Original Research Article

A New Approach to Bicomplex Jacobsthal Matrix and Bicomplex Jacobsthal-Lucas Matrix Components

.

ABSTRACT

In this present paper, we give a detailed study of a new generation of Bicomplex Jacobsthal matrix and Bicomplex Jacobsthal-Lucas matrix using Jacobsthal $\mathcal{F}$-matrix and Jacobsthal Lucas $\mathcal{F}'$-matrix. Also presented some formulas, facts, and properties about these matrices. In addition, a new vector called the bicomple Jacobsthal vectors $\vec{\mathcal{B}}_n$ and the bicomplex Jacobsthal-Lucas vectors $\vec{\mathcal{B}}'_n$ with matrix components is presented. Some properties of this vector apply to various properties of geometry which are not generally known in the geometry of space. Then, using this matrix representation, we give some identities.

Keywords: Bicomplex Jacobsthal matrix; Bicomplex Jacobsthal-Lucas matrix; Jacobsthal numbers; Jacobsthal-Lucas numbers.

1 Introduction

Bicomplex numbers are introduced in a book based on multicomplex spaces and function and are given from a number of different points of view of analysis. A bicomplex number is defined by

$$X = a + bi_1 + c\, i_2 + d\, i_3$$

where *a,b,c,d* are real numbers and +1, i_1, i_2, i_3 are units given by rules $i_1^2 = i_2^2 = -1$ and, $i_1 i_2 = i_2 i_1 = i_3$. Let X and X' be bicomplex numbers.
The addition and subtraction of X and X' are given by

$$X \mp X' = (a \mp a') + (b \mp b')i_1 + (c \mp c')\, i_2 + (d \mp d')i_3$$

and multiplication of these numbers as follows

$$\begin{aligned} XX' &= (a + b\, i_1 + ci_2 + d\, i_3)(a' + b'i_1 + c'i_2 + d'i_3) \\ &= (aa' - bb' - cc' + dd') + (ab' + ba' - cd' - dc')\, i_1 \\ &\quad + (ac' - ca' + bd' - db')\, i_2 + (ad' + da' + cb' + bc')\, i_3. \end{aligned}$$

The conjugates of the bicomplex number X are described by X^{i_1}, X^{i_2} and X^{i_3}. In that case, there are different conjugations as follows, [1]:

$$\begin{aligned} X^{i_1} &= a - bi_1 + ci_2 - di_3, \\ X^{i_2} &= a + bi_1 - ci_2 - di_3, \\ X^{i_3} &= a - bi_1 - ci_2 + di_3. \end{aligned} \qquad (1)$$

The Jacobsthal numbers J_n are defined for all integers $n \geq 0$ by the second order recurrence relation

$$J_{n+2} = J_{n+1} + 2J_n$$

and *initial conditions* $J_0 = 0$, $J_1 = 1$. The Jacobsthal-Lucas numbers j_n are defined for all integers $n \geq 0$ by the same second order recurrence relation as

$$j_{n+2} = j_{n+1} + 2j_n$$

but *initial conditions* $j_0 = 2$ and $j_1 = 1$.

The next Jacobsthal number is also given by the recursion formula:

$$J_{n+1} - 2J_n = (-1)^n \text{ by } J_{n+1} = 2^n - J_n.$$

The first recursion formula above is also satisfied by *the powers of* 2. The Jacobsthal number at a specific point in the sequence may be calculated directly using *the closed-form equation*:

$$J_n = \frac{2^n-(-1)^n}{3}. \tag{2}$$

The Jacobsthal numbers can be extended to *negative indices* using the recurrence relation or *the explicit formula*, giving

$$J_{-n} = \frac{(-1)^{n+1} J_n}{2}.$$

The following identity holds

$$2^n(J_{-n} + J_n) = 3J_n^2.$$

The following Jacobsthal-Lucas number also satisfies:

$$j_{n+1} - 2J_n = -3(-1)^n.$$

The Jacobsthal-Lucas number at a specific point in the sequence may be calculated directly using *the closed-form* equation:

$$j_n = 2^n + (-1)^n. \tag{3}$$

Different applications of Jacobsthal and Jacobsthal-Lucas numbers and their generalization have been studied by many mathematicians in almost all fields of science [2-11]. Jacobsthal $\mathcal{F}$-matrix and Jacobsthal Lucas $\mathcal{F}'$-matrix are presented in [12-17].

