Original Research Article

# Some Fixed Point Theorems in $e-$ Complete $E$-Metric Space

## Abstract

In this article, we show fixed point results for various contractions in $\mathtt{E}-$ metric space along with certain corollaries.. Also using implicit relation to find fixed point result in $\mathtt{E}-$ metric space .

*Keywords: Semi-interior points; Non-Solid Cones; e−Complete;* $\mathtt{E}-$*Metric Space.*

2010 Mathematics Subject Classification: 47H09; 47H10; 54H25.

## 1 Introduction and Preliminaries

The notion of Cone metric space introduced by Huang and Zhang [4] replacing the real numbers by an ordered Banach space, also they discovered fixed point theorems in cone metric space using contraction mapping. In 2008, Rezapour and Hamlbarani [12] redifined fixed point result without assumptions of normality of cone in Banach space. Al-Rawashdeh et.al. [1] introduced the concept of $\mathtt{E}-$metric space in 2012. TVS-valued cone Banach space were defined by Mehmood et al.[9] in 2014.

Typically, the Banach space $\mathtt{E}$ was taken into account with a defined order in relation to the positive solid cone $\mathtt{E}^+$ of $\mathtt{E}$, i.e., by assuming that the interior of $\mathtt{E}^+$ is not empty. The non-solid cones were just briefly mentioned in a few results [6, 7]. When non-solid cones are involved, [7] took into account the quasi interior points of P rather than interior points.

In 2017, Polyrakis [2] introduced the notions of semi-interior points and see some other article related to cone and $\mathtt{E}-$metric space [3, 5, 7]. Mehmood et al. [8] generalized the Banach,Kannan and Chatterjea contractions in the setting of $\mathtt{E}-$metric space via non-solid and possibly non-normal cones.

In this article, we develope ciric type and some other different kinds of contraction in the setting of $\mathtt{E}-$metric space. Also finally define some implicit relation in the setting of $\mathtt{E}-$metric space.

let $\mathtt{E}$ be an normed space with a norm $\|\cdot\|$, its ordered by the positive cone $\mathtt{E}^+$, then $x, y \in \mathtt{E}, x \preceq y$ iff $y - x \in \mathtt{E}^+$.

**Definition 1.1.** [1] An ordered space $\mathtt{E}$ is a vector space over the real numbers, with a partial order relation $\preceq$ such that

$(O1)$ for all $x, y$, and $z \in \mathtt{E}, x \preceq y$ implies $x + z \preceq y + z$,

$(O2)$ for all $a \in \mathbb{R}^+$and $x \in \mathtt{E}$ with $x \succeq 0_{\mathtt{E}}, ax \succeq 0_{\mathtt{E}}$.

Moreover if $\mathtt{E}$ is equipped with a norm $\|\cdot\|$, then $\mathtt{E}$ is called normed ordered space.

**Definition 1.2.** The positive cone $\mathtt{E}^+$of a normed ordered space $\mathtt{X}$ is called;

(a) normal, if there exists a constant $M > 0$, such that

$$0 \leq x \leq y \text{ implies } \|x\| \leq M\|y\|$$

for all such $x, y \in \mathtt{X}$.

(b) solid, if $\mathtt{E}^+$has non-empty interior,

(c) reflexive, iff $\mathtt{E}^+ \cap U$ is weakly compact, where $U$ is the unit ball in $\mathtt{X}$,

(d) strongly reflexive, iff $\mathtt{E}^+ \cap U$ is compact.

In this above definition, reflexive and strongly reflexive cones can be defined in [7].

**Definition 1.3.** [1] Let $\mathtt{X}$ be a nonempty set and let $\mathtt{E}$ be an ordered space, over the real scalars. An ordered $\mathtt{E}$-metric on $\mathtt{X}$ is an $\mathtt{E}$-valued function $d^{\mathtt{E}} : \mathtt{X} \times \mathtt{X} \to \mathtt{E}$ such that for all $x, y$ and $z \in \mathtt{X}$, we have

$(E1)$ $d^{\mathtt{E}}(x, y) \geq 0_{\mathtt{E}}$ and $d^{\mathtt{E}}(x, y) = 0_{\mathtt{E}}$ if and only if $x = y$,

$(E2)$ $d^{\mathtt{E}}(x, y) = d^{\mathtt{E}}(y, x)$,

$(E3)$ $d^{\mathtt{E}}(x, y) \preceq d^{\mathtt{E}}(x, z) + d^{\mathtt{E}}(z, y)$.

Then the pair $\left(\mathtt{X}, d^{\mathtt{E}}\right)$ is called $\mathtt{E}$-metric space. Now we recall the following definitions of convergent and Cauchy sequence in an $\mathtt{E}$-metric space with the assumption that int $\left(\mathtt{E}^+\right)$is nonempty.

