *In Example 2.7 of [2], (Example 5.9 of [3]) it has been shown that a strong reflexive cone* $\mathtt{E}^+$ *of* $L_1([0,1])$ *exists which generate a dense subspace* $\mathtt{X}$ *of* $L_1([0,1])$, *i.e.,*

$$\mathtt{X} = \mathtt{E}^+ - \mathtt{E}^+$$

and

$$L_1([0,1]) = \bar{\mathtt{X}}.$$

Let $V = co\left(\left(B^+(0,1)\right) \cup \left(-B^+(0,1)\right)\right)$ *and* E_1^+ *be a set of positive elements of* $L_1([0,1])$ *generated by the set* $\Omega = 3v + V$, *where* $v = \sum_{k=1}^\infty \alpha^{k-1} e_k$ *for* $\alpha \in (0,1)$, *where* $\{e_i\}$ *is the set of standard normalized basis. It has been shown in [2], that the cone* E^+ *has empty interior but has semi-interior points.*

Hence every interior point of the cone $\mathtt{E}^+$ *is the semi-interior point if int* $(\mathtt{E}^+)$ *is nonempty.*

Proposition 1.1. *[2] Any point that is semi-interior to a closed cone* $\mathtt{E}^+$ *in a complete ordered normed space* $\mathtt{E}$ *is also an interior point of* $\mathtt{E}^+$.

Definition 1.7. [8] Assume that $(\mathtt{E}^+)^\ominus$ is non-empty and $\mathtt{E}$ is the ordered normed space with $(\mathtt{X}, p^{\mathtt{E}})$ be an $\mathtt{E}-$ metric space. For any $(u_n) \in \mathtt{X}$ and $u \in \mathtt{X}$, then

(i) (u_n) e-converges to u when for every $e \ggg 0_E$, $\exists$ a natural number $u_n \xrightarrow{e} u$

(ii) (u_n) is an e-Cauchy sequence when for every $e \ggg 0_{\mathtt{E}}$, $\exists\ k \in \mathbb{N}\ \ni p^{\mathtt{E}}(u_n, u_m) \ll e$ for all $n, m \geq k$.

(iii) $(\mathtt{X}, p^{\mathtt{E}})$ is e-complete if every e-Cauchy sequence is e-convergent.

Theorem 1.3. *[8] Let $G : \mathtt{X} \to \mathtt{X}$ be a mapping and $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ be an e-complete $\mathtt{E}$-metric space with positive closed cone $\mathtt{E}^{+}$ satisfying*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq a p^{\mathtt{E}}(u, v)$$

$\forall u, v \in \mathtt{X}$ and $a \in [0, 1)$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$ and for each $u \in \mathtt{X}$, the sequence of iteration $(\mathtt{T}^{\mathrm{n}} u)_{n>0}$ converges to a fixed-point of T.

Corollary 1.4. *[8] Consider an e-complete $\mathtt{E}$-metric space $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ with positive closed cone $\mathtt{E}^{+} \ni \left(\mathtt{E}^{+}\right)^{\ominus} \neq \varphi$.*
For $e \ggg 0_{\mathtt{E}}, u_0 \in \mathtt{X}$, set $B(u_0, e) = \left\{u \in \mathtt{X} : p^{\mathtt{E}}(u, v) \preceq e\right\}$ and $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ be a mapping such that

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq a p^{\mathtt{E}}(u, v)$$

$\forall u, v \in B(u_0, e)$, where $a \in [0, 1)$ and $p^{\mathtt{E}}(u_0, \mathtt{G}u_0) \preceq (1 - a)e$. Then $\mathtt{G}$ has a unique fixed point in $B(u_0, e)$.

Corollary 1.5. *[8] Let $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ be an $e-$ complete $\mathtt{E}$-metric space with positive closed cone $\mathtt{E}^{+}$ such that $\left(\mathtt{E}^{+}\right)^{\ominus} \neq \varphi$. Let for some $n \in \mathbb{N}$, the mapping $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ satisfies*

$$p^{\mathtt{E}}\left(\mathtt{T}^{\mathrm{n}} u, \mathtt{T}^{\mathrm{n}} v\right) \preceq a p^{\mathtt{E}}(u, v)$$

for all $u, v \in \mathtt{X}$, where $a \in [0, 1)$ is a constant. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$.

