*Original Research Article*

# OPTIMAL INVESTMENT STRATEGIES FOR DC-PENSION FUND UNDER COMBINED STOCHASTIC VOLATILITY MODELS


**ABSTRACT:**

We investigate the optimal investment strategies of DC pension under stochastic volatility model using combined Heston-Hull-White (HHW) model with a constant income drawdown. The pension fund manager (PFM) aims to maximize the expected terminal utility of wealth in a complete market setting under constant relative risk aversion (CRRA). The goal of the PFM is to maintain the standard of living of the participants after retirement. We derive the HJB equation associated with the control problem and finally established the close form solution using stochastic dynamic programming principle (SDPP). The results show that the optimal investment and benefit payment strategies converge uniquely with time.



**KEYWORDS**: *DC pension fund, Optimal control strategies, Power utility, Stochastic optimal control, Hybrid model*


## 1. INTRODUCTION

There are two different methods to implement pension funds policies: Defined-Benefit plan (DB) and Defined-Contribution plan (DC). In DB, the benefits are fixed. The benefits accruable are defined in advanced and the contributions are systematically adjusted to guarantee that the fund balance is maintained in line with government policies. The participant of this type of scheme is responsible for all the associated financial risks. The defined-contribution scheme is designed so that the contributions are defined in advance and benefits on the return on the assets of the fund, all the associated financial risks are borne by the beneficiary.

“The demographic advancement that affects the sustainability of retirement income and the subsequent evolution of equity market has increased adoption of the subject. The benefit payments depend on the fund portfolio and the efficiency of the investment strategy” [21].

“In the classical work of Merton dynamics portfolio selection model, return rates and volatilities of risk assets are all assumed to be deterministic” [1] and[2]. However, many scholars started to investigate the problems of investment and consumption under different market environments [3],[4], [5], [6],[7],[8], and [9]. In these works, the optimal control theory is a dominant approach in solving problems associated to asset allocation and optimization for the pension scheme.

However, in the real world, the market tends to support the stochastic volatility. Recently, the studies on the stochastic volatility (SV) model is a useful tool in examining the stock price. See the works of [10], and [11] all used Heston’s volatility model to describe the price of the risky asset. By extension, Guan and Liang [12] and Hao and Xue-Yan [13] derived “the optimal investment and consumption problem of a DC pension fund with combined stochastic affine interest rate and stochastic volatility”.

In a more practical approach, we investigate the combined models to overcome the problem of stochastic price index persistent in the financial markets.

In this paper, we proposed the use of three (3) combined models of Heston, Hull and White (HHW) to model the dynamics of risky assets price movement and to obtain the optimal investment and payments for DC-pension fund. The model will enable us determine with certain accuracy, tractability and efficiency when pricing derivatives on equity returns and the effect of interest rates. As the pension management and the plan members are given more flexibilities to select without restrictions appropriate benefit outgo, the optimal benefit outgo is described as control variable in recent literatures. In Kapu and Orszag [14], the benefit outgo is dynamically chosen by the PFM to achieve the plan member’s objectives. In DiGiacinto et al [15] solve “the stochastic optimal problem with constraints on the control policies with some defined objective functions”.

The benefit payment policy has not gained popularity in a DC pension fund study. For the benefit payment policy, the retirement depends on traditional DC scheme as demonstrated in actuarial principles and concepts. Furthermore, this study will explore the optimal investment and benefit payments policies problem for a DC pension fund scheme under the income drawdown option associated with stochastic interest rates.

The novel of this work is the introduction of the Hesston-Hull-White model in the pension fund management. We analyze the optimal investment and benefit payments strategies with exact solution under Power utility function. The interesting challenges of the proposed model are the incorporation of the three parameter of the model: stochastic asset price risky diffusion, stochastic volatility diffusion, and the risk-free rate diffusion.

The paper is organized as follows. In section two (2), we setup the dynamics of the fund wealth process using the HHW model. In section three (3), we setup the Optimization procedure for the value function, given the investor (pension fund manager) option on the proportion of wealth to invest in risky asset and amount to withdraw from the pool on exit from the scheme and derive the HJB equation. In section four (4), the closed-form optimal investment and benefit payment policies for the income drawdown is established. Finally, in section five (5) we conclude the findings.