The n^{th}-power of the $\mathcal{F}$-matrix and $\mathcal{F}'$-matrix are

$$\mathcal{F}^n = \mathcal{F}_n = \begin{bmatrix} J_{n+1} & 2J_n \\ J_n & 2J_{n-1} \end{bmatrix} \text{ and } \mathcal{F}'_n = \begin{bmatrix} j_{n+1} & 2j_n \\ j_n & 2j_{n-1} \end{bmatrix} \tag{4}$$

where

$$\mathcal{F}^{n+1}\mathcal{F}^n = \mathcal{F}^{2n+1} = \mathcal{F}_{n+1}\mathcal{F}_n = \mathcal{F}_{2n+1} \text{ and also } \det(\mathcal{F}^n) = (-1)^n(-2)^n. \tag{5}$$

In this paper, we define the bicomplex Jacobsthal matrix and the bicomplex Jacobsthal-Lucas matrix by combining Jacobsthal numbers, Jacobsthal-Lucas numbers and bicomplex numbers. We present a matrix representation using some properties of these numbers.

A bicomplex Jacobsthal numbers and bicomplex Jacobsthal-Lucas numbers, respectively, are defined

$$X_n = J_n + J_{n+1}\, i_1 + J_{n+2}\, i_2 + J_{n+3}\, i_3 \quad \text{and} \quad X'_n = j_n + j_{n+1}\, i_1 + j_{n+2}\, i_2 + j_{n+3}\, i_3, \tag{6}$$

where J_n and $\mathrm{j_n}$ are the n^{th} Jacobsthal, Jacobsthal-Lucas numbers and $i_1^2 = i_2^2 = -1$, $i_3^2 = +1$. If we start from $n = 0$, the bicomplex Jacobsthal number and bicomplex Jacobsthal-Lucas number can be written, respectively, as;

$$X_0 = 1i_1 + 1i_2 + 3i_3;\ X_1 = 1 + 1i_1 + 3i_2 + 5i_3$$

and

$$X'_0 = 2 + 1i_1 + 5i_2 + 7i_3;\ \ X'_1 = 1 + 5i_1 + 7i_2 + 17i_3.$$

2 Bicomplex Jacobsthal Matrix and Bicomplex Jacobsthal-Lucas Matrix

For $n \geq 0$, the n^{th} bicomplex Jacobshtal matrix $\mathcal{B}_n$ and the n^{th} bicomplex Jacobsthal-Lucas matrix $\mathcal{B}'_n$ are de ined as

$$\mathcal{B}_n = \mathcal{F}_n + \mathcal{F}_{n+1}\, i_1 + \mathcal{F}_{n+2}\, i_2 + \mathcal{F}_{n+3}\, i_3 \text{ and } \mathcal{B}'_n = \mathcal{F}'_n + \mathcal{F}'_{n+1} i_1 + \mathcal{F}'_{n+2}\, i_2 + \mathcal{F}'_{n+3}\, i_3 \tag{7}$$

where i_1, i_2 and i_3 are *arbitrary units* which satisfy the relations;

$$i_1^2 = i_2^2 = -1,\ i_3^2 = +1. \tag{8}$$

Starting from $n = 1$, the bicomplex Jacobsthal matrix can be written as;

$$\begin{aligned}
\mathcal{B}_1 &= \mathcal{F}_1 + \mathcal{F}_2 i_1 + \mathcal{F}_3\, i_2 + \mathcal{F}_4\, i_3 \\
&= \begin{bmatrix} J_2 & 2J_1 \\ J_1 & 2J_0 \end{bmatrix} + \begin{bmatrix} J_3 & 2J_2 \\ J_2 & 2J_1 \end{bmatrix} i_1 + \begin{bmatrix} J_4 & 2J_3 \\ J_3 & 2J_2 \end{bmatrix} i_2 + \begin{bmatrix} J_5 & 2J_4 \\ J_4 & 2J_3 \end{bmatrix} i_3 \\
&= \begin{bmatrix} 1 + 3i_1 + 5i_2 + 11i_3 & 2 + 2i_1 + 6i_2 + 10i_3 \\ 1 + 1i_1 + 3i_2 + 5i_3 & 2i_1 + 2i_2 + 6i_3 \end{bmatrix} \\
&= \begin{bmatrix} X_2 & 2X_1 \\ X_1 & 2X_0 \end{bmatrix},
\end{aligned}$$