Theorem 1.6. *[8] Let $\left(\mathtt{X}, p^{\mathtt{E}}\right)$ be an $e-$ complete $\mathtt{E}$-metric space with positive closed cone $\mathtt{E}^{+}$ such that $\left(\mathtt{E}^{+}\right)^{\ominus} \neq \varphi$. A mapping $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq \theta\left[p^{\mathtt{E}}(\mathtt{G}u, u) + p^{\mathtt{E}}(\mathtt{G}v, v)\right]$$

$\forall u, v \in \mathtt{X}$ and some $\theta \in \left[0, \frac{1}{2}\right)$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$, and for each $u \in \mathtt{X}$, $(\mathtt{T}^{\mathrm{n}} u)_{n \geq 0}$ converges to a fixed point of $\mathtt{X}$.

2 Main Result

Theorem 2.1. *Consider an be a normed space $\mathtt{E}$ and it's ordered by a closed positive cone $\mathtt{E}^{+}$. A function $\mathtt{G} : \mathtt{X} \to \mathtt{X}$, where $\mathtt{X}$ is an $e-$complete $\mathtt{E}-$metric space with positive closed cone $\mathtt{E}^{+}$ such that $(\mathtt{E}^{+})^{\ominus} \neq \varphi$, it satisfies*

$$\min\{p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v), p^{\mathtt{E}}(u, \mathtt{G}u), p^{\mathtt{E}}(v, \mathtt{G}v)\} - \min\{p^{\mathtt{E}}(u, \mathtt{G}v), p^{\mathtt{E}}(v, \mathtt{G}u)\} \preceq a p^{\mathtt{E}}(u, v) \tag{2.1}$$

$\forall u, v \in \mathtt{X}$ & $a \in (0, 1)$ and $p^{\mathtt{E}}(u, u_n) \ll (1 - a)\frac{\tau}{m}$. Then $\{T^n u\}$ converges to a fixed point of $\mathtt{G}$

Proof. Consider the iterative sequence

$$u_{n+1} = \mathtt{G}u_n = T^n u_0$$

where $u_0 \in \mathtt{X}$, with $u_n \neq u_{n+1} \forall n \in \mathbb{N}$ for $n \in \mathbb{N}$. Substitute $u = u_n, v = u_{n+1}$ in equation (2.1), we get

$$\min\{p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n+1}), p^{\mathtt{E}}(u_n, \mathtt{G}u_n), p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_{n+1})\} - \min\{p^{\mathtt{E}}(u_n, \mathtt{G}u_{n+1}), p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_n)\} \preceq a p^{\mathtt{E}}(u_n, u_{n+1})$$
$$\min\{p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_{n+1}, u_n)\} - \min\{p^{\mathtt{E}}(u_n, u_n), p^{\mathtt{E}}(u_{n+1}, u_{n+1})\} \preceq a p^{\mathtt{E}}(u_n, u_{n+1}) \tag{2.2}$$

Consider the LHS of equation (2.2),

$$\min\{p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_{n+1}, u_n)\} - \min\{p^{\mathtt{E}}(u_n, u_n), p^{\mathtt{E}}(u_{n+1}, u_{n+1})\} = \min\{p^{\mathtt{E}}(u_{n+1}, u_{n+2}), p^{\mathtt{E}}(u_n, u_{n+1})\}$$
$$- \min\{p^{\mathtt{E}}(u_n, u_{n+2}), p^{\mathtt{E}}(u_{n+1}, u_{n+1})\}$$

Suppose if $\min\{p^{\mathtt{E}}(u_{n+1}, u_{n+2}), p^{\mathtt{E}}(u_n, u_{n+1})\} = p^{\mathtt{E}}(u_n, u_{n+1})$,
From equation (2.2)

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq a p^{\mathtt{E}}(u_n, u_{n+1})$$

It is not possible, since $a \in (0, 1)$

$$\therefore \min\{p^{\mathtt{E}}(u_{n+1}, u_{n+2}), p^{\mathtt{E}}(u_n, u_{n+1})\} = p^{\mathtt{E}}(u_{n+1}, u_{n+2}).$$

Again for equation (2.2), we get

$$p^{\mathtt{E}}(u_{n+1}, u_n) \preceq a p^{\mathtt{E}}(u_n, u_{n+1}). \tag{2.3}$$