## 2. MATHEMATICAL MODEL.

In this section, we consider the market structure and define the stochastic dynamics of the asset values and the contributions. We consider a complete and frictionless financial market which is continuously open over the fixed time interval $[0,T], where\ T > 0$ denotes the retirement time. We also consider the uncertainty involved in the financial market is defined and shaped by a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{P})\ and\ \mathcal{F} = f(t)_{t\geq 0}$, represents the information available before time $t$ in the market, and adapted from $\mathcal{F}(t)_{t\geq 0}$. $where\ \Omega\ is\ the\ real\ probability\ space\ and\ \mathcal{P}\ is\ the\ probability\ measure$

### A: The Financial Market:

We suppose that the market is composed of three kinds of financial assets namely: fixed income securities (bond/ bank account), equities (stocks) and real estate securities (property). These assets operate independently with their associated risks depending on market environments.

A1: **Fixed income securities**: The first asset in a financial market is the risk-free asset (bond/ bank account) denoted by $\mathcal{B}(t)$ and the price of this asset at time $t$ evolves according to the dynamics of differential equations:

$$d\mathcal{B}(t) = \mathcal{B}(t)t(t)dt \ \ , \ \mathcal{B}(0) = \mathcal{B}_0, r(t) > 0. \quad (1)$$

where $\mathcal{B}_0$ is the initial price of the risk-free asset, and $r(t)$ is the instantaneous rate of interest follows Hull-White model (1990). On the historical probability measure $\mathbb{P}$, the dynamics of $r(t)$ is given by the mean-reverting stochastic differential equation (SDE) as :

$$dr(t) = (\theta(t) - \beta r(t))dt + \sigma dW(t), r(0) = r_0. \quad (2)$$

Where $\sigma > 0$, $\theta(t)$, $and\ \beta$ denotes the interest rate volatility, the mean-reversion which is time dependent, and the reversion rate. $W(t)$ is a standard Brownian motion.

A2: **The Stock**: The stock is denoted by $S(t)$ whose dynamic follows SDE governed by

$$dS(t) = S(t)\theta_s dt + \sigma_s dW_s(t); \ \ S(0) = S_0 \quad (3)$$

Where $S_0, \theta_s(t),\ and\ \sigma_s$ denote; the initial stock price, the expected rate of return, and the stock volatility rate respectively.

A3: **The real estate security**: The real estate security exhibits a significant investment asset and a potential contributor to pension fund wealth. We denote property by $R(t)$ with the dynamics governed by SDE

$$dR(t) = R(t)\theta_R(t)dt + \sigma_R dW_R(t), \ R(0) = R_0 \quad (4)$$

Where $R(0) > 0, \theta_R(t)\ , and\ \sigma_s > 0$ are the initial returns expected rate of return, and the real estate price volatility respectively.

### B: Contributions to the Funds:

In the defined contribution (DC) management, the members will be continuously contributing the part/ proportion of their salaries to the retirement time $T$. The contributions process at time $t$ is given by the SDE.

$$dC(t) = C(t)\theta_c(t)dt + \sigma_C W_C(t), C(0) = C_0. \quad (5)$$

Where $\theta_C > 0, \sigma_C > 0,\ and\ C_0 > 0$ are respectively the rates of contribution, contributions volatility, and the initial contribution.

### C: Pension Wealth Process:

We consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a filtration$\{\mathcal{F}_t\}_{t\geq 0}\ , t \geq 0$.

The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by the trajectories of a 1-dimensional standard Brownian motion$B(t)\ , t \geq 0$. Given the financial market composed of two types assets: riskless and risky assets respectively. The riskless asset $S_0(t), t \geq 0$; evolves according to the dynamics.

$$dS_0(t) = rS_0(t)dt, S_0(0) = 1 \quad (6)$$

Where $r \geq 0$ is the instantaneous spot rate of return. The price of the risky asset $S_0(t), t \geq 0$ follows the Itô's process evolves and satisfies the SDE:

$$dS_1(t) = rS_1(t)dt + \sigma S_1(t)dB(t), \ \ S_1(0) = S_0 \quad (7)$$

$r$ is the rate of expected return and $\sigma > 0$ is the instantaneous rate of volatility.

We assume that the financial market consists of two assets namely: risky and risk-free. Risk-free is usually at time $t$ evolves as ODE

$$d\mathcal{B}(t) = \mathcal{B}(t)r(t)dt \ \ , \ r(0) = r_0 > 0 \quad (8)$$

The price of the risk-free asset is controlled by the ODE as stated in (8). The dynamics of a risky asset price evolve according to the combined models of SDEs as sated as:

$$\left.\begin{aligned} dS(t) &= S(t)\left\{r(t)dt + (v(t)^{\frac{1}{2}}dW_a(t)\right\}, \quad S(0) > 0, \\ dV(t) &= \kappa(\psi - v(t))dt + w(v(t)^{\frac{1}{2}})dW_b(t), \ V(0) > 0 \\ dR(t) &= \lambda(\theta(t) - R(t))dt + \sigma r^p dW_c(t), \qquad R(0) > 0 \end{aligned}\right\} \quad (9)$$

The parameters in (9) are described as follows:

(a) $S(t)$ is the price of asset evolution
(b) $V(t)$ is the volatility and evolves as CIR
(c) $R(t)$ is the risk neural interest rate given by Ornstein Uhlenbeck process.
(d) $W_a, W_b, W_c$ are the Wiener process caused by risky assets, volatility and interest rate processes.

