A NOVEL SET OF FUZZY f-DIVERGENCE MEASURE-RELATED INTUITIONISTIC FUZZY INFORMATION EQUALITIES AND INEQUALITIES

Abstract

In the literature on fuzzy information theory, there are numerous divergence metrics and fuzzy information. Disparities are crucial for determining relationships. Here, we'll discuss some fresh information inequalities related to fuzzy measures and how they apply to the detection of patterns. With the aid of the fuzzy f-divergence measure and Jensen's inequality, links between new and well-known fuzzy divergence measures were also created.

Keywords: f-divergence, Arithmetic-Geometric mean divergence, Harmonic mean divergence, Relative Arithmetic Geometric divergence, Theoretic exponential information distance measures, Relative J-divergence etc.

1. INTRODUCTION

The fuzzy sets (FS) that Zadeh described in 1965 have demonstrated useful applications in numerous academic domains. The concept of a fuzzy set is advantageous because it addresses ambiguity and uncertainty that the Cantorian set was unable to handle. According to fuzzy set theory, an element's membership in a fuzzy set is expressed as a single value between zero and one. Nevertheless, since there may be some hesitation degree, it may not always be the case that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree. As a result, Atanassov (1983, 1986) presented intuitionistic fuzzy sets (IFS), a generalization of fuzzy sets that include the degree of hesitation known as the hesitation margin (and is defined as 1 minus the sum of membership and non-membership degrees respectively). In many application domains, the idea of defining intuitionistic fuzzy sets as generalized fuzzy sets is quite fascinating and practical.

Since it incorporates the degree of belongingness, degree of non-belongingness, and the hesitation margin, intuitionistic fuzzy set knowledge, and semantic representation become more significant, resourceful, and relevant (Atanassov, 1994, 1999). Szmidt and Kacprzyk (2001) demonstrated that intuitionistic fuzzy sets can be helpful when a problem's description by a linguistic variable expressed in terms of a membership function only looks to be a crude approximation. IFS is a tool for more human consistent reasoning under imperfectly specified facts and imprecise information because of its flexibility in addressing ambiguity (Szmidt and Kacprzyk, 2004).

De et al. (2001) provided a three-step approach to medical diagnosis utilizing intuitionistic fuzzy sets, including symptom determination, medical knowledge formulation based on intuitionistic fuzzy relations, and diagnosis based on the composition of intuitionistic fuzzy relations. Since there is a chance of a non-null hesitation part existing at each point of evaluating an unidentified object, an intuitionistic fuzzy set is a tool for simulating real-world issues like sale analysis, new product marketing, financial services, negotiation processes, psychological investigations, etc (Szmidt and Kacprzyk, 1997, 2001). Rigid research based on the theory and applications of intuitionistic fuzzy sets was conducted by Atanassov (1999, 2012). The distance measurements approach is used in many IFS applications. Given its many real-world applications in pattern recognition, machine learning, decision making, and market forecasting, distance measure between intuitionistic fuzzy sets is a key idea in fuzzy mathematics. In recent years, numerous distance metrics between intuitionistic fuzzy sets have been suggested, investigated, and employed in medical diagnostics (Szmidt and Kacprzyk, 1997, 2000, Wang and Xin, 2005).

Using the f-divergence functional, Csiszar [2, 3] established a generalized measure of information in 1961.

$$I_f(P,Q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right) \tag{1.1}$$

Where $f: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is a convex function and $P, Q \in \Gamma n$. Here

$$\Gamma n = \{P = (p_1, p_2, \dots\dots\dots, p_n) | p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}, n \geq 2$$

be the collection of all discrete probability distributions with finite lengths. The literature on information theory and statistics includes a wide variety of information and divergence measures. Here, we'll provide a few examples of Csiszar f-divergence measure divergence measures.

Intuitionistic fuzzy relative Information [10]

$$K(A,B) = \sum_{i=1}^{n} \left[\mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + \nu_A(x_i) \log \frac{\nu_A(x_i)}{\nu_B(x_i)}\right] \tag{1.2}$$

Intuitionistic fuzzy Chi-square divergence [6]

$$\mathcal{X}^2(A,B) = \sum_{i=1}^{n} \left(\left(\frac{\mu_A^2(x_i)}{\mu_B(x_i)} - \mu_B(x_i)\right) + \left(\frac{\nu_A^2(x_i)}{\nu_B(x_i)} - \nu_B(x_i)\right)\right) \tag{1.3}$$

Intuitionistic fuzzy relative J-S divergence [10]

$$F(A,B) = \sum_{i=1}^{n} \mu_A(x_i) \log \left(\frac{2\mu_A(x_i)}{\mu_A(x_i) + \mu_B(x_i)}\right) + \sum_{i=1}^{n} \nu_A(x_i) \log \left(\frac{2\nu_A(x_i)}{\nu_A(x_i) + \nu_B(x_i)}\right) \tag{1.4}$$