**Definition 1.4.** [8] Let $\mathtt{E}$ be a normed ordered space and $\left(\mathtt{X}, d^{\mathtt{E}}\right)$ be an $\mathtt{E}$-metric space, then a sequence $(x_n)$ in $\mathtt{X}$ is called convergent to a point $x \in \mathtt{X}$ if for all $c \in \text{int}\left(\mathtt{E}^+\right)$, there exists a natural number $\mathbb{N}$ such that

$$d^{\mathtt{E}}\left(x_n, x\right) \ll c$$

for all $n \geq \mathbb{N}$ and we write

$$\lim_{n \to \infty} x_n = x,$$

or simply $x_n \to x$. The sequence $(x_n)$ is called Cauchy, if for all $c \in \text{int}\left(\mathtt{E}^+\right)$, there exists a natural number $N_1$ such that

$$d^{\mathtt{E}}\left(x_n, x_m\right) \ll c$$

for all $n, m \leq N_1$

**Definition 1.5.** [2] Let $\mathtt{E}$ be an ordered normed space ordered by the positive cone $\mathtt{E}^+$, we shall denote by $0_{\mathtt{E}}$ the zero of $\mathtt{E}$,

$$U = \{x \in \mathtt{E} : \|x\| \leq 1\}$$

be the closed unit ball of $\mathtt{E}$, and by $U^+$we mean the positive part of unit ball defined by the set

$$U^+ = U \cap \mathtt{E}^+.$$

**Definition 1.6.** [2] The point $x_0 \in \mathtt{E}^+$is a semi-interior point of $\mathtt{E}^+$ if there exists a real number $\rho > 0$ such that

$$x_0 - \rho U^+ \subseteq \mathtt{E}^+$$

Clearly any interior point of $\mathtt{E}^+$is semi-interior point. The set of all semi-interior points of $\mathtt{E}^+$is denoted by $\left(\mathtt{E}^+\right)^{\ominus}$, and for $x, y \in \mathtt{E}^+$, $x \lll y$ if and only if $y - x \in \left(\mathtt{E}^+\right)^{\ominus}$.

**Example 1.1.** *[2] Let $\mathtt{X_n}$ be the space $\mathbb{R}^2$ ordered by the pointwise ordering and with norm $\|x_n\|_n$ having as unit ball, the polygon $D_n$ of $\mathbb{R}^2$ with vertices $(1,0),(0,1),(-n,n),(-1,0),(0,-1),(n,-n)$. The norm is given by the formula:*

$$\|(x,y)\|_n = \begin{cases} |x|+|y|, & \text{if } xy \geq 0 \\ \max\{|x|,|y|\} - \frac{n-1}{n}\min\{|x|,|y|\}, & \text{if } xy < 0. \end{cases}$$

*Suppose that $\mathtt{E}$ is the space of all the sequences $(x_n)$ such that $x_n = (x_1^n, x_2^n) \in \mathtt{X_n}$ with $\|x_n\|_n \leq \lambda(x)$ for any $n \in \mathbb{N}$ and the real number $\lambda > 0$, depending upon $x$. Assume $\mathtt{E}$ is ordered by cone $\mathtt{E}^+ = \{x = (x_n) \in \mathtt{X} : x_n \in \mathbb{R}_+^2 \text{ for any } n\}$ and equipped with the norm*

$$\|x\|_\infty = \sup_{n\in\mathbb{N}} \|x_n\|_n .$$

*Also suppose that $\mathtt{X} = \mathtt{E}^+ - \mathtt{E}^+$ is the subspace of $\mathtt{E}$ generated by $\mathtt{E}^+$ and ordered by $\mathtt{E}^+$. Let $1 = (y_n)$, where $y_n = (1,1)$ for any $n \in \mathbb{N}$. Then 1 is not an interior point of $\mathtt{E}^+$. Indeed for any $m \in \mathbb{N}$, take $y = (y_n) \in X$ such that,*

$$y = (y_n) = \begin{cases} \mathfrak{y}_\mathfrak{n} = (-2,2), & \text{for } n = m \\ y_n = (0,0), & \text{for } n \neq m \end{cases}$$

*then it is easy to show that $\|y\|_\infty = \frac{2}{m}$ and $1 + y \notin \mathtt{E}^+$, even it can be shown that any $x \in \mathtt{E}^+$ is not an interior point of $\mathtt{E}^+$, therefore $\mathtt{E}^+$ has an empty interioc. The positive part of closed unit ball is*

$$U^+ = \{x \in \mathtt{E}^+ : \|x\|_\infty \leq 1\}$$

*moreover for any $x = (x_n) \in U^+$ it is easy to see that $\|x_n\|_n = x_n^1 + x_n^2 \leq 1$, so*

$$1 - U^+ \subseteq \mathtt{E}^+$$

*and 1 is a semi-interior point of $\mathtt{E}^+$. Also the space $\mathtt{X}$ is not complete space (using Proposition 2.4 of [5]).*

**Example 1.2.** *In Example 2.7 of [2], (Example 5.9 of [3] ) it has been shown that a strong reflexive cone $\mathtt{E}^+$ of $L_1([0,1])$ exists which generate a dense subspace $\mathtt{X}$ of $L_1([0,1])$, i.e.,*

$$\mathtt{X} = \mathtt{E}^+ - \mathtt{E}^+$$

*and*

$$L_1([0,1]) = \bar{\mathtt{X}}.$$

*Let $V = co\left(\left(B^+(0,1)\right) \cup \left(-B^+(0,1)\right)\right)$ and $E_1^+$ be a set of positive elements of $L_1([0,1])$ generated by the set $\Omega = 3y + V$, where $y = \sum_{k=1}^{\infty} \alpha^{k-1} e_k$ for $\alpha \in (0,1)$, where $\{e_i\}$ is the set of standard normalized basis. It has been shown in [5], that the cone $E^+$ has empty interior but has semi-interior points.*

*Clearly every interior point of the cone $\mathtt{E}^+$ is the semi-interior point if int $(\mathtt{E}^+)$ is nonempty.*

**Proposition 1.1.** *(Proposition 2.4 of [2]) If $\mathtt{E}$ is a complete ordered normed space with generating and closed cone $\mathtt{E}^+$, then any semi-interior point of $\mathtt{E}^+$ is an interior point of $\mathtt{E}^+$*