From (9) $r > 0$ a constant interest rate, correlation $dW_a(t)dW_b(t) = \sigma_{a,b}(t)d(t)$ and $|\sigma_{a,b}| < 1$. The variance process, $(t)$ , of the risky asset $S(t)$ is a mean-reverting square root process, in which $\kappa > 0$ determines the speed of adjustment of the volatility towards its theoretical mean $\psi > 0$ and $w > 0$ is the second-order volatility (i.e. the validity of the volatility). Note that (9) is not in the class of affine processes. If $p = 0$ in (9), then it represents Heston-Hull-White (HHW or 2HW) model and for $p = \frac{1}{2}$ it becomes the Heston-Cox-Ingersoll-Ross (HCIR) model.
Furthermore, the correlations of both models are given by $dW_a dW_b = \sigma_{a,b}dt$, $dW_a dW_c = \sigma_{a,c}dt$ , $dW_b dW_c = \sigma_{b,c}dt$. We define for lucidity the hybrid SDE models that fit into the class of affine diffusion process (see [20]). For processes within this class a closed-form solution of the characteristic function exists.
Consider a system of SDE defined as

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dW(t) \quad (10)$$

(10) is affine if $\mu X(t) = a_0 + a_1 X(t) \quad for\ any\ (a_0, a_1) \in R^n \times R^{n\times n}$ (11)

$$(\sigma(X(t))\sigma(X(t))^T)_{i,j} = (C_0)_{i,j} + (C_1)_{i,j}^T X(T), \quad for\ (C_0, C_1) \in R^{n\times n} \times R^{n\times n\times n} \quad (12)$$

$$r(X(t)) = r_0 + r_1^T X(t) \quad for\ (r_0, r_1) \in R \times R^n \quad (13)$$

$for\ i,j = 1, \dots, n, with\ r(X(t))$ as the interest rate.
Here, we investigate the effect of hybrid models Heston-Hull-White (HHW) on the optimal investment strategies of a DC-pension scheme.

**Heston Hybrid models With Stochastic Interest Rate**

Consider the state vector $(t) = [S(t), V(t)]^T$ , under the risk-neutral pricing measure, the Heston stochastic volatility model is specified by the following SDEs:

$$dS(t) = S(t)\left[\sigma dt + \sqrt{V(t)}dW_x(t)\right], S(0) > 0 dV(t) = \kappa(\Lambda - V(t))dt + r\sqrt{V(t)}dW_v(t),\ V(0) > 0, \quad (14)$$

$with\ r > 0$ a constant interest rate.
The full hybrid models under consideration is stated as: Given a constant interest rate, $r$, may be insufficient for pricing interest rate sensitive products. We extend the state vector with additional stochastic dynamics:
$X(t) = [S(t), V(t), r(t)]^T$ which corresponds to a hybrid stochastic volatility equity model with a stochastic interest rate process, $r(t)$. Usually the investment problem for the DC pension plan member, the wealth accumulated is usually used to purchase an annuity.
Suppose that $x_0$ is the initial wealth of the member's pension account, and denote an evolving investment strategy by a pair of stochastic processes

$\pi = \{(\pi_s(t), (\pi_B(t)\}, t \in [0,T]$ where $\pi_s(t)$ and $\pi_B(t)$ (15)

are the proportions of wealth invested in stock (bond) and the proportion of wealth invested in risk-free asset at time $t\ is\ \pi_0(t) = 1 - \pi_B(t) - \pi_S(t)$. We denote $X^\pi(t)$ as the wealth amount in the DC pension account at time $t$ underinvestment strategy $\pi$ is stated in (16).

$$dX^\pi(t) = X^\pi(t)\pi_0(t)\frac{dS_0(t)}{S_0} + X^\pi(t)\pi_B\frac{dB_k(t)}{B_k(t)} + X^\pi(t)\pi_s(t)\frac{dS(t)}{S(t)} + C(t)dt \quad (16)$$

In this study, we consider the pension fund member who wishes to participate in income drawdown payment policy at the end of the contract.
This approach allows the retiree to withdraw a percentage of income. At retirement, the pension fund wealth depends on the investment returns and benefit payments.
If $X(t)$ is the wealth, and $\pi(t)$ is the proportion of the wealth invested in the risky asset at time $t$, the balance from the portfolio for the riskless asset investment is $X(t) - \pi(t)$
$Let\ \psi(t)$ be the fixed benefit payments with the dynamics of the pension fund wealth according to the SDE model

$$dX(t) = [X(t)r(t) - \psi(t)]dt + \pi(t)(v(t))^{1/2}dW(t), \quad X(0) = X_0 > 0 \quad (17)$$

where $X(0)$ is the available wealth at time $t$, $\pi(t)$ is the wealth invested in the risky asset and $dW(t)$ is the 1-dimensional Brownian motion.