Intuitionistic fuzzy Hellinger discrimination [5]

$$h(P,Q) = 1 - B(P,Q) = \frac{1}{2} \sum_{i=1}^{n} \left(\sqrt{p_i} - \sqrt{q_i}\right)^2 \tag{1.5}$$

Where $B(A,B) = \sum_{i=1}^{n} \left(\sqrt{\frac{\mu_A(x_i)}{\mu_B(x_i)}} + \sqrt{\frac{\nu_A(x_i)}{\nu_B(x_i)}}\right)$ is known as Bhattacharya distance measure [1]

Intuitionistic fuzzy Verma [12] distance measure (Motivated by Kullback and Liebler [8])

$$V_a(A,B) = \sum_{i=1}^{n} \mu_B(x_i) \ln\left(\frac{\mu_B(x_i)+a\mu_A(x_i)}{\mu_A(x_i)}\right) + \sum_{i=1}^{n} \nu_B(x_i) \ln\left(\frac{\nu_B(x_i)+a\nu_A(x_i)}{\nu_A(x_i)}\right) - \left(\mu_A(x_i) + \nu_A(x_i)\right) \ln(1+a), \quad a > 0 \tag{1.6}$$

and another intuitionistic fuzzy Verma [13] distance measure, due to general rule, is given by

$$V_a(A,B) = \sum_{i=1}^{n} \left(\ln\left(\frac{1+a\mu_A(x_i)}{1+a\mu_B(x_i)}\right) - \ln\frac{\mu_A(x_i)}{\mu_B(x_i)}\right).\mu_B(x_i)\left(1 + a\mu_B(x_i)\right) + \sum_{i=1}^{n} \left(\ln\left(\frac{1+a\nu_A(x_i)}{1+a\nu_B(x_i)}\right) - \ln\frac{\nu_A(x_i)}{\nu_B(x_i)}\right).\nu_B(x_i)\left(1 + a\nu_B(x_i)\right), \quad a > 0 \tag{1.7}$$

Intuitionistic fuzzy J-divergence measure [4, 7]

$$J(A,B) = \sum_{i=1}^{n} \left(\mu_A(x_i) - \mu_B(x_i)\right) \log\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right) + \sum_{i=1}^{n} \left(\nu_A(x_i) - \nu_B(x_i)\right) \log\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}\right) \tag{1.8}$$

Intuitionistic fuzzy relative J-divergence measure [6, 7]

$$J_R(A,B) = \sum_{i=1}^{n} \left(\mu_A(x_i) - \mu_B(x_i)\right) \log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right) + \sum_{i=1}^{n} \left(\nu_A(x_i) - \nu_B(x_i)\right) \log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_B(x_i)}\right) \tag{1.9}$$

Intuitionistic fuzzy relative Jensen-Shannon divergence measure [9]

$$F(A,B) = \sum_{i=1}^{n} \mu_A(x_i) \log\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \sum_{i=1}^{n} \nu_A(x_i) \log\left(\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right) \tag{1.10}$$

and $$F(B,A) = \sum_{i=1}^{n} \mu_B(x_i) \log\left(\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \sum_{i=1}^{n} \nu_B(x_i) \log\left(\frac{2\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right) \tag{1.11}$$

Intuitionistic fuzzy Jensen-Shannon [9] divergence measure

$$I(A,B) = \frac{1}{2}[F(B,A) + F(A,B)] = \frac{1}{2}\left[\sum_{i=1}^{n} \left(\mu_B(x_i) \log\left(\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \nu_B(x_i) \log\left(\frac{2\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right)\right] - \frac{1}{2}\left[\sum_{i=1}^{n} \left(\mu_A(x_i) \log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_A(x_i)}\right) + \nu_A(x_i) \log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right)\right)\right] \tag{1.12}$$

Intuitionistic fuzzy arithmetic-geometric mean divergence [10]

$$T(A,B) = \sum_{i=1}^{n} \left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right) \log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\sqrt{\mu_A(x_i).\mu_B(x_i)}}\right) + \sum_{i=1}^{n} \left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2}\right) \log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\sqrt{\nu_A(x_i).\nu_B(x_i)}}\right) \tag{1.13}$$

Intuitionistic fuzzy arithmetic mean divergence [11]

$$\boldsymbol{A}(A,B) = \sum_{i=1}^{n}\left[\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right) + \left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2}\right)\right] \quad (1.14)$$

Intuitionistic fuzzy harmonic mean divergence [11]

$$H(A,B) = \sum_{i=1}^{n}\left[\left(\frac{2\mu_A(x_i).\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \left(\frac{2\nu_A(x_i).\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right] \quad (1.15)$$

Intuitionistic fuzzy relative arithmetic-geometric divergence [10]

$$G(B,A) = \sum_{i=1}^{n}\left[\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right) + \left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2}\right)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_B(x_i)}\right)\right] \quad (1.16)$$