where X_0, X_1 and X_2 are the bicomplex Jacobsthal numbers.

3 Some Identities on Bicomplex Jacobsthal Matrix

Identity 3.1. Let $n \geq 1$ be integer. Then, from the equality (1), we can give the following relation between bicomplex Jacobsthal matrices

$$\mathcal{B}_n - \mathcal{B}_{n+1} i_1 - \mathcal{B}_{n+2}\ i_2 + \mathcal{B}_{n+3} i_3 = 45\left(\mathcal{F}_{n-1} + 2(-1)^n\,\mathcal{F}'_{n-1}\begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}\right).$$

Proof. We will give the proof of identity

$$\mathcal{B}_n - \mathcal{B}_{n+1} i_1 - \mathcal{B}_{n+2}\ i_2 + \mathcal{B}_{n+3} i_3.$$

We have,

$$\begin{aligned}
\mathcal{B}_n - \mathcal{B}_{n+1} i_1 - \mathcal{B}_{n+2}\ i_2 + \mathcal{B}_{n+3} i_3 &= \mathcal{F}_n + \mathcal{F}_{n+1}\, i_1 + \mathcal{F}_{n+2}\, i_2 + \mathcal{F}_{n+3}\, i_3\\
&\quad - (\mathcal{F}_{n+1} + \mathcal{F}_{n+2}\, i_1 + \mathcal{F}_{n+3} i_2 + \mathcal{F}_{n+4}\, i_3\,) i_1\\
&\quad - (\mathcal{F}_{n+2} + \mathcal{F}_{n+3}\, i_1 + \mathcal{F}_{n+4} i_2 + \mathcal{F}_{n+5}\, i_3\,) i_2\\
&\quad + (\mathcal{F}_{n+3} + \mathcal{F}_{n+4}\, i_1 + \mathcal{F}_{n+5}\, i_2 + \mathcal{F}_{n+6}\, i_3) i_3\\
&= \mathcal{F}_n + \mathcal{F}_{n+2} - \mathcal{F}_{n+4} - \mathcal{F}_{n+6}
\end{aligned}$$

If we use the equality (4), then we can write as

$$= \begin{bmatrix}J_{n+1} & 2J_n\\ J_n & 2J_{n-1}\end{bmatrix} - \begin{bmatrix}J_{n+3} & 2J_{n+2}\\ J_{n+2} & 2J_{n+1}\end{bmatrix} - \begin{bmatrix}J_{n+5} & 2J_{n+4}\\ J_{n+4} & 2J_{n+3}\end{bmatrix} + \begin{bmatrix}J_{n+7} & 2J_{n+6}\\ J_{n+6} & 2J_{n+5}\end{bmatrix}$$

$$= \begin{bmatrix}J_{n+1} - J_{n+3} - J_{n+5} + J_{n+7} & 2(J_n - J_{n+2} - J_{n+4} + J_{n+6})\\ J_n - J_{n+2} - J_{n+4} + J_{n+6} & 2(J_{n-1} - J_{n+1} - J_{n+3} + J_{n+5})\end{bmatrix},$$

from the equalities (2) and (3), we get;

$$= 15\begin{bmatrix}2^{n+1} & 2^{n+1}\\ 2^n & 2^n\end{bmatrix}$$

$$= 15\left(\begin{bmatrix}3J_n & 3J_n\\ 3J_{n-1} & 3J_{n-1}\end{bmatrix} + \begin{bmatrix}j_n & j_n\\ j_{n-1} & j_{n-1}\end{bmatrix}\right)$$