## 3 The Optimization Framework

The objectives of Pension Fund Manager (PFM) among others is to ensure the best strategy in the investment process, with a define target of fund solvency. Again, the PFM is to select investment strategy to minimize the expected value of utility loss function.
The optimization framework is defined as

$$\hat{Q} = (t, v, r, x) = \inf_{\pi,p} E\left[\int_0^\infty \exp(-\beta t)\, U(p(t), X(t))\, dt\right] \quad (18)$$

$where\ \beta$ is the discount rate, $p(t)$ is the payment at a given time $t$, and $\pi, p$ are the parameters for both investment and outputs strategies.
However, during the transitory stage for $(s, x) \in C$ the problem consists of maximizing over the set of admissible strategies above the solvency level
$l(t) \equiv l \coloneqq l(T)\ \ for\ t \geq T$.
Given

$$\Theta_{ad}(S, x) = \{\theta: [S,T] \times \Omega \to [0,1] - prog.meas.wrt\ \{\mathcal{F}_t\}_{t\in[S,T]} | X(t; s, x, \theta(\cdot) \geq l(t), t \in [S,T]\} \quad (19)$$

**Proposition 1**: $Let\ (s, x) \in C\ and\ let\ X(t) = X(t; s, x, 0)$;

$the\ X(t) - l(t) \geq (x - l(s)) \exp(r(t-s)), t \in [S,T]$ (20)

Recall that for $(s,x) \in C, \theta(\cdot) \in \Theta_{ad}(S,x)$ , we define

$$\hat{Q}(s,x;\ \theta(\cdot)) = E\left[\int_0^\infty \exp(-\rho t)\, U(t, X(t;s,x,\theta(\cdot))dt + f(X(T;s,x,\theta(\cdot))\right] \quad (21)$$

The value function for $(s,x) \in C$ is defined as $V(s,x) = \underset{\theta(\cdot)\in\Theta_{ad}(s,x)}{Sup} \hat{Q}(s,x;\theta(\cdot))$

**Definition 1**:

$Let\ \hat{Q}(t,v,r,x) \in C^{(1,2,2,2)}$ be the space of the value function $\hat{Q}(t,v,r,x)$ with its partial derivatives are continuous on $[0,T] \times R \times R \times R$. The stochastic optimal control for optimal strategies $\Phi = (\pi, p)$ and the value function is given by

$$\hat{Q}(t,v,r,x) = \underset{\pi,\rho}{Max}\ E\{\exp(-\beta t)\,[U(p(t))] | v(t) = v, r(t) = r, X(t) = x\ \} \quad (22)$$

With boundary condition$\hat{Q}(t,v,r,x) = U(x)$. From (22) $v(t), r(t)\ and\ X(t)$ are the volatility of the risky asset, interest rate and fund wealth respectively.

The function of PFM is to find an optimal strategy $\Phi^* = (\pi^*(t), p^*(t))$ such that

$$\hat{Q}(t,v,r,x) = \hat{Q}_{\Phi^*}(t,v,r,x)$$

**The Hamilton-Jacobi-Bellman (HJB) Equation**

The associated HJB equation for the optimal value function by (22) is stated as:

**Theorem 1**:

$Let\ \hat{Q}(t,v,r,x) \in C^{(1,2,2,2)}([0,T] \times R \times R \times R)\ where\ C^{(1,2,2,2)}([0,T] \times R \times R \times R)$ is the space function that are continuously differentiable with respect to$t \in [0,T]$ and twice continuously differentiable with respect to $(v,r,x) \in (R \times R \times R)$.