Intuitionistic fuzzy triangular discrimination divergence measure

$$\Delta_1(A,B) = \sum_{i=1}^{n}\left(\frac{\left(\mu_A(x_i)-\mu_B(x_i)\right)^3}{\mu_A(x_i)+\mu_B(x_i)} + \frac{\left(\nu_A(x_i)-\nu_B(x_i)\right)^3}{\nu_A(x_i)+\nu_B(x_i)}\right) \quad (1.17)$$

Intuitionistic fuzzy logarithmic mean divergence [11]

$$L(A,B) = \sum_{i=1}^{n}\left(\frac{\mu_A(x_i)-\mu_B(x_i)}{\log\mu_A(x_i)-\log\mu_B(x_i)} + \frac{\mu_A(x_i)-\mu_B(x_i)}{\log\mu_A(x_i)-\log\mu_B(x_i)}\right) \quad (1.18)$$

Intuitionistic fuzzy Kumar-Johnson [8] distance measures

$$\psi M(A,B) = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)^2}{\left(\mu_A(x_i).\mu_B(x_i)\right)^{3/2}} + \frac{\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)^2}{\left(\nu_A(x_i).\nu_B(x_i)\right)^{3/2}}\right) \quad (1.19)$$

Intuitionistic fuzzy theoretic exponential information distance measures

$$\phi D(A,B) = \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)^2}{\left(\mu_A(x_i)\mu_B(x_i)\right)^{3/2}}.\,e^{\frac{\mu_A(x_i)}{\mu_B(x_i)}} + \frac{\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)^2}{\left(\nu_A(x_i)\nu_B(x_i)\right)^{3/2}}.\,e^{\frac{\nu_A(x_i)}{\nu_B(x_i)}}\right) \quad (1.20)$$

and

$$\phi D_\rho(A,B) = \frac{1}{4}\sum_{i=1}^{n}\left[\begin{array}{c}\frac{\left(\mu_A(x_i)-\mu_B(x_i)\right)\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)\left(2\mu_A^3(x_i)+5\mu_A^2(x_i)\mu_B(x_i)-2\mu_A(x_i)\mu_B^2(x_i)+3\mu_B^3(x_i)\right)}{\mu_B(x_i)\left(\mu_A(x_i)\mu_B(x_i)\right)^{5/2}}.\,e^{\frac{\mu_A(x_i)}{\mu_B(x_i)}} \\ \frac{\left(\nu_A(x_i)-\nu_B(x_i)\right)\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)\left(2\nu_A^3(x_i)+5\nu_A^2(x_i)\nu_B(x_i)-2\nu_A(x_i)\nu_B^2(x_i)+3\nu_B^3(x_i)\right)}{\nu_B(x_i)\left(\nu_A(x_i)\nu_B(x_i)\right)^{5/2}}.\,e^{\frac{\nu_A(x_i)}{\nu_B(x_i)}}\end{array}\right]$$

(1.21)

2. OUR WORK

Proposition 2.1 Let $(A, B) \in \Gamma n \times \Gamma n$ and if $K(B, A)$, $F(A, B)$, $J(A, B)$ shows K-L, Relative J-S, J-divergence measures respectively. Then we have the following new equality

$$2D_1(A, B) = J(A, B) - K(B, A) + F(B, A) - F(A, B).$$

Proof: If we take the convex function,

$$\begin{aligned} F(y) &= \frac{(y-1)^2}{(y+1)^1} + \frac{1}{3}\frac{(y-1)^4}{(y+1)^3} + \frac{1}{5}\frac{(y-1)^6}{(y+1)^5} + \frac{1}{7}\frac{(y-1)^8}{(y+1)^7} + \cdots \\ &= (y-1)\left[\left(\frac{y-1}{y+1}\right)^1 + \frac{1}{3}\left(\frac{y-1}{y+1}\right)^3 + \frac{1}{5}\left(\frac{y-1}{y+1}\right)^5 + \frac{1}{7}\left(\frac{y-1}{y+1}\right)^7 + \cdots\right] \end{aligned}$$

$$i.e. \qquad F(y) = (y-1)\frac{1}{2}\log\left[\frac{1+\left(\frac{y-1}{y+1}\right)}{1-\left(\frac{y-1}{y+1}\right)}\right] = (y-1)\frac{1}{2}\log y \qquad (2.1.1)$$

Next we get the subsequent f-divergence measure if we put $y = \frac{p_i}{q_i}$ in (2.1.1) $i.e.$

$$D_1(P, Q) = \sum_{i=1}^{n} q_i \left(\frac{p_i}{q_i} - 1\right)\frac{1}{2}\log\left(\frac{p_i}{q_i}\right).$$