$$= 15\left(3\begin{bmatrix}J_n & J_n + 2J_{n-1} - 2J_{n-1}\\ J_{n-1} & J_{n-1} + 2J_{n-2} - 2J_{n-2}\end{bmatrix} + \begin{bmatrix}j_n & j_n\\ j_{n-1} & j_{n-1}\end{bmatrix}\right)$$

$$= 15\left(3\begin{bmatrix}J_n & 2J_{n-1}\\ J_{n-1} & 2J_{n-2}\end{bmatrix} + 3\begin{bmatrix}0 & J_n - 2J_{n-1}\\ 0 & J_{n-1} - 2J_{n-2}\end{bmatrix} + \begin{bmatrix}j_n & 2j_{n-1}\\ j_{n-1} & 2j_{n-2}\end{bmatrix} + \begin{bmatrix}0 & j_n - 2j_{n-1}\\ 0 & j_{n-1} - 2j_{n-2}\end{bmatrix}\right)$$

$$= 15\left(3\mathcal{F}_{n-1} + 3\begin{bmatrix}0 & J_n - 2J_{n-1}\\ 0 & J_{n-1} - 2J_{n-2}\end{bmatrix} + \mathcal{F}'_{n-1} + \begin{bmatrix}0 & j_n - 2j_{n-1}\\ 0 & j_{n-1} - 2j_{n-2}\end{bmatrix}\right)$$

$$= 45\mathcal{F}_{n-1} + 15\mathcal{F}'_{n-1}\left(3\begin{bmatrix}0 & -3(-1)^n\\ 0 & -3(-1)^{n-1}\end{bmatrix} + \begin{bmatrix}0 & 3(-1)^n\\ 0 & 3(-1)^{n-1}\end{bmatrix}\right)$$

$$= 45\left(\mathcal{F}_{n-1} + 2(-1)^n\,\mathcal{F}'_{n-1}\begin{bmatrix}0 & -1\\ 0 & 1\end{bmatrix}\right).$$

Identity 3.2. For $n \geq 1$

$$\mathcal{B}_n\mathcal{B}_n^{i_3} + \mathcal{B}_{n-1}\mathcal{B}_{n-1}^{i_3} = -93\left(\mathcal{F}'_{2n-2} + \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\right)$$

where $\mathcal{B}_n^{i_3}$ is the conjugation with respect to *the imaginary unit* i_3.

Proof. Now we will prove the identity $\mathcal{B}_n \times \mathcal{B}_n^{i_3} + \mathcal{B}_{n-1} \times \mathcal{B}_{n-1}^{i_3}$.
By using the equalities (1), (4) and (6), we get

$$=\mathcal{F}_{n-1}^2 + 2\,\mathcal{F}_n^2 - 2\,\mathcal{F}_{n+2}^2 - \mathcal{F}_{n+3}^2$$

$$= \mathcal{F}_{2n-2} + 2\mathcal{F}_{2n} - 2\mathcal{F}_{2n+4} - \mathcal{F}_{2n+6}$$

$$= \begin{bmatrix} J_{2n-1} & 2J_{2n-2} \\ J_{2n-2} & 2J_{2n-3} \end{bmatrix} + 2\begin{bmatrix} J_{2n+1} & 2J_{2n} \\ J_{2n} & 2J_{2n-1} \end{bmatrix} - 2\begin{bmatrix} J_{2n+5} & 2J_{2n+4} \\ J_{2n+4} & 2J_{2n+3} \end{bmatrix} - \begin{bmatrix} J_{2n+7} & 2J_{2n+6} \\ J_{2n+6} & 2J_{2n+5} \end{bmatrix}$$

$$= \begin{bmatrix} J_{2n-1} + 2J_{2n+1} - 2J_{2n+5} + J_{2n+7} & 2(J_{2n-2} + 2J_{2n} - 2J_{2n+4} + J_{2n+6}) \\ J_{2n-2} + 2J_{2n} - 2J_{2n+4} + J_{2n+6} & 2(J_{2n-3} + 2J_{2n-1} - 2J_{2n+3} + J_{2n+5}) \end{bmatrix}.$$