**Definition 1.7.** [8] Let $\mathtt{E}$ be a ordered normed space with assumption that $(\mathtt{E}^+)^\ominus$ is nonempty and $(\mathtt{X}, d^{\mathtt{E}})$ be an $\mathtt{E}$-metric space. Let $(x_n)$ be a sequence in $\mathtt{X}$ and $x \in \mathtt{X}$. Then

(i) $(x_n)$ $e$-converges to $x$ whenever for every $e : 0_E$, there exists a natural number $x_n \xrightarrow{e} x$

(ii) $(x_n)$ is an $e$-Cauchy sequence whenever for every $e \not\succ 0_{\mathtt{E}}$, there exists a natural number $k$ such that $d^{\mathtt{E}}(x_n, x_m) \ll e$ for all $n, m \geq k$.

(iii) $(\mathtt{X}, d^{\mathtt{E}})$ is $e$-complete if every $e$-Cauchy sequence is $e$-convergent.

**Theorem 1.3.** *[8] Let $\left(\mathtt{X}, d^{\mathtt{E}}\right)$ be an e-complete* $\mathtt{E}$*-metric space with closed positive cone* $\mathtt{E}^{+}$ *such that* $\left(\mathtt{E}^{+}\right)^{\ominus} \neq \varphi$. *Let* $\mathtt{T}: \mathtt{X} \rightarrow \mathtt{X}$ *be a mapping satisfying*

$$d^{\mathtt{E}}(\mathtt{T}x, \mathtt{T}y) \preceq \lambda d^{\mathtt{E}}(x, y)$$

*for all* $x, y \in \mathtt{X}$ *and some* $\lambda \in [0,1)$. *Then* $\mathtt{T}$ *has a unique fixed point in* $\mathtt{X}$, *and for each* $x \in \mathtt{X}$, *the iterative sequence* $(\mathtt{T}^{\mathtt{n}} x)_{n>0}$ *converges to the fixed point of* $T$.

**Corollary 1.4.** *[8] Let $\left(\mathtt{X}, d^{\mathtt{E}}\right)$ be an e-complete* $\mathtt{E}$*-metric space with closed positive cone* $\mathtt{E}^{+}$ *such that* $\left(\mathtt{E}^{+}\right)^{\ominus} \neq \varphi$. *For* $e \ggg 0_{\mathtt{E}}$ *and* $x_0 \in \mathtt{X}$, *set* $B(x_0, e) = \left\{x \in \mathtt{X} : d^{\mathtt{E}}(x, y) \preceq e\right\}$. *Let* $\mathtt{T}: \mathtt{X} \rightarrow \mathtt{X}$ *be a mapping such that*

$$d^{\mathtt{E}}(\mathtt{T}x, \mathtt{T}y) \preceq \lambda d^{\mathtt{E}}(x, y)$$

*for all* $x, y \in B(x_0, e)$, *where* $\lambda \in [0,1)$ *is a constant and* $d^{\mathtt{E}}(x_0, \mathtt{T}x_0) \preceq (1-\lambda)e$. *Then* $\mathtt{T}$ *has a unique fixed point in* $B(x_0, e)$.

**Corollary 1.5.** *[8] Let $\left(\mathtt{X}, d^{\mathtt{E}}\right)$ be a complete* $\mathtt{E}$*-metric space with closed positive cone* $\mathtt{E}^{+}$ *such that* $\left(\mathtt{E}^{+}\right)^{\ominus} \neq \varphi$. *Let for some* $n \in \mathbb{N}$, *the mapping* $\mathtt{T}: \mathtt{X} \rightarrow \mathtt{X}$ *satisfies*

$$d^{\mathtt{E}}\left(\mathtt{T}^{\mathtt{n}} x, \mathtt{T}^{\mathtt{n}} y\right) \preceq \lambda d^{\mathtt{E}}(x, y)$$

*for all* $x, y \in \mathtt{X}$, *where* $\lambda \in [0,1)$ *is a constant. Then* $\mathtt{T}$ *has a unique fixed point in* $\mathtt{X}$.

**Theorem 1.6.** *[8] Let $\left(\mathtt{X}, d^{\mathtt{E}}\right)$ be a complete* $\mathtt{E}$*-metric space with closed positive cone* $\mathtt{E}^{+}$ *such that* $\left(\mathtt{E}^{+}\right)^{\ominus} \neq \varphi$. *Let* $\mathtt{T}: \mathtt{X} \rightarrow \mathtt{X}$ *be a mapping satisfying*

$$d^{\mathtt{E}}(\mathtt{T}x, \mathtt{T}y) \preceq \lambda \left[d^{\mathtt{E}}(\mathtt{T}x, x) + d^{\mathtt{E}}(\mathtt{T}y, y)\right]$$

*for all* $x, y \in \mathtt{X}$ *and some* $\lambda \in \left[0, \frac{1}{2}\right)$. *Then* $\mathtt{T}$ *has a unique fixed point in* $\mathtt{X}$, *and for each* $x \in \mathtt{X}$, *the iterative sequence* $(\mathtt{T}^{\mathtt{n}} x)_{n \geq 0}$ *converges to the fixed point of* $\mathtt{X}$.