Applying intuitionistic fuzzy in above equation, we achieve

$$D_1(A, B) = \sum_{i=1}^{n} \mu_B(x_i)\left(\frac{\mu_A(x_i)}{\mu_B(x_i)} - 1\right)\frac{1}{2}\log\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)$$

$$+ \sum_{i=1}^{n} \nu_B(x_i)\left(\frac{\nu_A(x_i)}{\nu_B(x_i)} - 1\right)\frac{1}{2}\log\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}\right)$$

$$i.e. \quad 2D_1(A, B) = \sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i)\right)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}$$

$$+ \sum_{i=1}^{n}\left(\nu_A(x_i) - \nu_B(x_i)\right)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}$$

$$i.e. \quad 2D_1(A, B) = \sum_{i=1}^{n} \mu_A(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} - \sum_{i=1}^{n} \mu_B(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}$$

$$+ \sum_{i=1}^{n} \nu_A(x_i)\log\frac{\nu_A(x_i)}{\nu_B(x_i)} - \sum_{i=1}^{n} \nu_B(x_i)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}$$

$$= \sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i) + \mu_B(x_i)\right)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} - \sum_{i=1}^{n} \mu_B(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}$$

$$+ \sum_{i=1}^{n}\left(\nu_A(x_i) - \nu_B(x_i) + \nu_B(x_i)\right)\log\frac{\nu_A(x_i)}{\nu_B(x_i)} - \sum_{i=1}^{n} \nu_B(x_i)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}$$

$$= \sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i)\right)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} + \sum_{i=1}^{n} \mu_B(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)} - \sum_{i=1}^{n} \mu_B(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}$$

$$+\sum_{i=1}^{n}\left(\nu_A(x_i)-\nu_B(x_i)\right)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}+\sum_{i=1}^{n}\nu_B(x_i)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}-\sum_{i=1}^{n}\nu_B(x_i)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}$$

$$=J(A,B)-\sum_{i=1}^{n}\mu_B(x_i)\log\frac{\mu_B(x_i)}{\mu_A(x_i)}-\sum_{i=1}^{n}\mu_B(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}$$

$$-\sum_{i=1}^{n}\nu_B(x_i)\log\frac{\nu_B(x_i)}{\nu_A(x_i)}-\sum_{i=1}^{n}\nu_B(x_i)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}$$

$$=J(A,B)-K(B,A)-\sum_{i=1}^{n}\mu_B(x_i)\log\frac{\mu_A(x_i)}{\mu_B(x_i)}-\sum_{i=1}^{n}\nu_B(x_i)\log\frac{\nu_A(x_i)}{\nu_B(x_i)}$$

$$=J(A,B)-K(B,A)-\sum_{i=1}^{n}\mu_B(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}.\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right)$$

$$-\sum_{i=1}^{n}\nu_B(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_B(x_i)}.\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)$$

$$=J(A,B)-K(B,A)+\sum_{i=1}^{n}\mu_B(x_i)\log\left(\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right)-\sum_{i=1}^{n}\mu_B(x_i)\log\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right)$$

$$+\sum_{i=1}^{n}\nu_B(x_i)\log\left(\frac{2\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)-\sum_{i=1}^{n}\nu_B(x_i)\log\left(\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)$$

$$=J(A,B)-K(B,A)+F(B,A)-\sum_{i=1}^{n}\frac{\mu_B(x_i)}{\mu_A(x_i)}\left(\mu_A(x_i)\log\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right)\right)$$

$$-\sum_{i=1}^{n}\frac{\nu_B(x_i)}{\nu_A(x_i)}\left(\nu_A(x_i)\log\left(\frac{2\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right)$$

$$=J(A,B)-K(B,A)+F(B,A)-F(A,B).$$

Hence the required equality.

Proposition 2.2 Let $(A,B)\in\Gamma n\times\Gamma n$ and if $\boldsymbol{A}(A,B)$, $G(A,B)$, $F(A,B)$, $\mathcal{X}^2(A,B)$ represents AMD, RAGD, RJSD, Chi-Square distance measures respectively then show that the new equality relation

$$\frac{4D_1(P,Q).\boldsymbol{A}(A,B)-G(P,Q)-G(Q,P)}{A(P,Q)}=F(Q,P)+F(P,Q)$$

and inequality relation

$$4D_1(P,Q).\boldsymbol{A}(A,B)-G(P,Q)-G(Q,P)<[\mathcal{X}^2(Q,P)+\mathcal{X}^2(P,Q)]\boldsymbol{A}(A,B)$$

Proof: If we put $y=\frac{p_i+q_i}{2q_i}$ in the convex function (2.1.1) then we get the subsequent f-divergence measure

$$D_1(P,Q)=(y-1)\frac{1}{2}\log y=\left(\frac{p_i+q_i}{2q_i}-1\right)\frac{1}{2}\log\left(\frac{p_i+q_i}{2q_i}\right)$$

$$2D_1(P,Q) = \frac{1}{2}\sum_{i=1}^{n} (p_i - q_i)\log\left(\frac{p_i+q_i}{2q_i}\right).$$