Then, $\hat{Q}(t,v,r,x)$ satisfies the following HJB equation:

$$0 = \underset{\pi,p}{Sup}\Big\{\exp(-\beta t)\, U(p(t)) + \hat{Q}_t + (xr - p(t))\hat{Q}_t + \kappa(\psi - r)\hat{Q}_v + \lambda(\theta - r)\hat{Q}_r + \tfrac{1}{2}(\pi^2 v)\hat{Q}_{xx} + \tfrac{1}{2}(w^2 v)\hat{Q}_{vv} + \tfrac{1}{2}(\sigma^2)\hat{Q}_{rr} + \pi p_1 w v \hat{Q}_{xv} + \sigma\pi p_2 v^{\frac{1}{2}}\hat{Q}_{xr}\Big\} \quad (23)$$

With boundary condition$\hat{Q}(T,v,r,x) = U(x)$, where$\hat{Q}_t, \hat{Q}_x, \hat{Q}_v, \hat{Q}_r, \hat{Q}_{xx}, \hat{Q}_{vv}, \hat{Q}_{rr}, \hat{Q}_{xv}, \hat{Q}_{xr}$ Denote partial derivatives of first and second order with respect to the variables $t, x, v, r$ respectively.

Proof: For any stopping time where $\in [R_{t,T}]$ , where $R_{t,T}$ is denoted by the set of stopping time valued in $[t,T]$, we assume that

$$\hat{Q}(t,v,r,x) = \underset{\pi}{sup} E\left[\hat{Q}(\tau, v(\tau), r(\tau), x(\tau,\pi)\right] \quad (24)$$

Consider time interval indexed by $\tau = t + s$ with a constant control policy defined by $\Pi(t) = \Pi = (\pi, p)$, for some arbitrary $(\pi, \varphi) \in \Pi$ in the context of (24) of the dynamic programming principle.

$$\hat{Q}(\tau + s, v(\tau + s), r(\tau + s), x(\tau + s, \Pi)) =$$
$$\hat{Q}(t,v,r,x) + \int_t^{\tau+s}(\hat{Q}_t + \hat{Q}\eta^\pi)(i, v(i), r(i), x(i,\pi))di + submartingale \quad (25)$$

by definition

$$\hat{Q}\eta^\pi = \exp(-\beta t)\, U(p(t) + \hat{Q}_t + (xr - p(t))\hat{Q}_x + \kappa(\psi - v)\hat{Q}_v +$$
$$\lambda(\theta - r)\hat{Q}_r + \tfrac{1}{2}(\pi^2 v)\hat{Q}_{xx} + \tfrac{1}{2}(\omega^2 v)\hat{Q}_{vv} + \tfrac{1}{2}(\sigma^2)\hat{Q}_{rr} + \pi\rho_1\omega v\hat{Q}_{xv} + \sigma\pi\rho_2 V^{\frac{1}{2}}\hat{Q}_{xr} \quad (26)$$

using (26) implies : $\int_t^{\tau+s}(\hat{Q}_t + \hat{Q}\eta^\pi)(i, v(i), r(i), x(i,\pi))di \geq 0$ (27)

By mean-value theorem: $\hat{Q}(t,v,r,x) = \hat{Q}\eta^\pi \leq 0$ (28)

Similarly , suppose $(\pi, p) \in \Pi$ is an optimal control policy, then

$$\hat{Q}(t,v,r,x) = E\left[\hat{Q}(\tau + s), v(\tau + s), r(\tau + s), x(\tau + s), \pi^*\right] \quad (29)$$

By conjuncture: The HJB equation defined by (23) has a classical solution $\hat{Q}$ satisfying $\hat{Q}_x > 0\ and\ \hat{Q}_{xx} < 0$.Then the first-order condition for maximizing the quantity in HJB equation yields the optimal control policies

$$(\pi^*(t), p^*(t)) \text{ as } \pi^*(t) = \left(\frac{\sigma\rho_2 V^{\frac{1}{2}}\hat{Q}_{xr}}{V\hat{Q}_{xx}}\right) \text{ and } p^*(t) = \frac{1}{U}\left(\frac{\hat{Q}_x}{e^{-\beta t}}\right) \quad (30)$$

Substituting (30) into (23) gives

$$\hat{Q}_t + xr\hat{Q}_x + e^{-\beta t}U(p(t)) - p(t)\hat{Q}_x + \kappa(\psi - v)\hat{Q}_v + \lambda(\theta - r)\hat{Q}_r +$$
$$\tfrac{1}{2}\omega^2 v\hat{Q}_{rv} + \tfrac{1}{2}\sigma^2\hat{Q}_{rr} - \tfrac{1}{2}\left(\frac{v\rho_1^2\hat{Q}_{xv^2}}{\hat{Q}_{xx}}\right) - \tfrac{1}{2}\left(\rho_2^2\sigma^2\hat{Q}_{xr^2}\right) - \frac{\sigma\omega\rho_1\rho_2 V^{\frac{1}{2}}\hat{Q}_{xr}\hat{Q}_{xv}}{\hat{Q}_{xx}} \quad (31)$$

(31) is transformed into a non-linear second-order PDE.