# 2 Main Result

**Theorem 2.1.** *Let* $\mathtt{E}$ *be a normed space and it's ordered by a closed positive cone* $\mathtt{E}^{+}$. *A mapping* $\mathtt{T}: \mathtt{X} \rightarrow \mathtt{X}$, *where* $X$ *is an* $e-$*complete* $\mathtt{E}-$*metric space with closed positive cone* $\mathtt{E}^{+}$ *such that* $(\mathtt{E}^{+})^{\ominus} \neq \varphi$ *and it satisfies*

$$\min\{d^{\mathtt{E}}(\mathtt{T}x, \mathtt{T}y), d^{\mathtt{E}}(x, \mathtt{T}x), d^{\mathtt{E}}(y, \mathtt{T}y)\} - \min\{d^{\mathtt{E}}(x, \mathtt{T}y), d^{\mathtt{E}}(y, \mathtt{T}x)\} \preceq a d^{\mathtt{E}}(x, y) \tag{2.1}$$

$\forall x, y \in \mathtt{X}$ & $a \in (0,1)$ *and* $d^{\mathtt{E}}(x, x_n) \ll (1-a)\frac{\tau}{m}$. *Then* $\{T^n x\}$ *converges to a fixed point of* $\mathtt{T}$

*Proof.* Consider the iterative sequence

$$x_{n+1} = \mathtt{T}x_n = T^n x_0$$

where $x_0 \in \mathtt{X}$, with $mathit x_n \neq x_{n+1} \forall n \in \mathbb{N}$ for $n \in \mathbb{N}$. Substitute $x = x_n, y = x_{n+1}$ in equation (2.1), we get

$$\min\{d^{\mathtt{E}}(\mathtt{T}x_n, \mathtt{T}x_{n+1}), d^{\mathtt{E}}(x_n, \mathtt{T}x_n), d^{\mathtt{E}}(x_{n+1}, \mathtt{T}x_{n+1})\} - \min\{d^{\mathtt{E}}(x_n, \mathtt{T}x_{n+1}), d^{\mathtt{E}}(x_{n+1}, \mathtt{T}x_n)\} \preceq a d^{\mathtt{E}}(x_n, x_{n+1})$$

$$\min\{d^{\mathtt{E}}(x_{n+1}, \mathtt{T}x_n), d^{\mathtt{E}}(x_n, x_{n+1}), d^{\mathtt{E}}(x_{n+1}, x_n)\} - \min\{d^{\mathtt{E}}(x_n, x_n), d^{\mathtt{E}}(x_{n+1}, x_{n+1})\} \preceq a d^{\mathtt{E}}(x_n, x_{n+1}) \tag{2.2}$$

Consider the LHS of equation (2.2),

$$\min\{d^{\mathtt{E}}(x_{n+1}, \mathtt{T}x_n), d^{\mathtt{E}}(x_n, x_{n+1}), d^{\mathtt{E}}(x_{n+1}, x_n)\} - \min\{d^{\mathtt{E}}(x_n, x_n), d^{\mathtt{E}}(x_{n+1}, x_{n+1})\} = \min\{d^{\mathtt{E}}(x_{n+1}, x_{n+2}), d^{\mathtt{E}}(x_n, x_{n+1})\} - \min\{d^{\mathtt{E}}(x_n, x_{n+2}), d^{\mathtt{E}}(x_{n+1}, x_{n+1})\}$$

Suppose if $\min\{d^{\mathtt{E}}(x_{n+1}, x_{n+2}), d^{\mathtt{E}}(x_n, x_{n+1})\} = d^{\mathtt{E}}(x_n, x_{n+1})$,
From equation (2.2)
$d^{\mathtt{E}}(x_n, x_{n+1}) \preceq a d^{\mathtt{E}}(x_n, x_{n+1})$
It is not possible, since $a \in (0,1)$
$\therefore \min\{d^{\mathtt{E}}(x_{n+1}, x_{n+2}), d^{\mathtt{E}}(x_n, x_{n+1})\} = d^{\mathtt{E}}(x_{n+1}, x_{n+2})$.
Again for equation (2.2), we get

$$d^{\mathtt{E}}(x_{n+1}, x_n) \preceq a d^{\mathtt{E}}(x_n, x_{n+1}). \tag{2.3}$$

Now,

$$\begin{aligned} d^{\mathtt{E}}(x_n, x_{n+1}) &= d^{\mathtt{E}}(\mathtt{T}x_n, Tx_{n-1}) \\ &\preceq a d^{\mathtt{E}}(x_n, x_{n-1}) \\ &\vdots \\ &\preceq a^n d^{\mathtt{E}}(x_1, x_0). \end{aligned}$$

Now for $n > m$, consider

$$\begin{aligned} d^{\mathtt{E}}(x_m, x_n) &\preceq d^{\mathtt{E}}(x_m, x_{m+1}) + d^{\mathtt{E}}(x_{m+1}, x_{m+2}) + \cdots + d^{\mathtt{E}}(x_{n-1}, x_n) \\ &\preceq \left(a^m + a^{m+1} + \cdots + a^{n-1}\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq a^m \left(1 + a + a^2 + \cdots + a^{n-m-1}\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq a^m \left(\frac{1 - a^{n-m}}{1-a}\right) d^{\mathtt{E}}(x_1, x_0) \end{aligned}$$