Applying intuitionistic fuzzy in above equation, we achieve

$$D_1(A,B) = \left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)} - 1\right)\frac{1}{2}\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right) + \left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)} - 1\right)\frac{1}{2}\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right)$$

$$i.e.\ \ 2D_1(A,B) = \frac{1}{2}\sum_{i=1}^{n}\left(\mu_A(x_i) - \mu_B(x_i)\right)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right)$$

$$+\frac{1}{2}\sum_{i=1}^{n}\left(\nu_A(x_i) - \nu_B(x_i)\right)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right)$$

$$i.e.\ \ 2D_1(A,B) = \frac{1}{2}\left[\sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right) - \sum_{i=1}^{n}\mu_B(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=1}^{n}\nu_A(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right) - \sum_{i=1}^{n}\nu_B(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right)\right]$$

$$i.e.\ \ 2D_1(A,B) = \frac{1}{2}\left[\begin{array}{r} \sum_{i=1}^{n}\mu_B(x_i)\log\left(\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) - \sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \\ \sum_{i=1}^{n}\nu_B(x_i)\log\left(\frac{2\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right) - \sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{2\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right) + \\ \sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_A(x_i)}\right) - \sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_A(x_i)}\right) + \\ \sum_{i=1}^{n}\nu_A(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right) - \sum_{i=1}^{n}\nu_A(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right) \end{array}\right]$$

$$i.e.\ \ 2D_1(A,B) = \frac{1}{2}\left[\sum_{i=1}^{n}\mu_B(x_i)\log\left(\frac{2\mu_B(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) - \sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_A(x_i)}\right)\right] +$$

$$\frac{1}{2}\left[\sum_{i=1}^{n}\nu_B(x_i)\log\left(\frac{2\nu_B(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right) - \sum_{i=1}^{n}\nu_A(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_A(x_i)}\right) + \sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=1}^{n}\nu_A(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right) + \sum_{i=1}^{n}\nu_A(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_B(x_i)}\right)\right]$$

$$i.e.\ \ 2D_1(A,B) = \frac{1}{2}[F(B,A) + F(A,B)] + \frac{1}{2}\left[\sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_A(x_i)}.\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}.\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_B(x_i)}\right)\right]$$

$$i.e.\ \ 2D_1(A,B) = \frac{1}{2}[F(B,A) + F(A,B)] + \frac{1}{2}\left[\sum_{i=1}^{n}\mu_A(x_i)\log\left(\frac{\left(\mu_A(x_i)+\mu_B(x_i)\right)^2}{4\mu_A(x_i).\mu_B(x_i)}\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=1}^{n}\nu_A(x_i)\log\left(\frac{\left(\nu_A(x_i)+\nu_B(x_i)\right)^2}{4\nu_A(x_i).\nu_B(x_i)}\right)\right]$$

$$=\frac{1}{2}[F(B,A)+F(A,B)]+\left(\sum_{i=1}^{n}\mu_A(x_i)\right)\left(\sum_{i=1}^{n}\frac{2}{\mu_A(x_i)+\mu_B(x_i)}\right)$$

$$\left(\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right)\log\left(\frac{\left(\mu_A(x_i)+\mu_B(x_i)\right)^2}{4\mu_A(x_i).\mu_B(x_i)}\right)\right)+\left(\sum_{i=1}^{n}\nu_A(x_i)\right)\left(\sum_{i=1}^{n}\frac{2}{\nu_A(x_i)+\nu_B(x_i)}\right)$$

$$\left(\sum_{i=1}^{n}\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2}\right)\log\left(\frac{\left(\nu_A(x_i)+\nu_B(x_i)\right)^2}{4\nu_A(x_i).\nu_B(x_i)}\right)\right)$$

$$=\frac{1}{2}[F(B,A)+F(A,B)]$$

$$+\frac{1}{2A(A,B)}\left[\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_A(x_i)}\right)+\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2}\right)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\mu_B(x_i)}\right)\right]$$

$$+\frac{1}{2A(A,B)}\left[\sum_{i=1}^{n}\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2}\right)\log\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2\nu_A(x_i)}\right)+\sum_{i=1}^{n}\left(\frac{\nu_A(x_i)+\nu_B(x_i)}{2}\right)\log\left(\frac{\mu_A(x_i)+\mu_B(x_i)}{2\nu_B(x_i)}\right)\right]$$

Thus, $$2D_1(A,B)=\frac{1}{2}[F(B,A)+F(A,B)]+\frac{1}{2\boldsymbol{A}(A,B)}[G(A,B)+G(B,A)]$$

i.e. $$\frac{4D_1(A,B).\boldsymbol{A}(A,B)-G(A,B)-G(B,A)}{\boldsymbol{A}(A,B)}=F(B,A)+F(A,B) \qquad (2.2.1)$$

Which is the required equality.

But, $F(B,A)+F(A,B)<\mathcal{X}^2(B,A)+\mathcal{X}^2(A,B)$

From (2.2.1) we get the inequality

$$4D_1(A,B).\boldsymbol{A}(A,B)-G(A,B)-G(B,A)<[\mathcal{X}^2(B,A)+\mathcal{X}^2(A,B)]\boldsymbol{A}(A,B).$$

Hence the required result.