## 4. Explicit Solution for The Value Function and Optimal Strategies:

We consider the power utility function type to obtain the optimal control strategies. The utility function under consideration is of the type:

$$U(x) = \frac{x^q}{q}, q < 1, q \neq 0 \quad (32)$$

The solution of (31) is of the form:

$$\hat{Q}(t,v,r,x) = e^{-\beta t}\frac{x^q}{q}\Psi(t,v,r) \text{ with BC } \Psi(T,v,r) = 1, \quad (33)$$

with partial derivatives for (33):

$$\hat{Q}_t = (\Psi_t - \Psi\beta)\frac{e^{-\beta t}x^q\Psi}{q} \quad ; \quad \hat{Q}_x = e^{-\beta t}x^{q-1}\Psi \quad ; \quad \hat{Q}_r = \frac{e^{-\beta t}\Psi_r}{q}$$
$$\hat{Q}_v = \frac{e^{-\beta t}x^q\Psi_{rr}}{q} \quad ; \quad \hat{Q}_{xx} = (q-1)e^{-\beta t}x^{q-2}\Psi \, ; \quad \hat{Q}_{rr} = \frac{e^{-\beta t}x^q\Psi_{rr}}{q}$$
$$\hat{Q}_{vv} = \frac{e^{-\beta t}x^q\Psi_{vv}}{q} \quad ; \quad \hat{Q}_{xv} = e^{-\beta t}x^{q-1}\Psi_v \, ; \quad \hat{Q}_{xr} = e^{-\beta t}x^{q-1}\Psi_r \tag{34}$$

Substituting (34) into (31)

$$\Rightarrow \quad \frac{e^{-\beta t}x^q(\Psi_t - \Psi\beta)}{q} + re^{-\beta t}x^q\Psi + \kappa(\psi - v)\frac{e^{-\beta t}x^q\Psi_v}{q} + \lambda(\theta - r)\frac{e^{-\beta t}x^q\Psi_r}{q}$$
$$\frac{1}{2}\omega^2 v\frac{e^{-\beta t}x^q}{q}\Psi_{vv} + \frac{1}{2}\sigma^2\frac{e^{-\beta t}x^q}{q}\Psi_{rr} - \frac{1}{2}\frac{\rho_2^2\sigma^2 e^{-\beta t}x^q\Psi r^2}{(q-1)\Psi} - \frac{1}{2}\omega^2\rho_1^2 v x^q e^{-\beta t}\Psi v^2$$
$$\frac{-\omega\rho_1\rho_2\sigma v^{\frac{1}{2}}x^q e^{-\beta t}\Psi_r\Psi_v}{(q-1)\Psi} + \frac{(1-q)e^{-\beta t}x^q}{q}\Psi\frac{q}{q-1} = 0 \tag{35}$$

Further simplification and elimination of $x$, yields the non-linear second –order PDE:

$$\Psi_t + (qr - \beta)\Psi + \kappa(\Psi - v)\Psi_v + \lambda(\theta - r)\Psi_r + \frac{1}{2}\omega^2 v\Psi_{vv} + \frac{1}{2}\sigma^2\Psi_{rr} -$$
$$\frac{1}{2}\frac{q\omega^2\rho_1{}^2 v\Psi v^2}{(q-1)\Psi} - \frac{1}{2}\frac{q\rho_2{}^2\sigma^2\Psi r^2}{(q-1)\Psi} - \frac{q\omega\rho_1\rho_2\sigma v^{\frac{1}{2}}\Psi_r\Psi_v}{(q-1)\Psi} + (1-q)\Psi^{\frac{q}{q-1}} = 0 \tag{36}$$

By conjecture, the solution to (36) is of the form with their respective partial derivatives.

$$\Psi(t, v, r) = [\mathrm{G}(t, v, r)]^{1-q} \tag{37}$$
$$\Psi_t = (1-q)G^{-q}G_t \quad ; \quad \Psi_v = (1-q)G^{-q}G_v \; ; \; \Psi_r = (1-q)G^{-q}G_r \, ;$$
$$\Psi_{vv} = (1-q)(G^{-q}G_{vv} - qG^{-q-1}G_v^2) \; ; \quad \Psi_{rr} = (1-q)(G^{-q}G_{rr} - qG^{-q-1}G_r^2) \tag{38}$$