For any $\tau \ggg 0_{\mathtt{E}}, \exists \rho > 0$ such that $\tau - \rho U^+ \subseteq \mathtt{E}^+$ and a natural number $n_1$ such that $a^m \left(\frac{1-a^{n-m}}{1-a}\right) d^{\mathtt{E}}(x_1, x_0) \in \frac{\rho}{2} U^+$for any $m, n \geq n_1$, therefore $\tau - \frac{a^{n-m}}{1-a} d^{\mathtt{E}}(x_1, x_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$, hence

$$d^{\mathtt{E}}(x_n, x_m) \preceq a^m \left(\frac{1 - a^{n-m}}{1-a}\right) d^{\mathtt{E}}(x_1, x_0) \ll \tau \text{ for all } n, m \geq n_1$$

Hence $(x_n)$ is an $e$-Cauchy sequence, since $X$ is $e$-complete $\exists\ x \in X$ such that $x_n \xrightarrow{e} x$. For a given $\tau \ggg 0_E$, choose $n_2 \in \mathbb{N}$, such that $d^{\mathtt{E}}(x, x_n) \ll (1-a)\frac{\tau}{m}$ for all $n \geq n_2$. Consider for all $n \geq n_2$

$$\begin{aligned} d^{\mathtt{E}}(x, Tx) &\preceq d^{\mathtt{E}}(x, x_n) + d^{\mathtt{E}}(x_n, Tx) \\ &= d^{\mathtt{E}}(x, x_n) + d^{\mathtt{E}}(Tx_{n-1}, Tx) \\ &\preceq d^{\mathtt{E}}(x, x_n) + a d^{\mathtt{E}}(x_{n-1}, x) \\ &\ll (1-a)\frac{\tau}{m} + a\frac{\tau}{m} = \frac{\tau}{m} \end{aligned}$$

Therefore $d^{\mathtt{E}}(x, \mathtt{T}x) \ll \frac{\tau}{m}$ for any $\frac{\tau}{m} \ggg 0_{\mathtt{E}}$ and $m \in \mathbb{N}$, therefore $\frac{\tau}{m} - d^{\mathtt{E}}(x, Tx) \in E^+$ for all $m \in \mathbb{N}$, which implies $-d^{\mathtt{E}}(x, Tx) \in E^+$, but $d^{\mathtt{E}}(x, Tx) \in E^+$, therefore

$$d^{\mathtt{E}}(x, \mathtt{T}x) = 0_E$$

Hence $x = Tx$. Let $y \in X$ be such that $x = \mathtt{T}x \text{and} y = Ty$, then consider

$$d^{\mathtt{E}}(x, y) = d^{\mathtt{E}}(Tx, Ty) \preceq a d^{\mathtt{E}}(x, y),$$

$d^{\mathtt{E}}(x, y) = 0_{\mathtt{E}}$
Hence $x = y$. □

**Theorem 2.2.** *Let* $(\mathtt{X}, d^{\mathtt{E}})$ *be an* $e-$*complete* $\mathtt{E}-$ *Metric space with closed positive cone* $\mathtt{E}^+$ *such that* $(\mathtt{E}^+)^{\ominus} \neq \phi$. *A mapping* $\mathtt{T} : \mathtt{X} \to \mathtt{X}$ *and thereexist a non-negative number* $b_1, b_2, b_3$ *satisfying* $b_1 + b_2 + b_3 < 1$ *such that for each* $x, y \in \mathtt{X}$,

$$d(\mathtt{T}x, \mathtt{T}y) \preceq b_1 d(x, \mathtt{T}y) + b_2 d(y, \mathtt{T}x) + b_3 d(x, y) \tag{2.4}$$

*Then* $\mathtt{T}$ *has a unique fixed point in* $\mathtt{X}$.

*Proof.* Consider $x_0 \in \mathtt{X}, x_{n+1} = \mathtt{T}x_n = \mathtt{T}^n x_0 \forall n \in \mathbb{N}$. For any $n \in \mathbb{N}, x_n \neq x_{n+1}$

$$\begin{aligned} d^{\mathtt{E}}(x_n, x_{n+1}) &= d^{\mathtt{E}}(\mathtt{T}x_n, \mathtt{T}x_{n-1}) \\ &\preceq b_1 d^{\mathtt{E}}(x_n, \mathtt{T}x_{n-1}) + b_2 d^{\mathtt{E}}(x_{n-1}, \mathtt{T}x_n) + b_3 d^{\mathtt{E}}(x_n, x_{n-1}) \\ &= b_1 d^{\mathtt{E}}(x_n, x_n) + b_2 d^{\mathtt{E}}(x_{n-1}, x_{n+1}) + b_3 d^{\mathtt{E}}(x_n, x_{n-1}) \\ &\preceq b_2 d^{\mathtt{E}}(x_{n-1}, x_n) + b_2 d^{\mathtt{E}}(x_n, x_{n+1}) + b_3 d^{\mathtt{E}}(x_n, x_{n-1}) \\ (1 - b_2) d^{\mathtt{E}}(x_n, x_{n+1}) &\preceq (b_2 + b_3) d^{\mathtt{E}}(x_{n-1}, x_n) \end{aligned}$$

Therfore,

$$d^{\mathtt{E}}(x_n, x_{n+1}) \preceq \left(\frac{b_2 + b_3}{1 - b_2}\right) d^{\mathtt{E}}(x_{n-1}, x_n) \tag{2.5}$$