Proposition 2.3 Let $(A,B)\in\Gamma n\times\Gamma n$ and if $\mathcal{X}^2(A,B)$, $\psi M(A,B)$, $\phi D_\rho(A,B)$ shows chi-square, Kumar-Johnson, theoretic exponential information distance measures respectively then we have the following new inequality

$$\frac{1}{2}[\mathcal{X}^2(A,B)+\mathcal{X}^2(B,A)]\leq\frac{1}{2}.\psi M(A,B)\leq\phi D_\rho(A,B)$$

Proof: Ofcourse, $$\left(\sqrt{y}-1\right)^2\geq 0$$

$$i.e.\qquad \sqrt{y}+\frac{1}{\sqrt{y}}\geq 2$$

$$i.e. \qquad \frac{1}{2}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right) \geq 1$$

Obviously, $$\frac{1}{2}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right) \leq \left[\frac{1}{2}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)\right]^2 \leq \frac{1}{4}(y+1).\frac{(2y^3+5y^2-2y+3)}{y^2}.e^y$$

$$i.e. \qquad \frac{1}{2}.\frac{(y-1)^2}{\sqrt{y}}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right) \leq \frac{1}{4}.\frac{(y-1)^2}{\sqrt{y}}\left(\sqrt{y}+\frac{1}{\sqrt{y}}\right)^2 \leq \frac{1}{4}(y+1).\frac{(y-1)^2}{\sqrt{y}}.\frac{(2y^3+5y^2-2y+3)}{y^2}.e^y$$

$$i.e. \qquad \frac{1}{2}.\frac{(y+1)(y-1)^2}{y} \leq \frac{1}{4}.\frac{(y^2-1)^2}{y^{3/2}} \leq \frac{1}{4}.\frac{(y-1)(y^2-1)(2y^3+5y^2-2y+3)}{y^{5/2}}.e^y$$

Substituting y by $\frac{p_i}{q_i}$, we achieve the following result

$$\frac{1}{2}.\frac{\left(\frac{p_i}{q_i}+1\right)\left(\frac{p_i}{q_i}-1\right)^2}{\frac{p_i}{q_i}} \leq \frac{1}{4}.\frac{\left[\left(\frac{p_i}{q_i}\right)^2-1\right]^2}{\left(\frac{p_i}{q_i}\right)^{3/2}} \leq \frac{1}{4}.\frac{\left(\frac{p_i}{q_i}-1\right)\left(\left(\frac{p_i}{q_i}\right)^2-1\right)\left(2\left(\frac{p_i}{q_i}\right)^3+5\left(\frac{p_i}{q_i}\right)^2-2\frac{p_i}{q_i}+3\right)}{\left(\frac{p_i}{q_i}\right)^{5/2}}.e^{p_i/q_i}$$

Applying intuitionistic fuzzy in above equation, we achieve

$$\frac{1}{2}.\left[\frac{\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}+1\right)\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}-1\right)^2}{\frac{\mu_A(x_i)}{\mu_B(x_i)}}+\frac{\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}+1\right)\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}-1\right)^2}{\frac{\nu_A(x_i)}{\nu_B(x_i)}}\right] \leq \frac{1}{4}.\left[\frac{\left[\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^2-1\right]^2}{\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^{3/2}}+\frac{\left[\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}\right)^2-1\right]^2}{\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}\right)^{3/2}}\right]$$

$$\leq \frac{1}{4}.\left[\begin{array}{l}\frac{\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}-1\right)\left(\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^2-1\right)\left(2\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^3+5\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^2-2\frac{\mu_A(x_i)}{\mu_B(x_i)}+3\right)}{\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^{5/2}}.e^{\mu_A(x_i)/\mu_B(x_i)} \\ \frac{\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}-1\right)\left(\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}\right)^2-1\right)\left(2\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}\right)^3+5\left(\frac{\nu_A(x_i)}{\nu_B(x_i)}\right)^2-2\frac{\nu_A(x_i)}{\nu_B(x_i)}+3\right)}{\left(\frac{\mu_A(x_i)}{\mu_B(x_i)}\right)^{5/2}}.e^{\nu_A(x_i)/\nu_B(x_i)}\end{array}\right]$$

$$i.e. \qquad \frac{1}{2}\sum_{i=1}^{n}\left[\frac{\left(\mu_A(x_i)+\mu_B(x_i)\right)\left(\mu_A(x_i)-\mu_B(x_i)\right)^2}{\mu_A(x_i)\left(\mu_B(x_i)\right)^2}+\frac{\left(\nu_A(x_i)+\nu_B(x_i)\right)\left(\nu_A(x_i)-\nu_B(x_i)\right)^2}{\nu_A(x_i)\left(\nu_B(x_i)\right)^2}\right]$$