Now,

$$\begin{aligned} p^{\mathtt{E}}(u_n, u_{n+1}) &= p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n-1}) \\ &\preceq a p^{\mathtt{E}}(u_n, u_{n-1}) \\ &\vdots \\ &\preceq a^n p^{\mathtt{E}}(u_1, u_0). \end{aligned}$$

Now for $n > m$, consider

$$\begin{aligned} p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\ &\preceq \left(a^m + a^{m+1} + \cdots + a^{n-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq a^m \left(1 + a + a^2 + \cdots + a^{n-m-1}\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq a^m \left(\frac{1 - a^{n-m}}{1 - a}\right) p^{\mathtt{E}}(u_1, u_0) \end{aligned}$$

For any $\tau \ggg 0_{\mathtt{E}}, \exists \rho > 0$ such that $\tau - \rho U^+ \subseteq \mathtt{E}^+$ and $n_1 \in \mathbb{N}$ such that $a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \in \frac{\rho}{2} U^+$for any $m, n \geq n_1$, therefore $\tau - \frac{a^{n-m}}{1-a} p^{\mathtt{E}}(u_1, u_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$,

$$p^{\mathtt{E}}(u_n, u_m) \preceq a^m \left(\frac{1 - a^{n-m}}{1 - a}\right) p^{\mathtt{E}}(u_1, u_0) \ll \tau \text{ for all } n, m \geq n_1$$

Hence (u_n) is an e-Cauchy sequence, since u is e-complete $\exists \; u \in \mathtt{X}$ such that $u_n \xrightarrow{e} u$. For a given $\tau \ggg 0_E$, choose $n_2 \in \mathbb{N}$, such that $p^{\mathtt{E}}(u, u_n) \ll (1-a)\frac{\tau}{m}$ for all $n \geq n_2$. Consider for all $n \geq n_2$

$$\begin{aligned} p^{\mathtt{E}}(u, \mathtt{G}u) &\preceq p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(u_n, \mathtt{G}u) \\ &= p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(Tu_{n-1}, \mathtt{G}u) \\ &\preceq p^{\mathtt{E}}(u, u_n) + a p^{\mathtt{E}}(u_{n-1}, u) \\ &\ll (1-a)\frac{\tau}{m} + a\frac{\tau}{m} = \frac{\tau}{m} \end{aligned}$$

Therefore $p^{\mathtt{E}}(u, \mathtt{G}u) \ll \frac{\tau}{m}$ for any $\frac{\tau}{m} \ggg 0_{\mathtt{E}}$ and $m \in \mathbb{N}$, therefore $\frac{\tau}{m} - p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$ for all $m \in \mathbb{N}$, which implies $-p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$, but $p^{\mathtt{E}}(u, \mathtt{G}u) \in E^+$, therefore

$$p^{\mathtt{E}}(u, \mathtt{G}u) = 0_E$$

Hence $u = Tu$. Let $v \in \mathtt{X}$ be such that $u = \mathtt{G}u$ and $v = \mathtt{G}v$, then consider

$$p^{\mathtt{E}}(u, v) = p^{\mathtt{E}}(Tu, \mathtt{G}v) \preceq a p^{\mathtt{E}}(u, v),$$

$p^{\mathtt{E}}(u, v) = 0_{\mathtt{E}}$
Hence $u = v$. □

Theorem 2.2. *Let* $(\mathtt{X}, p^{\mathtt{E}})$ *be an* $e-$*complete* $\mathtt{E}-$ *metric space with closed positive cone* $\mathtt{E}^{+}$ *such that* $(\mathtt{E}^{+})^{\ominus} \neq \phi$. *A mapping* $\mathtt{G} : \mathtt{X} \to \mathtt{X}$ *and there exist a non-negative number* b_1, b_2, b_3 *satisfying* $b_1 + b_2 + b_3 < 1$ *such that for each* $u, v \in \mathtt{X}$,

$$p^{\mathtt{E}}(\mathtt{G}u, \mathtt{G}v) \preceq b_1 p^{\mathtt{E}}(u, \mathtt{G}v) + b_2 p^{\mathtt{E}}(v, \mathtt{G}u) + b_3 p^{\mathtt{E}}(u, v) \tag{2.4}$$

Then $\mathtt{G}$ *has a unique fixed point in* $\mathtt{X}$.