Here $r = \dfrac{b - 2 + b_3}{1 - b_2} < 1$ and From equation (2.5) Continuing the above proces, we get

$$d^{\mathtt{E}}(x_n, x_{n+1}) \preceq r^n d^{\mathtt{E}}(x_0, x_1) \tag{2.6}$$

For $n > m$,

$$\begin{aligned} d^{\mathtt{E}}(x_m, x_n) &\preceq d^{\mathtt{E}}(x_m, x_{m+1}) + d^{\mathtt{E}}(x_{m+1}, x_{m+2}) + \cdots + d^{\mathtt{E}}(x_{n-1}, x_n) \\ &\preceq \left(r^m + r^{m+1} + \cdots + r^{n-1}\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq r^m \left(1 + r + r^2 + \cdots + r^{n-m-1}\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq r^m \left(\frac{1 - r^{n-m}}{1 - r}\right) d^{\mathtt{E}}(x_1, x_0) \end{aligned}$$

for $n > m$, consider

$$\begin{aligned} d^{\mathtt{E}}(x_m, x_n) &\preceq d^{\mathtt{E}}(x_m, x_{m+1}) + d^{\mathtt{E}}(x_{m+1}, x_{m+2}) + \cdots + d^{\mathtt{E}}(x_{n-1}, x_n) \\ &\preceq \left(r^m + r^{m+1} + \cdots + r^{n-1}\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq r^m \left(1 + r + r^2 + \cdots + r^{n-m-1}\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq r^m \left(\frac{1 - r^{n-m}}{1 - r}\right) d^{\mathtt{E}}(x_1, x_0) \end{aligned}$$

For any $\tau \ggg 0_{\mathtt{E}}, \exists \rho > 0$ such that $\tau - \rho U^+ \subseteq \mathtt{E}^+$ and a natural number $n_3$ such that $r^m \left(\frac{1-r^{n-m}}{1-r}\right) d^{\mathtt{E}}(x_1, x_0) \in \frac{\rho}{2} U^+$ for any $m, n \geq n_3$, therefore $\tau - \frac{r^{n-m}}{1-r} d^{\mathtt{E}}(x_1, x_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$, hence

$$d^{\mathtt{E}}(x_n, x_m) \preceq r^m \left(\frac{1 - r^{n-m}}{1 - r}\right) d^{\mathtt{E}}(x_1, x_0) \ll \tau \text{ for all } n, m \geq n_3$$

Hence $(x_n)$ is an $e$-Cauchy sequence, since $X$ is $e$-complete $\exists\ x \in X$ such that $x_n \xrightarrow{e} x$. For a given $\tau \ggg 0_E$, choose $n_4 \in \mathbb{N}$, such that $d^{\mathtt{E}}(x, x_n) \ll (1 - r)\frac{\tau}{m}$ for all $n \geq n_4$. Consider for all $n \geq n_4$

$$\begin{aligned} d^{\mathtt{E}}(x, Tx) &\preceq d^{\mathtt{E}}(x, x_n) + d^{\mathtt{E}}(x_n, Tx) \\ &= d^{\mathtt{E}}(x, x_n) + d^{\mathtt{E}}(Tx_{n-1}, Tx) \\ &\preceq d^{\mathtt{E}}(x, x_n) + r d^{\mathtt{E}}(x_{n-1}, x) \\ &\ll (1 - r)\frac{\tau}{m} + r\frac{\tau}{m} = \frac{\tau}{m} \end{aligned}$$

Therefore $d^{\mathtt{E}}(x,\mathtt{T}x) \ll \frac{\tau}{m}$ for any $\frac{\tau}{m} \ggg 0_{\mathtt{E}}$ and $m \in \mathbb{N}$, therefore $\frac{\tau}{m} - d^{\mathtt{E}}(x,Tx) \in E^+$ for all $m \in \mathbb{N}$, which implies $-d^{\mathtt{E}}(x,Tx) \in E^+$, but $d^{\mathtt{E}}(x,Tx) \in E^+$, therefore

$$d^{\mathtt{E}}(x,\mathtt{T}x) = 0_E$$

Hence $x = Tx$. Let $y \in X$ be such that $x = \mathtt{T}x$ and $y = Ty$, then consider

$$d^{\mathtt{E}}(x,y) = d^{\mathtt{E}}(Tx,Ty) \preceq r d^{\mathtt{E}}(x,y),$$

$d^{\mathtt{E}}(x,y) = 0_{\mathtt{E}}$
Hence $x = y$. □

**Corollary 2.3.** *Let $(\mathtt{X}, d^{\mathtt{E}})$ be an e−complete $\mathtt{E}-$ Metric space with closed positive cone $\mathtt{E}^+$ such that $(\mathtt{E}^+)^{\ominus} \neq \phi$. A self mapping $\mathtt{T}$ of $\mathtt{X}$ satisfies*

$$d^{\mathtt{E}}(\mathtt{T}x,\mathtt{T}y) \preceq a \max\left\{d^{\mathtt{E}}(x,y), d^{\mathtt{E}}(x,\mathtt{T}x), d^{\mathtt{E}}(y,\mathtt{T}y), d^{\mathtt{E}}(x,\mathtt{T}y), d^{\mathtt{E}}(y,\mathtt{T}x)\right\}. \tag{2.7}$$

*For all $x,y \in \mathtt{X}$ & $a \in (0,1)$. Then $\mathtt{T}$ has a unique fixed point in $\mathtt{X}$.*

**Corollary 2.4.** *Let $(\mathtt{X}, d^{\mathtt{E}})$ be an e−complete $\mathtt{E}-$ Metric space with closed positive cone $\mathtt{E}^+$ such that $(\mathtt{E}^+)^{\ominus} \neq \phi$. A self mapping $\mathtt{T}$ of $\mathtt{X}$ satisfies*

$$d^{\mathtt{E}}(\mathtt{T}x,\mathtt{T}y) \preceq a \max\left\{d^{\mathtt{E}}(x,y), d^{\mathtt{E}}(x,\mathtt{T}x), d^{\mathtt{E}}(y,\mathtt{T}y), \frac{d^{\mathtt{E}}(x,\mathtt{T}y) + d^{\mathtt{E}}(y,\mathtt{T}x)}{2}\right\}. \tag{2.8}$$