Put (38) into (36) yields

$$\begin{cases} (1-q)G^{-q}G_t + (qr - \beta)G^{1-q} + \kappa(\psi - v)(1-q)G^{-q}G_v \\ +\lambda(\theta - r)(1-q)G^{-q}G_r + \frac{1}{2}\omega^2 v(1-q)(G^{-q}G_{vv} - qG^{-q-1}G_v^2) \\ +\frac{1}{2}\sigma^2(1-q)(G^{-q}G_{rr} - qG^{-q-1}G_r^2) - \frac{1}{2}\frac{q\omega^2\rho_1^2 v(1-q)^2(G^{-q})^2 G_v^2}{(q-1)G^{1-q}} \\ -\frac{1}{2}\frac{q\rho_2^2\sigma^2(1-q)^2(G^{-q})^2 G_r^2}{(q-1)G^{1-q}} - \frac{q\omega\rho_1\rho_2\sigma\sqrt{v}(1-q)^2(G^{-q})^2 G_r G_v}{(q-1)G^{1-q}} + (1-q)(G^{1-q})^{\frac{q}{q-1}} = 0 \end{cases} \tag{39}$$

$$\begin{cases} G_t + \frac{qr-\beta}{(1-q)}G + \kappa(\psi - v)G_v + \lambda(\theta - r)G_r + \frac{1}{2}\omega^2 vG_{vv} + \frac{1}{2}\sigma^2 G_{rr} \\ -\frac{1}{2}\omega^2 vq(1-\rho_1^2)\frac{G_v^2}{G} - \frac{1}{2}\sigma^2 q(1-\sigma_2^2)\frac{G_r^2}{G} + q\omega\rho_1\rho_2\sigma\sqrt{v}\frac{G_r G_v}{G} + 1 = 0 \, , \; G(T, v, r) = 1 \end{cases} \tag{40}$$

By straight assumption and noting that (40) is a non-linear 2nd order PDE, the solution of (40) is of the form

$$G(t, v, r) = \alpha^{\frac{1}{1-q}}\int_t^T \hat{G}(s, v, r)ds + (1-\alpha)^{\frac{1}{(1-q)}}\hat{G}(t, v, r) \tag{41}$$

Further simplifications (40) and (41) is given as:

$$\begin{cases} \Big\{\hat{G}_t + \frac{qr-\beta}{(1-q)}\hat{G} + \kappa(\psi - v)\hat{G}_r + \frac{1}{2}\omega^2 v\hat{G}_{vv} + \frac{1}{2}\sigma^2\hat{G}_{rr} \\ -\frac{1}{2}\omega^2 vq(1-\rho_1^2)\frac{\hat{G}_v^2}{\hat{G}} - \frac{1}{2}\sigma^2 q(1-\rho_2^2)\frac{\hat{G}_r^2}{\hat{G}} + q\omega\rho_1\rho_2\sigma\sqrt{v}\frac{\hat{G}_r\hat{G}_v}{\hat{G}} = 0 \\ \hat{G}(T, v, r) = 1 \end{cases} \tag{42}$$

From (42), it follows that its solution is of the form:

$$\hat{G}(t, v, r) = \exp a(t) + b(t)v + c(t)r \; , a(T) = b(T) = c(T) = 0 \tag{43}$$

Subsequent substitutions and simplifications, we arrive at:

$$\begin{cases} (a'(t) + b'(t) + c'(t)r) + \frac{qr-\beta}{(1-q)} + \kappa(\psi - v)b(t) + \lambda(\theta - r)c(t) + \\ \frac{1}{2}\omega^2 vb^2(t) - \frac{1}{2}\omega^2 vq(1-\rho_1^2)b^2(t) + \frac{1}{2}\sigma^2 c^2(t) \\ -\frac{1}{2}\sigma^2 q(1-\rho_2^2)c^2(t) + q\omega\rho_1\rho_2\sigma\sqrt{v}b(t)c(t) = 0 \end{cases} \tag{44}$$

Separating the (44) into the terms of $a'(t), b'(t), c'(t)$ yields:

$$c'(t) - \lambda c(t) + \frac{q}{(1-q)} = 0 \; , \; c(T) = 0 \tag{45}$$
$$b'(t) - \frac{1}{2}\omega^2 q(1-\rho_1^2)\big(b(t)\big)^2 - \kappa b(t) = 0, \quad b(T) = 0 \tag{46}$$
$$a'(t) + \kappa\psi b(t) + \lambda\theta c(t) + \frac{1}{2}\omega^2 v\big(b(t)\big)^2 + \frac{1}{2}\sigma^2\big(1 - q(1-\rho_2^2)\big)c^2(t)$$
$$+q\omega\rho_1\rho_2\sigma\sqrt{v}b(t)c(t) - \frac{\beta}{(1-q)} = 0, \;\; a(T) = 0. \tag{47}$$

From (45), we have $c(t) = \frac{q\left(e^{-\lambda(T-t)}-1\right)}{\lambda(q-1)}$ (48)

$$\Rightarrow \; b'(t) = \frac{1}{2}\omega^2 q(1-\rho_1^2)(b(t))^2 + \kappa b(t) \tag{49}$$