Proof. Consider $u_0 \in \mathtt{X}, u_{n+1} = \mathtt{G}u_n = \mathtt{T^n}u_0 \forall n \in \mathbb{N}$. For any $n \in \mathbb{N}, u_n \neq u_{n+1}$

$$\begin{aligned}
p^{\mathtt{E}}(u_n, u_{n+1}) &= p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n-1}) \\
&\preceq b_1 p^{\mathtt{E}}(u_n, \mathtt{G}u_{n-1}) + b_2 p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_n) + b_3 p^{\mathtt{E}}(u_n, u_{n-1}) \\
&= b_1 p^{\mathtt{E}}(u_n, u_n) + b_2 p^{\mathtt{E}}(u_{n-1}, u_{n+1}) + b_3 p^{\mathtt{E}}(u_n, u_{n-1}) \\
&\preceq b_2 p^{\mathtt{E}}(u_{n-1}, u_n) + b_2 p^{\mathtt{E}}(u_n, u_{n+1}) + b_3 p^{\mathtt{E}}(u_n, u_{n-1}) \\
(1 - b_2) p^{\mathtt{E}}(u_n, u_{n+1}) &\preceq (b_2 + b_3) p^{\mathtt{E}}(u_{n-1}, u_n)
\end{aligned}$$

Therfore,

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq \left(\frac{b_2 + b_3}{1 - b_2}\right) p^{\mathtt{E}}(u_{n-1}, u_n) \tag{2.5}$$

Here $s = \dfrac{b_2 + b_3}{1 - b_2} < 1$ and From equation (2.5) Continuing the above process, we get

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq s^n p^{\mathtt{E}}(u_0, u_1) \tag{2.6}$$

For $n > m$,

$$\begin{aligned}
p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\
&\preceq \left(s^m + s^{m+1} + \cdots + s^{n-1}\right) p^{\mathtt{E}}(u_1, u_0) \\
&\preceq s^m \left(1 + s + s^2 + \cdots + s^{n-m-1}\right) p^{\mathtt{E}}(u_1, u_0) \\
&\preceq s^m \left(\frac{1 - s^{n-m}}{1 - s}\right) p^{\mathtt{E}}(u_1, u_0)
\end{aligned}$$

for $n > m$, consider

$$\begin{aligned}
p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\
&\preceq \left(s^m + s^{m+1} + \cdots + s^{n-1}\right) p^{\mathtt{E}}(u_1, u_0) \\
&\preceq s^m \left(1 + s + s^2 + \cdots + s^{n-m-1}\right) p^{\mathtt{E}}(u_1, u_0) \\
&\preceq s^m \left(\frac{1 - s^{n-m}}{1 - s}\right) p^{\mathtt{E}}(u_1, u_0)
\end{aligned}$$

For any $\tau \ggg 0_{\mathtt{E}}, \exists \rho > 0$ such that $\tau - \rho U^{+} \subseteq \mathtt{E}^{+}$ and $n_3 \in \mathbb{N} \ni s^m \left(\frac{1-s^{n-m}}{1-s}\right) p^{\mathtt{E}}(u_1, u_0) \in \frac{\rho}{2} U^{+}$for any $m, n \geq n_3$, therefore $\tau - \frac{r^{n-m}}{1-r} p^{\mathtt{E}}(u_1, u_0) - \frac{\rho}{2} U^{+} \subseteq \tau - \rho U^{+} \subseteq \mathtt{E}^{+}$, hence

$$p^{\mathtt{E}}(u_n, u_m) \preceq s^m \left(\frac{1 - s^{n-m}}{1 - s}\right) p^{\mathtt{E}}(u_1, u_0) \ll \tau \text{ for all } n, m \geq n_3$$

Hence (u_n) is an e-Cauchy sequence, since u is e-complete $\exists \; u \in \mathtt{X}$ such that $u_n \xrightarrow{e} u$. Given $\tau \ggg 0_E$, choose $n_4 \in \mathbb{N}$, such that $p^{\mathtt{E}}(u, u_n) \ll (1 - s)\frac{\tau}{m} \; \forall \; n \geq n_4$.

$$\begin{aligned}
p^{\mathtt{E}}(u, \mathtt{G}u) &\preceq p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(u_n, \mathtt{G}u) \\
&= p^{\mathtt{E}}(u, u_n) + p^{\mathtt{E}}(T u_{n-1}, \mathtt{G}u) \\
&\preceq p^{\mathtt{E}}(u, u_n) + r p^{\mathtt{E}}(u_{n-1}, u) \\
&\ll (1 - s)\frac{\tau}{m} + s\frac{\tau}{m} = \frac{\tau}{m}
\end{aligned}$$