*For all $x,y \in \mathtt{X}$ & $a \in (0,1)$. Then $\mathtt{T}$ has a unique fixed point in $\mathtt{X}$.*

**Definition 2.1.** Let $\Psi$ be the class of all real valued continuous functions $\psi : (\mathbb{R}^+)^6 \to \mathbb{R}^+$ is non-decreasing. Then it satisfies the condition
For any $x,y \in \mathbb{R}^+$,
If it is either $x \preceq \psi\left(y,x,0,y+x,y,\frac{x+y}{2}\right)$ or $x \preceq \psi\left(y,y,x,y,x,y,\frac{x}{2}\right)$ or $x \preceq \psi\left(x,y,x,x,y,y\right)$
$\exists$ a real number $0 < a < 1$ such that $x \preceq ay$

**Theorem 2.5.** *Let $(\mathtt{X}, d^{\mathtt{E}})$ be an e−complete $\mathtt{E}-$ metricspace with closed positive cone such that $(\mathtt{E}^+)^{\ominus} \neq \phi$. A continuous self mapping $\mathtt{T}$ of $\mathtt{X}$ satisfies*

$$d^{\mathtt{E}}(\mathtt{T}x,\mathtt{T}y) \preceq \psi\left(d^{\mathtt{E}}(x,y), d^{\mathtt{E}}(x,\mathtt{T}x), d^{\mathtt{E}}(x,\mathtt{T}y), d^{\mathtt{E}}(y,\mathtt{T}x), d^{\mathtt{E}}(y,\mathtt{T}y), \frac{d^{\mathtt{E}}(x,\mathtt{T}x) + d^{\mathtt{E}}(y,\mathtt{T}y)}{2}\right). \tag{2.9}$$

*$\forall x,y \in \mathtt{X}$. Then $T$ has a unique fixed point in $X$.*

*Proof.* For any $x_0 \in \mathtt{X}$, and $n \geq 1$ such that $x_1 = \mathtt{T}x_0$ Continuing this process upto $(n+1)$ terms, we get
$x_{n+1} = \mathtt{T}x_n = \mathtt{T}^{n+1}x_0$
Consider

$$\begin{aligned}
d^{\mathtt{E}}(x_n,x_{n+1}) &= d^{\mathtt{E}}(\mathtt{T}x_n,\mathtt{T}x_{n-1}) \\
&\preceq \psi\left(d^{\mathtt{E}}(x_n,x_{n-1}), d^{\mathtt{E}}(x_n,\mathtt{T}x_n), d^{\mathtt{E}}(x_n,\mathtt{T}x_{n-1}), d^{\mathtt{E}}(x_{n-1},\mathtt{T}x_n), d^{\mathtt{E}}(x_{n-1},\mathtt{T}x_{n-1}), \frac{d^{\mathtt{E}}(x_n,\mathtt{T}x_n) + d^{\mathtt{E}}(x_{n-1},\mathtt{T}x_{n-1})}{2}\right) \\
&= \psi\left(d^{\mathtt{E}}(x_n,x_{n-1}), d^{\mathtt{E}}(x_n,x_{n+1}), d^{\mathtt{E}}(x_n,x_n), d^{\mathtt{E}}(x_{n-1},x_{n+1}), d^{\mathtt{E}}(x_{n-1},x_n), \frac{d^{\mathtt{E}}(x_n,x_{n+1}) + d^{\mathtt{E}}(x_{n-1},x_n)}{2}\right) \\
&\preceq \psi\left(d^{\mathtt{E}}(x_n,x_{n-1}), d^{\mathtt{E}}(x_n,x_{n+1}), 0, (d^{\mathtt{E}}(x_{n-1},x_n) + d^{\mathtt{E}}(x_n,x_{n+1})), d^{\mathtt{E}}(x_{n-1},x_n), \frac{d^{\mathtt{E}}(x_n,x_{n+1}) + d^{\mathtt{E}}(x_{n-1},x_n)}{2}\right)
\end{aligned}$$

From definition of 2.1

$$d^{\mathtt{E}}(x_n,x_{n+1}) \preceq a d^{\mathtt{E}}(x_n,x_{n-1})$$

Similarly,

$$d^{\mathtt{E}}(x_n, x_{n-1}) \preceq a d^{\mathtt{E}}(x_{n-1}, x_{n-2})$$

Then

$$d^{\mathtt{E}}(x_n, x_{n+1}) \preceq a d^{\mathtt{E}}(x_n, x_{n-1}) \preceq a^2 d^{\mathtt{E}}(x_{n-1}, x_{n-2})$$

On Continuing this inequality, we get

$$d^{\mathtt{E}}(x_n, x_{n+1}) \preceq a^n d^{\mathtt{E}}(x_0, x_1)$$

Next for $n > m$,

$$\begin{aligned} d^{\mathtt{E}}(x_m, x_n) &\preceq d^{\mathtt{E}}(x_m, x_{m+1}) + d^{\mathtt{E}}(x_{m+1}, x_{m+2}) + \cdots + d^{\mathtt{E}}(x_{n-1}, x_n) \\ &\preceq \left(\sum_{k=m}^{n-1} a^k\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq a^m \left(\sum_{k=0}^{n-m-1} a^k\right) d^{\mathtt{E}}(x_1, x_0) \\ &\preceq a^m \left(\frac{1-a^{n-m}}{1-a}\right) d^{\mathtt{E}}(x_1, x_0) \\ &= a^m M l \end{aligned}$$