If we use the equalities $J_{n+r} - J_{n-r} = \frac{1}{3}\left(2^{n-r}(2^{2r} - 1)\right)$, $j_{n+1} - 2j_n = 3(-1)^{n+1}$ and $j_n = 2^n + (-1)^n$ in [4], we have;

$$= -93\begin{bmatrix} 2^{2n-1} & 2^{2n-1} \\ 2^{2n-2} & 2^{2n-2} \end{bmatrix}$$

$$= -93\begin{bmatrix} j_{2n-1} - (-1)^{2n-1} & j_{2n-1} - (-1)^{2n-1} \\ j_{2n-2} - (-1)^{2n-2} & j_{2n-2} - (-1)^{2n-2} \end{bmatrix}$$

$$= -93\begin{bmatrix} j_{2n-1} - (-1)^{2n-1} & j_{2n-1} + 2j_{2n-2} - 2j_{2n-2} - (-1)^{2n-1} \\ j_{2n-2} - (-1)^{2n-2} & j_{2n-2} + 2j_{2n-3} - 2j_{2n-3} - (-1)^{2n-2} \end{bmatrix}$$

$$= -93\left(\begin{bmatrix} j_{2n-1} & 2j_{2n-2} \\ j_{2n-2} & 2j_{2n-3} \end{bmatrix} + \begin{bmatrix} -(-1)^{2n-1} & 3(-1)^{2n-1} - (-1)^{2n-1} \\ -(-1)^{2n-2} & 3(-1)^{2n-2} - (-1)^{2n-2} \end{bmatrix}\right)$$

$$= -93\left(\mathcal{F}'_{2n-2} + \begin{bmatrix} 1 & -2 \\ -1 & 2 \end{bmatrix}\right).$$

Identity 3.3.

$$\mathcal{B}_{n+1}^2 - \mathcal{B}_n^2 = 4(X_3 + X'_3 + 3 - 14i_1 - 12i_2)\mathcal{F}_{2n}\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

Proof. By using the equalities (7) and (8),

$$\mathcal{B}_{n+1}^2 = \mathcal{F}_{2n+2} - \mathcal{F}_{2n+4} - \mathcal{F}_{2n+6} + \mathcal{F}_{2n+8} + 2(\mathcal{F}_{n+1}\mathcal{F}_{n+2} - \mathcal{F}_{n+3}\mathcal{F}_{n+4})i_1$$
$$+2(\mathcal{F}_{n+1}\mathcal{F}_{n+3} - \mathcal{F}_{n+2}\mathcal{F}_{n+4})i_2 + 2(\mathcal{F}_{n+1}\mathcal{F}_{n+4} + \mathcal{F}_{n+2}\mathcal{F}_{n+3})i_3$$

$$=\mathcal{F}_{2n+2} - \mathcal{F}_{2n+4} - \mathcal{F}_{2n+6} + \mathcal{F}_{2n+8} + 2(\mathcal{F}_{2n+3} - \mathcal{F}_{2n+7})i_1$$
$$+2(\mathcal{F}_{2n+4} - \mathcal{F}_{2n+6})i_2 + 4i_3\,\mathcal{F}_{2n+5},$$

$$= \mathcal{F}_{2n}(\mathcal{F}_2 - \mathcal{F}_4 - \mathcal{F}_6 + \mathcal{F}_8) + 2\mathcal{F}_{2n}(\mathcal{F}_3 - \mathcal{F}_7)i_1$$

$$+2\mathcal{F}_{2n}(\mathcal{F}_4 - \mathcal{F}_6)i_2 + 4\mathcal{F}_{2n+5}i_3$$

$$= \mathcal{F}_{2n}\left(\begin{bmatrix}120 & 120\\ 60 & 60\end{bmatrix} + 2\begin{bmatrix}-80 & -80\\ -40 & -40\end{bmatrix} i_1 + \begin{bmatrix}-64 & -64\\ -32 & -32\end{bmatrix} i_2 + \begin{bmatrix}84 & 88\\ 44 & 40\end{bmatrix} i_3\right)$$

$$= \mathcal{F}_{2n}\begin{bmatrix}120 - 160i_1 - 64i_2 + 84i_3 & 120 - 160i_1 - 64i_2 + 88i_3\\ 60 - 80i_1 - 32i_2 + 44i_3 & 60 - 80i_1 - 32i_2 + 40i_3\end{bmatrix}.$$