Therefore $p^{\mathtt{E}}(u,\mathtt{G}u) \ll \frac{\tau}{m}$, $\forall\ \frac{\tau}{m} \ggg 0_{\mathtt{E}}$ and $m \in \mathbb{N}$, $\frac{\tau}{m} - p^{\mathtt{E}}(u,\mathtt{G}u) \in E^{+}$ for all $m \in \mathbb{N}$, which implies $-p^{\mathtt{E}}(u,\mathtt{G}u) \in E^{+}$, but $p^{\mathtt{E}}(u,\mathtt{G}u) \in E^{+}$,

$$p^{\mathtt{E}}(u,\mathtt{G}u) = 0_E$$

Hence $u = Tu$. Let $v \in \mathtt{X}$ be such that $u = \mathtt{G}u$ and $v = \mathtt{G}v$, then consider

$$p^{\mathtt{E}}(u,v) = p^{\mathtt{E}}(Tu,\mathtt{G}v) \preceq s p^{\mathtt{E}}(u,v),$$

$p^{\mathtt{E}}(u,v) = 0_{\mathtt{E}}$
Hence $u = v$. □

Corollary 2.3. *Let $(\mathtt{X}, p^{\mathtt{E}})$ be an e−complete $\mathtt{E}-$ Metric space with closed positive cone $\mathtt{E}^{+}$ such that $(\mathtt{E}^{+})^{\ominus} \neq \phi$. A self mapping $\mathtt{G}$ of $\mathtt{X}$ satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u,\mathtt{G}v) \preceq a \max\left\{p^{\mathtt{E}}(u,v), p^{\mathtt{E}}(u,\mathtt{G}u), p^{\mathtt{E}}(v,\mathtt{G}v), p^{\mathtt{E}}(u,\mathtt{G}v), p^{\mathtt{E}}(v,\mathtt{G}u)\right\}. \quad (2.7)$$

For all $u, v \in \mathtt{X}$ & $a \in (0,1)$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$.

Corollary 2.4. *Let $(\mathtt{X}, p^{\mathtt{E}})$ be an e−complete $\mathtt{E}-$ Metric space with closed positive cone $\mathtt{E}^{+}$ such that $(\mathtt{E}^{+})^{\ominus} \neq \phi$. A self mapping $\mathtt{G}$ of $\mathtt{X}$ satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u,\mathtt{G}v) \preceq a \max\left\{p^{\mathtt{E}}(u,v), p^{\mathtt{E}}(u,\mathtt{G}u), p^{\mathtt{E}}(v,\mathtt{G}v), \frac{p^{\mathtt{E}}(u,\mathtt{G}v) + p^{\mathtt{E}}(v,\mathtt{G}u)}{2}\right\}. \quad (2.8)$$

For all $u, v \in \mathtt{X}$ & $a \in (0,1)$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$.

Definition 2.1. Let Ψ be the class of all real valued continuous functions $\psi : (\mathbb{R}^{+})^{6} \to \mathbb{R}^{+}$ is non-decreasing. Then it satisfies
For any $u, v \in \mathbb{R}^{+}$,
If it is either $u \preceq \psi\left(v, u, 0, v+u, v, \frac{u+v}{2}\right)$ or $u \preceq \psi\left(v, v, u, v, u, v, \frac{u}{2}\right)$ or $u \preceq \psi\left(u, v, u, u, v, v\right)$
$\exists$ a real number $0 < a < 1$ such that $u \preceq av$

Theorem 2.5. *Let $(\mathtt{X}, p^{\mathtt{E}})$ be an e−complete $\mathtt{E}-$ metric space with closed positive cone such that $(\mathtt{E}^{+})^{\ominus} \neq \phi$. A continuous self mapping $\mathtt{G}$ of $\mathtt{X}$ satisfies*

$$p^{\mathtt{E}}(\mathtt{G}u,\mathtt{G}v) \preceq \psi\left(p^{\mathtt{E}}(u,v), p^{\mathtt{E}}(u,\mathtt{G}u), p^{\mathtt{E}}(u,\mathtt{G}v), p^{\mathtt{E}}(v,\mathtt{G}u), p^{\mathtt{E}}(v,\mathtt{G}v), \frac{p^{\mathtt{E}}(u,\mathtt{G}u) + p^{\mathtt{E}}(v,\mathtt{G}v)}{2}\right). \quad (2.9)$$

$\forall u, v \in \mathtt{X}$. Then $\mathtt{G}$ has a unique fixed point in $\mathtt{X}$.