Where $M = \left(\frac{1-a^{n-m}}{1-a}\right)$ and $l = d^{\mathtt{E}}(x_1, x_0)$. Let $\tau \ggg 0_{\mathtt{E}}$ be given, choose $\rho > 0 \ni \tau - \rho U^+ \subseteq \mathtt{E}^+$ and a natural number $n_5$ such that $a^m \left(\frac{1-a^{n-m}}{1-a}\right) d^{\mathtt{E}}(x_1, x_0) \in \frac{\rho}{2} U^+$for any $m, n \geq n_5$, therefore $\tau - \frac{a^{n-m}}{1-a} d^{\mathtt{E}}(x_1, x_0) - \frac{\rho}{2} U^+ \subseteq \tau - \rho U^+ \subseteq \mathtt{E}^+$, hence

$$d^{\mathtt{E}}(x_n, x_m) \preceq a^m \left(\frac{1-a^{n-m}}{1-a}\right) d^{\mathtt{E}}(x_1, x_0) \ll \tau \text{ for all } n, m \geq n_6$$

Hence $(x_n)$ is an $e$-Cauchy sequence, since $X$ is $e$-complete $\exists\ x \in X$ such that $x_n \xrightarrow{e} x$. For a given $\tau \ggg 0_E$, choose $n_6 \in \mathbb{N}$, such that $d^{\mathtt{E}}(x, x_n) \ll (1-a)\frac{\tau}{m}$ for all $n \geq n_6$. Consider for all $n \geq n_6 \ni$

$$\begin{aligned} d^{\mathtt{E}}(x, \mathtt{T}x) &\preceq d^{\mathtt{E}}(x, x_{n+1}) + d^{\mathtt{E}}(x_{n+1}, \mathtt{T}x) \\ &= d^{\mathtt{E}}(x, x_{n+1}) + d^{\mathtt{E}}(\mathtt{T}x_n, \mathtt{T}x) \\ &\preceq d^{\mathtt{E}}(x, x_{n+1}) + \psi\left(d^{\mathtt{E}}(x_n, x), d^{\mathtt{E}}(x_n, \mathtt{T}x_n), d^{\mathtt{E}}(x_n, \mathtt{T}x), d^{\mathtt{E}}(x, \mathtt{T}x_n), d^{\mathtt{E}}(x, \mathtt{T}x), \frac{d^{\mathtt{E}}(x_n, \mathtt{T}x_n) + d^{\mathtt{E}}(x, \mathtt{T}x)}{2}\right) \\ &= d^{\mathtt{E}}(x, x_{n+1}) + \psi\left(d^{\mathtt{E}}(x_n, x), d^{\mathtt{E}}(x_n, x_{n+1}), d^{\mathtt{E}}(x_n, \mathtt{T}x), d^{\mathtt{E}}(x, x_{n+1}), d^{\mathtt{E}}(x, \mathtt{T}x), \frac{d^{\mathtt{E}}(x_n, x_{n+1}) + d^{\mathtt{E}}(x, \mathtt{T}x)}{2}\right) \end{aligned}$$

Taking $x_n \xrightarrow{e} x$, we get

$$d^{\mathtt{E}}(x, \mathtt{T}x) \preceq 0 + \psi\left(0, 0, d^{\mathtt{E}}(x, \mathtt{T}x), 0, d^{\mathtt{E}}(x, \mathtt{T}x), \frac{d^{\mathtt{E}}(x, \mathtt{T}x)}{2}\right)$$

From the definition 2.1, we have

$$d^{\mathtt{E}}(x, \mathtt{T}x) \preceq 0_{\mathtt{E}}.$$

Here, $d^{\mathtt{E}}(x, \mathtt{T}x) \notin \mathtt{E}^+$.
$\therefore$ *Only possible* $d^{\mathtt{E}}(x, \mathtt{T}x) = 0$. Hence, $\mathtt{T}x = x$.

Suppose $\mathtt{T}$ has another fixed point say $y = \mathtt{T}y$, then

$$\begin{aligned} d^{\mathtt{E}}(x,y) &= d^{\mathtt{E}}(\mathtt{T}x,\mathtt{T}y) \\ &\preceq \psi\left(d^{\mathtt{E}}(x,y), d^{\mathtt{E}}(x,\mathtt{T}x), d^{\mathtt{E}}(x,\mathtt{T}y), d^{\mathtt{E}}(y,\mathtt{T}x), d^{\mathtt{E}}(y,\mathtt{T}y), \frac{d^{\mathtt{E}}(x,\mathtt{T}x)+d^{\mathtt{E}}(y,\mathtt{T}y)}{2}\right) \\ &= \psi\left(d^{\mathtt{E}}(x,y), 0, d^{\mathtt{E}}(x,y), d^{\mathtt{E}}(x,y), 0, 0\right) \end{aligned}$$

From the definition of 2.1, we get
$d^{\mathtt{E}}(x,y) \preceq 0_{\mathtt{E}}, -d^{\mathtt{E}}(x,y) \notin \mathtt{E}^{+}$
Therefore $d^{\mathtt{E}}(x,y) = 0_{\mathtt{E}}$
Hence $x = y$. □

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