Similarly, we can get

$$\mathcal{B}_n^2 = \mathcal{F}_{2n}\begin{bmatrix}80 - 40i_1 - 16i_2 + 4i_3 & 80 - 40i_1 - 16i_2 + 8i_3\\ 40 - 20i_1 - 8i_2 + 4i_3 & 40 - 20i_1 - 8i_2\end{bmatrix}$$

$$\mathcal{B}_{n+1}^2 - \mathcal{B}_n^2 = (20 - 60i_1 - 24i_2 + 40i_3)\mathcal{F}_{2n}\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}$$

$$= 4(X_3 + X'_3 + 3 - 14i_1 - 12i_2)\mathcal{F}_{2n}\begin{bmatrix}2 & 2\\ 1 & 1\end{bmatrix}.$$

Identity 3.4. For $n \geq 0$, The n^{th} negabicomplex Jacobsthal matrix is

$$\mathcal{B}_{-n} = \left(\frac{\begin{bmatrix}2J_{n-1} & -2J_n\\ -J_n & J_{n+1}\end{bmatrix}}{2^n(-1)^n}\right)\begin{bmatrix}X_1 & 2X_1\\ X_0 & X_1 - (1 + i_2 + 3i_3)\end{bmatrix}$$

Proof. We will give proof of identity $\mathcal{B}_{-n}$. We have

$$\mathcal{B}_{-n} = \mathcal{F}_{-n} + \mathcal{F}_{-n+1}\, i_1 + \mathcal{F}_{-n+2}\, i_2 + \mathcal{F}_{-n+3}\, i_3$$

$$\mathcal{B}_{-n} = \mathcal{F}_{-n}(\mathcal{F}_0 + \mathcal{F}_1\, i_1 + \mathcal{F}_2\, i_2 + \mathcal{F}_3\, i_3)$$

$$= (\mathcal{F}_n)^{-1}(\mathcal{F}_0 + \mathcal{F}_1\, i_1 + \mathcal{F}_2\, i_2 + \mathcal{F}_3\, i_3)$$

$$= \left(\begin{bmatrix}J_{n+1} & 2J_n\\ J_n & 2J_{n-1}\end{bmatrix}\right)^{-1}\begin{bmatrix}1 + i_1 + 2i_2 + 3i_3 & i_1 + i_2 + 2i_3\\ i_1 + i_2 + 2i_3 & 1 + i_2 + i_3\end{bmatrix}$$

$$= \left(\frac{\begin{bmatrix}2J_{n-1} & -2J_n\\ -J_n & J_{n+1}\end{bmatrix}}{|\mathcal{F}_n|}\right)\begin{bmatrix}X_1 & 2X_1\\ X_0 & X_1 - (1 + i_2 + 3i_3)\end{bmatrix}$$

$$= \left(\frac{\begin{bmatrix}2J_{n-1} & -2J_n\\ -J_n & J_{n+1}\end{bmatrix}}{(-1)^n 2^n}\right)\begin{bmatrix}X_1 & 2X_1\\ X_0 & X_1 - (1 + i_2 + 3i_3)\end{bmatrix}$$

where X_1 and X_0 are bicomplex Jacobsthal numbers.

4 Some Applications on Bicomplex Jacobsthal Matrix

Let $\mathcal{B}_n$ be the n^{th} bicomplex Jacobsthal matrix, for $n \geq 0$, these number is 2^{th} linear recurrence sequence. Then, we suppose the sets of $\mathbb{C}_2$ and $\mathbb{C}'_2$ are

$$\mathbb{C}_2 = \{ \mathcal{B}_n \mid \mathcal{B}_n = \mathcal{F}_n + \mathcal{F}_{n+1}\, i_1 + \mathcal{F}_{n+2}\, i_2 + \mathcal{F}_{n+3}\, i_3 ,\quad \mathcal{F}_n \text{ is } n^{th}\ \mathcal{F}\text{-matrix} \},$$

and

$$\mathbb{C}_2' = \{ \mathcal{B}_n \mid \mathcal{B}_n = \begin{bmatrix} \alpha_n & \beta_n \\ \beta_n & \alpha_n \end{bmatrix} ; \alpha_n, \beta_n \epsilon \mathbb{C} \}$$