Proof. For any $u_0 \in \mathtt{X}$, and $n \geq 1$ such that $u_1 = \mathtt{G}u_0$ Continuing this process upto $(n+1)$ terms, we get
$u_{n+1} = \mathtt{G}u_n = \mathtt{T}^{\mathtt{n+1}} u_0$
Consider

$$\begin{aligned}
p^{\mathtt{E}}(u_n, u_{n+1}) &= p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u_{n-1}) \\
&\preceq \psi\left(p^{\mathtt{E}}(u_n, u_{n-1}), p^{\mathtt{E}}(u_n, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, \mathtt{G}u_{n-1}), p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_n), p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_{n-1}), \frac{p^{\mathtt{E}}(u_n, \mathtt{G}u_n) + p^{\mathtt{E}}(u_{n-1}, \mathtt{G}u_{n-1})}{2}\right) \\
&= \psi\left(p^{\mathtt{E}}(u_n, u_{n-1}), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_n, u_n), p^{\mathtt{E}}(u_{n-1}, u_{n+1}), p^{\mathtt{E}}(u_{n-1}, u_n), \frac{p^{\mathtt{E}}(u_n, u_{n+1}) + p^{\mathtt{E}}(u_{n-1}, u_n)}{2}\right) \\
&\preceq \psi\left(p^{\mathtt{E}}(u_n, u_{n-1}), p^{\mathtt{E}}(u_n, u_{n+1}), 0, (p^{\mathtt{E}}(u_{n-1}, u_n) + p^{\mathtt{E}}(u_n, u_{n+1})), p^{\mathtt{E}}(u_{n-1}, u_n), \frac{p^{\mathtt{E}}(u_n, u_{n+1}) + p^{\mathtt{E}}(u_{n-1}, u_n)}{2}\right)
\end{aligned}$$

From definition of 2.1

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq a p^{\mathtt{E}}(u_n, u_{n-1})$$

Similarly,

$$p^{\mathtt{E}}(u_n, u_{n-1}) \preceq a p^{\mathtt{E}}(u_{n-1}, u_{n-2})$$

Then

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq a p^{\mathtt{E}}(u_n, u_{n-1}) \preceq a^2 p^{\mathtt{E}}(u_{n-1}, u_{n-2})$$

On Continuing this inequality, we get

$$p^{\mathtt{E}}(u_n, u_{n+1}) \preceq a^n p^{\mathtt{E}}(u_0, u_1)$$

Next for $n > m$,

$$\begin{aligned} p^{\mathtt{E}}(u_m, u_n) &\preceq p^{\mathtt{E}}(u_m, u_{m+1}) + p^{\mathtt{E}}(u_{m+1}, u_{m+2}) + \cdots + p^{\mathtt{E}}(u_{n-1}, u_n) \\ &\preceq \left(\sum_{k=m}^{n-1} a^k\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq a^m \left(\sum_{k=0}^{n-m-1} a^k\right) p^{\mathtt{E}}(u_1, u_0) \\ &\preceq a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \\ &= a^m M l \end{aligned}$$

Where $M = \left(\frac{1-a^{n-m}}{1-a}\right)$ and $l = p^{\mathtt{E}}(u_1, u_0)$. Let $\tau \ggg 0_{\mathtt{E}}$ be given, choose $\rho > 0 \ni \tau - \rho U^+ \subseteq \mathtt{E}^+$ and $n_5 \in \mathbb{N}$ such that $a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \in \frac{\rho}{2} U^+$for any $m, n \geq n_5$, $\tau - \frac{a^{n-m}}{1-a} p^{\mathtt{E}}(u_1, u_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$,

$$p^{\mathtt{E}}(u_n, u_m) \preceq a^m \left(\frac{1-a^{n-m}}{1-a}\right) p^{\mathtt{E}}(u_1, u_0) \lll \tau \; \forall \; n, m \geq n_6$$