$$\leq \frac{1}{4}\sum_{i=1}^{n}\left[\frac{\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)^2}{\mu_B(x_i)\left(\mu_A(x_i)\mu_B(x_i)\right)^{3/2}}+\frac{\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)^2}{\nu_B(x_i)\left(\nu_A(x_i).\nu_B(x_i)\right)^{3/2}}\right]$$

$$\le \frac{1}{4}\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\left(\mu_A(x_i)-\mu_B(x_i)\right)\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)\left(2\mu_A^3(x_i)+5\mu_A^2(x_i)\mu_B(x_i)-2\mu_A(x_i)\mu_B^2(x_i)+3\mu_B^3(x_i)\right)}{\mu_B(x_i)\left(\mu_A(x_i)\mu_B(x_i)\right)^{5/2}}.e^{\mu_A(x_i)/\mu_B(x_i)} \\ \frac{\left(\nu_A(x_i)-\nu_B(x_i)\right)\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)\left(2\nu_A^3(x_i)+5\nu_A^2(x_i)\nu_B(x_i)-2\nu_A(x_i)\nu_B^2(x_i)+3\nu_B^3(x_i)\right)}{\nu_B(x_i)\left(\nu_A(x_i)\nu_B(x_i)\right)^{5/2}}.e^{\nu_A(x_i)/\nu_B(x_i)}\end{array}\right]$$

$i.e.$ $$\frac{1}{2}\sum_{i=1}^{n}\left[\frac{\left(\mu_A(x_i)+\mu_B(x_i)\right)\left(\mu_A(x_i)-\mu_B(x_i)\right)^2}{\mu_A(x_i)\mu_B(x_i)}+\frac{\left(\nu_A(x_i)+\nu_B(x_i)\right)\left(\nu_A(x_i)-\nu_B(x_i)\right)^2}{\nu_A(x_i)\nu_B(x_i)}\right]$$

$$\le \frac{1}{4}\sum_{i=1}^{n}\left[\frac{\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)^2}{\left(\mu_A(x_i)\mu_B(x_i)\right)^{3/2}}+\frac{\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)^2}{\left(\nu_A(x_i).\nu_B(x_i)\right)^{3/2}}\right]$$

$$\le \frac{1}{4}\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\left(\mu_A(x_i)-\mu_B(x_i)\right)\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)\left(2\mu_A^3(x_i)+5\mu_A^2(x_i)\mu_B(x_i)-2\mu_A(x_i)\mu_B^2(x_i)+3\mu_B^3(x_i)\right)}{\left(\mu_A(x_i)\mu_B(x_i)\right)^{5/2}}.e^{\mu_A(x_i)/\mu_B(x_i)} \\ +\frac{\left(\nu_A(x_i)-\nu_B(x_i)\right)\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)\left(2\nu_A^3(x_i)+5\nu_A^2(x_i)\nu_B(x_i)-2\nu_A(x_i)\nu_B^2(x_i)+3\nu_B^3(x_i)\right)}{\left(\nu_A(x_i)\nu_B(x_i)\right)^{5/2}}.e^{\nu_A(x_i)/\nu_B(x_i)}\end{array}\right]$$

$i.e.$ $$\frac{1}{2}\left[\sum_{i=1}^{n}\left(\left(\frac{\mu_A^2(x_i)}{\mu_B(x_i)}-\mu_B(x_i)\right)+\left(\frac{\nu_A^2(x_i)}{\nu_B(x_i)}-\nu_B(x_i)\right)\right)\right]$$

$$+\frac{1}{2}\left[\sum_{i=1}^{n}\left(\left(\frac{\mu_B^2(x_i)}{\mu_A(x_i)}-\mu_A(x_i)\right)+\left(\frac{\nu_B^2(x_i)}{\nu_A(x_i)}-\nu_A(x_i)\right)\right)\right]$$

$$\le \frac{1}{4}\sum_{i=1}^{n}\left(\frac{\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)^2}{\left(\mu_A(x_i)\mu_B(x_i)\right)^{3/2}}+\frac{\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)^2}{\left(\nu_A(x_i)\nu_B(x_i)\right)^{3/2}}\right)$$

$$\le \frac{1}{4}\sum_{i=1}^{n}\left[\begin{array}{l}\frac{\left(\mu_A(x_i)-\mu_B(x_i)\right)\left(\mu_A^2(x_i)-\mu_B^2(x_i)\right)\left(2\mu_A^3(x_i)+5\mu_A^2(x_i)\mu_B(x_i)-2\mu_A(x_i)\mu_B^2(x_i)+3\mu_B^3(x_i)\right)}{\left(\mu_A(x_i)\mu_B(x_i)\right)^{5/2}}.e^{\mu_A(x_i)/\mu_B(x_i)} \\ +\frac{\left(\nu_A(x_i)-\nu_B(x_i)\right)\left(\nu_A^2(x_i)-\nu_B^2(x_i)\right)\left(2\nu_A^3(x_i)+5\nu_A^2(x_i)\nu_B(x_i)-2\nu_A(x_i)\nu_B^2(x_i)+3\nu_B^3(x_i)\right)}{\left(\nu_A(x_i)\nu_B(x_i)\right)^{5/2}}.e^{\nu_A(x_i)/\nu_B(x_i)}\end{array}\right]$$

$i.e.$ $$\frac{1}{2}\left[\mathcal{X}^2(A,B)+\mathcal{X}^2(B,A)\right]\le \frac{1}{2}.\psi M(A,B)\le \phi D_\rho(A,B).$$

Hence the inequality.