Then there is an *isomorphism* between $\mathbb{C}_2$ and $\mathbb{C}_2'$, in that case, we can write

$$\mathcal{B}_n = (\mathcal{F}_n , \mathcal{F}_{n+1} , \mathcal{F}_{n+2} , \mathcal{F}_{n+3}) \rightarrow \mathcal{B}_n = \begin{bmatrix} \mathcal{F}_n + \mathcal{F}_{n+1} i_1 & \mathcal{F}_{n+2} + \mathcal{F}_{n+3} i_1 \\ \mathcal{F}_{n+2} + \mathcal{F}_{n+3} i_1 & \mathcal{F}_n + \mathcal{F}_{n+1} i_1 \end{bmatrix}.$$

Thus, we can write

$$\mathbb{C}_2' = \{ \mathcal{B}_n \mid \mathcal{B}_n = \alpha_n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \beta_n \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \alpha_n, \beta_n \ \ is\ n^{th}\ complex\ Jacobsthal\ matrix \}$$

and

$$\mathcal{B}_n = \mathcal{F}_n U_1 + \mathcal{F}_{n+1} U_2 + \mathcal{F}_{n+2} U_3 + \mathcal{F}_{n+3} U_4,$$

where

$$\mathcal{B}_n = \mathcal{F}_n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{F}_{n+1} \begin{bmatrix} i_1 & 0 \\ 0 & i_1 \end{bmatrix} + \mathcal{F}_{n+2} \begin{bmatrix} 0 & i_2 \\ i_2 & 0 \end{bmatrix} + \mathcal{F}_{n+3} \begin{bmatrix} 0 & i_3 \\ i_3 & 0 \end{bmatrix}.$$

Since $det\mathcal{B}_n \neq 0$, there is the inverse of matrix $\mathcal{B}_n$ and it is in $\mathbb{C}_2'$.

Definition 4.1. The n^{th} *bicomple Jacobsthal vectors* $\vec{\mathcal{B}}_n$ and the n^{th} *bicomplex Jacobsthal-Lucas vectors* $\vec{\mathcal{B}}_n'$ with matrix components are defined as

$$\vec{\mathcal{B}}_n = \mathcal{F}_{n+1}\, i_1 + \mathcal{F}_{n+2}\, i_2 + \mathcal{F}_{n+3}\, i_3$$

and

$$\vec{\mathcal{B}}_n' = \mathcal{F}_{n+1}' i_1 + \mathcal{F}_{n+2}'\ i_2 + \mathcal{F}_{n+3}' i_3 ,$$

respectively.

Theorem 4.1. Let $\vec{\mathcal{B}}_n$ and $\vec{\mathcal{B}}_{n+1}$ be bicomplex Jacobsthal vectors. *The cross product* is defined as

$$\vec{\mathcal{B}}_n \times \vec{\mathcal{B}}_{n+1} = det \begin{bmatrix} i_1 & i_2 & i_3 \\ \mathcal{F}_{n+1} & \mathcal{F}_{n+2} & \mathcal{F}_{n+3} \\ \mathcal{F}_{n+2} & \mathcal{F}_{n+3} & \mathcal{F}_{n+4} \end{bmatrix}$$

where in *the permanent* of $\vec{\mathcal{B}}_{\mathrm{n}} \times \vec{\mathcal{B}}_{\mathrm{n+1}}$, *the signatures of the permutations* are not taken into account, [15].