(49) characterizes the general Riccati equation and its solution will be derived as :

$$b^2(t) + \frac{2\kappa b(t)}{\omega^2 q(1-\rho_1^2)} = 0 \tag{50}$$

$Let\ \mathfrak{D} = B^2 - 4AC = \frac{4\kappa^2}{\omega^2 q^2\left(1-\rho_1^2\right)^2}$ , $where\ A, B, C$ are the usual quadratic function with value set as

$$A = 1\,, B = \frac{2\kappa}{\omega^2 q(1-\rho_1^2)} \quad , \; and \;\; C = 0 \tag{51}$$

$Let\ the\ roots\ for\ which\ \mathfrak{D} > 0$ with distinct real roots denoted as

$$\mathfrak{d}_1\ and\ \mathfrak{d}_2\ as\ \mathfrak{d}_{1,2} = \frac{-B \pm \sqrt{\mathfrak{D}}}{2A} \text{ and } \mathfrak{d}_1 = 0\,,\ \mathfrak{d}_2 = \frac{2\kappa}{\omega^2 q(1-\rho_1^2)} \tag{52}$$

Further more, using (49), (50) and (52) yields another set of equations as

$$b'(t) = \frac{1}{2}\omega^2 q(1-\rho_1^2)(b(t)-\mathfrak{d}_1)(b(t)-\mathfrak{d}_2), \quad b(T) = 0 \tag{53}$$

By integration with respect to $t$ results to the equation:

$$\frac{1}{(\mathfrak{d}_1-\mathfrak{d}_2)}\left(\int_t^T \frac{1}{(b(t)-\mathfrak{d}_1)} - \frac{1}{(b(t)-\mathfrak{d}_2)}\right) db(s) = \frac{1}{2}\omega^2 q(1-\rho_1^2) \tag{54}$$

Further integration (54) with boundary condition $(T) = 0$ , we have

$$b(t) = \frac{\mathfrak{d}_1\mathfrak{d}_2 - \mathfrak{d}_1\mathfrak{d}_2 e^{\frac{1}{2}\omega^2 q(1-\rho_1^2)(\mathfrak{d}_1-\mathfrak{d}_2)(T-t)}}{\left(\mathfrak{d}_1 - \mathfrak{d}_2 e^{\frac{1}{2}\omega^2 q(1-\rho_1^2)(\mathfrak{d}_1-\mathfrak{d}_2)(T-t)}\right)} = 0 \tag{55}$$

Using (47) and (55), we obtain

$$a(t) = \int_t^T \lambda\theta_c(s)ds + \int_t^T \sigma^2\left(1-q(1-\rho_2^2)\right)c^2(s)ds - \int_t^T \frac{\beta}{1-q}ds, \;\; a(T) = 0 \tag{56}$$

Put $\alpha = 1$ in (41) and (43) yields

$$G(t,v,r) = \int_t^T \hat{G}(t,v,r)ds = \int_t^T e^{a(s)+rc(s)}ds = \frac{1}{(a(t)+rc(t))}\left(e^{(a(t)+rc(t)(T-t)} - 1\right)$$

(57)

## 5. Formulation of the Optimal Control Strategies:

The optimal control strategies and the value function under HHW model with the CRRA utility function are defined by optimal control strategy $\Phi^* = (\pi^*(t), p^*(t))$ such that

$$\hat{Q}(t,v,r,x) = \hat{Q}_{\Phi^*}(t,v,r,x):$$

$$\pi^*(t) = \frac{\left(\sigma\rho_2\sqrt{v}c(t)\right)}{v}X(t) \tag{58}$$

$$p^*(t) = \frac{(a(t)+rc(t))}{\left(e^{(a(t)+rc(t))(T-t)}-1\right)}X(t) \tag{59}$$

$$\hat{Q}(t,v,r,x) = \frac{e^{-\beta t}x^q e^{(a(t)+rc(t))}}{q} \tag{60}$$

Where $\pi^*(t)\ and\ p^*(t)$ are the optimal investment and the out payment strategies according to the value function defined by $\hat{Q}(t,v,r,x)$.

The above formulation indicates that the optimal control policies are governed by the Fund wealth and risk aversion levels, and the interest rate process.

## 6. Conclusion

In this paper, we emphasized on the income drawdown policy associated with the optimal investment and benefit payment cash outflow for a DC pension fund. We also examined the volatility and interest rate dynamics that affect the optimal control strategies, and the closed-form solution using the HJB methods. Finally, the pension fund managers have to control the investment and the benefit payments policies to achieve the goals of the pension fund members with a view of maintaining a minimum life style on exit from the pension scheme.

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