Hence (u_n) is an e-Cauchy sequence, since u is e-complete $\exists \; u \in \mathtt{X}$ such that $u_n \xrightarrow{e} u$. For a given $\tau \ggg 0_E$, choose $n_6 \in \mathbb{N}$, such that $p^{\mathtt{E}}(u, u_n) \lll (1-a)\frac{\tau}{m} \; \forall \; n \geq n_6 \ni$

$$\begin{aligned} p^{\mathtt{E}}(u, \mathtt{G}u) &\preceq p^{\mathtt{E}}(u, u_{n+1}) + p^{\mathtt{E}}(u_{n+1}, \mathtt{G}u) \\ &= p^{\mathtt{E}}(u, u_{n+1}) + p^{\mathtt{E}}(\mathtt{G}u_n, \mathtt{G}u) \\ &\preceq p^{\mathtt{E}}(u, u_{n+1}) + \psi\left(p^{\mathtt{E}}(u_n, u), p^{\mathtt{E}}(u_n, \mathtt{G}u_n), p^{\mathtt{E}}(u_n, \mathtt{G}u), p^{\mathtt{E}}(u, \mathtt{G}u_n), p^{\mathtt{E}}(u, \mathtt{G}u), \frac{p^{\mathtt{E}}(u_n, \mathtt{G}u_n) + p^{\mathtt{E}}(u, \mathtt{G}u)}{2}\right) \\ &= p^{\mathtt{E}}(u, u_{n+1}) + \psi\left(p^{\mathtt{E}}(u_n, u), p^{\mathtt{E}}(u_n, u_{n+1}), p^{\mathtt{E}}(u_n, \mathtt{G}u), p^{\mathtt{E}}(u, u_{n+1}), p^{\mathtt{E}}(u, \mathtt{G}u), \frac{p^{\mathtt{E}}(u_n, u_{n+1}) + p^{\mathtt{E}}(u, \mathtt{G}u)}{2}\right) \end{aligned}$$

Taking $u_n \xrightarrow{e} u$, we get

$$p^{\mathtt{E}}(u, \mathtt{G}u) \preceq 0 + \psi\left(0, 0, p^{\mathtt{E}}(u, \mathtt{G}u), 0, p^{\mathtt{E}}(u, \mathtt{G}u), \frac{p^{\mathtt{E}}(u, \mathtt{G}u)}{2}\right)$$

From the definition 2.1, we have

$$p^{\mathtt{E}}(u, \mathtt{G}u) \preceq 0_{\mathtt{E}}.$$

Here, $p^{\mathtt{E}}(u, \mathtt{G}u) \notin \mathtt{E}^+$.
$\therefore$ *Only possible* $p^{\mathtt{E}}(u, \mathtt{G}u) = 0$. Hence, $\mathtt{G}u = u$.
Suppose $\mathtt{G}$ has another fixed point say $v = \mathtt{G}v$, then

$$
\begin{aligned}
p^{\mathtt{E}}(u,v) &= p^{\mathtt{E}}(\mathtt{G}u,\mathtt{G}v)\\
&\preceq \psi\left(p^{\mathtt{E}}(u,v), p^{\mathtt{E}}(u,\mathtt{G}u), p^{\mathtt{E}}(u,\mathtt{G}v), p^{\mathtt{E}}(v,\mathtt{G}u), p^{\mathtt{E}}(v,\mathtt{G}v), \frac{p^{\mathtt{E}}(u,\mathtt{G}u)+p^{\mathtt{E}}(v,\mathtt{G}v)}{2}\right)\\
&= \psi\left(p^{\mathtt{E}}(u,v), 0, p^{\mathtt{E}}(u,v), p^{\mathtt{E}}(u,v), 0, 0\right)
\end{aligned}
$$

From the definition of 2.1, we get
$p^{\mathtt{E}}(u,v) \preceq 0_{\mathtt{E}}, -p^{\mathtt{E}}(u,v) \notin \mathtt{E}^+$
$\therefore p^{\mathtt{E}}(u,v) = 0_{\mathtt{E}}$
Hence $u = v$. □

3 Conclusion

In the article that is being presented, we describe fixed point results with various types of contraction in an $\mathtt{E}-$metric space. We also established a new implicit relation, which enables the contraction to reach a fixed point result more conveniently. The main result of my article can be extended to random cone metric space and integral type fixed point results.

Acknowledgement

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