Proposition 2.4 Let $(P, Q) \in \Gamma n \times \Gamma n$ and if $B(P, Q)$, $H(P, Q)$, $\mathcal{X}^2(P, Q)$ shows Bhattacharya, Harmonic Mean, Chi-Square distance measures respectively then show that the following new inequality

$$\mathcal{X}^2(A, B) > B(A, B) > H(A, B)$$

Proof: Considering an inequality

$$y + 1 > \sqrt{y} > \frac{y}{y+1}$$

obviously $$y^2 - 1 > \sqrt{y} > \frac{2y}{y+1}.$$

Substituting y by $\frac{p_i}{q_i}$, we achieve the following result

$$\left(\frac{{p_i}^2}{{q_i}^2} - 1\right) > \sqrt{{p_i}/{q_i}} > \frac{2\,{p_i}/{q_i}}{{p_i}/{q_i} + 1}$$

Applying intuitionistic fuzzy in above equation, we achieve as follows:

$$\left(\frac{\mu_A^2(x_i)}{\mu_B^2(x_i)} - 1\right) + \left(\frac{\nu_A^2(x_i)}{\nu_B^2(x_i)} - 1\right) > \sqrt{\frac{\mu_A(x_i)}{\mu_B(x_i)}} + \sqrt{\frac{\nu_A(x_i)}{\nu_B(x_i)}} > \frac{2.\frac{\mu_A(x_i)}{\mu_B(x_i)}}{\frac{\mu_A(x_i)}{\mu_B(x_i)}+1} + \frac{2.\frac{\nu_A(x_i)}{\nu_B(x_i)}}{\frac{\nu_A(x_i)}{\nu_B(x_i)}+1}$$

$$i.e.\ \left(\frac{\mu_A^2(x_i)}{\mu_B^2(x_i)} - 1\right) + \left(\frac{\nu_A^2(x_i)}{\nu_B^2(x_i)} - 1\right) > \sqrt{\frac{\mu_A(x_i)}{\mu_B(x_i)}} + \sqrt{\frac{\nu_A(x_i)}{\nu_B(x_i)}} > \frac{2.\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)} + \frac{2.\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}$$

$$i.e.\ \sum_{i=1}^{n}\left[\mu_B(x_i)\left(\frac{\mu_A^2(x_i)}{\mu_B^2(x_i)} - 1\right) + \nu_B(x_i)\left(\frac{\nu_A^2(x_i)}{\nu_B^2(x_i)} - 1\right)\right]$$

$$> \sum_{i=1}^{n}\left[\mu_B(x_i)\sqrt{\frac{\mu_A(x_i)}{\mu_B(x_i)}} + \nu_B(x_i)\sqrt{\frac{\nu_A(x_i)}{\nu_B(x_i)}}\right]$$

$$> \sum_{i=1}^{n}\left[\mu_B(x_i)\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \nu_B(x_i)\left(\frac{2.\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right]$$

$$i.e.\ \sum_{i=1}^{n}\left[\mu_B(x_i)\left(\frac{\mu_A^2(x_i)}{\mu_B^2(x_i)} - 1\right) + \nu_B(x_i)\left(\frac{\nu_A^2(x_i)}{\nu_B^2(x_i)} - 1\right)\right]$$

$$> \sum_{i=1}^{n}\left[\sqrt{\mu_A(x_i).\mu_B(x_i)} + \sqrt{\nu_A(x_i).\nu_B(x_i)}\right]$$

$$> \sum_{i=1}^{n}\left[\mu_B(x_i)\left(\frac{2\mu_A(x_i)}{\mu_A(x_i)+\mu_B(x_i)}\right) + \nu_B(x_i)\left(\frac{2.\nu_A(x_i)}{\nu_A(x_i)+\nu_B(x_i)}\right)\right]$$

$$i.e.\quad \mathcal{X}^2(A, B) > B(A, B) > H(A, B)$$

Hence the desired result.

CONCLUSIONS

Using fuzzy Csiszar's f-divergence and fuzzy new f-divergence measure properties with the demonstration of validity, we have achieved some series of fuzzy divergence measures in this study. We have suggested a generalized series of Kulback-Leibler, arithmetic divergence measures, etc. combinations. Inequalities on both new and well-known fuzzy divergence metrics have also been derived by us.

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