Proof. Now, if we calculate $\vec{\mathcal{B}}_n \times \vec{\mathcal{B}}_{n+1}$, we obtain

$$= (\mathcal{F}_{n+2}\mathcal{F}_{n+4} + \mathcal{F}_{n+3}\mathcal{F}_{n+3}) i_1 - (\mathcal{F}_{n+1}\mathcal{F}_{n+4} + \mathcal{F}_{n+2}\mathcal{F}_{n+3}) i_2 + (\mathcal{F}_{n+1}\mathcal{F}_{n+3} + \mathcal{F}_{n+2}\mathcal{F}_{n+2}) i_3$$

$$=2\mathcal{F}_{2n+6}i_1 - 2\mathcal{F}_{n+5}i_2 + 2\mathcal{F}_{n+4}i_3$$

$$=2(\mathcal{F}_{2n+6}i_1 - \mathcal{F}_{2n+5}i_2 + \mathcal{F}_{2n+4}i_3 + \mathcal{F}_{2n+3}i_2 + \mathcal{F}_{2n+2}i_1 - \mathcal{F}_{2n+3}i_2 - \mathcal{F}_{2n+2}i_1)$$

$$=2(\mathcal{F}_{2n+2}i_1 + \mathcal{F}_{n+3}i_2 + \mathcal{F}_{n+4}i_3) + 2(\mathcal{F}_{2n+6}-\mathcal{F}_{2n+2})i_1 - 2(\mathcal{F}_{n+5} + \mathcal{F}_{n+3})i_2$$

Finally, we have

$$= \mathcal{F}_{2n}\left(\begin{bmatrix} 40 & 40 \\ 20 & 20 \end{bmatrix} i_1 - \begin{bmatrix} 25 & 28 \\ 14 & 12 \end{bmatrix} i_2\right) + \vec{\mathcal{B}}_{2n+1},$$

where $\vec{\mathcal{B}}_{2n+1}$ is bicomplex Jacobsthal vector.

5 Conclusion

In this study, we firstly introduced bicomplex Jacobsthal numbers and bicomplex Jacobsthal-Lucas numbers and matrices. Then, we give some identities that hold an important place in the literature on these matrices. We have also given bicomplex Jacobsthal vector $\vec{\mathcal{B}}_n$ with $\mathcal{F}$-matrix component to exert in geometry.

References

[1] Price GB. An Introduction to Multi-complex Spaces and Functions. Marcel Dekker Inc., New York; 1991.

[2] Horadam AF. Jacobsthal and Pell Curves. The Fibonacci Quarterly. 1988;26(1):79-83.

[3] Horadam AF. Jacobsthal Representation Numbers. Fib. Quart. 1996; 34:40-54.

[4] Koken F, Bozkurt D. On The Jacobsthal Numbers by Matrix Methods. Int. J. Contemp. Math. Sciences. 2008;3:605-614.

[5] Iyer MR. Some Results on Fibonacci Quaternions. The Fibonacci Quarterly. 1969;7(2):201-210.

[6] Knuth D. Negafibonacci Numbers and Hyperbolic Plane. Annual Meeting of the Math. Association of America; 2013.

[7] Horadam AF. A Generalized Fibonacci Sequence. American Math. Monthly. 1961;68:455-459.

[8] Lucas E. Theorie des Nombers. Gauthiers-Villars, Pa r i s; 1961.

[9] Koshy T. Fibonacci and Lucas Numbers with Applications. A Wiley-Interscience Publication, New York; 2001.

[10] Halıcı S. On Bicomplex Jacobsthal-Lucas Numbers. Journal of Mathematical Sciences and Modelling. 2020;3(3):139-143.

[11] Basin SL, Verner E, Hoggatt JR. A Primer on the Fibonacci Sequence-Part II. Fib. Quart. 1963;1(2):61-68.

[12] Brenner JL. June Meeting of the Pacific Northwest Section. 1. Lucas Matrix. Amer. Math. Monthly. 1951;58:220-221.

[13] Babadag F. Bicomplex Numbers: Further Contributions to a Fibonacci and Fibonacci-Lucas Matrices Oriented Approach. BEU Journal of Science. 2021;10(4):1386-1393.

[14] Verner E, Hoggatt JR. Belated Acknowledgement. Fib. Quart. 1968;6(3):85-85.

[15] Honsberger R. The Matrix Q. Mathematical Gems III. Washington, DC: Math. Assoc. Amer. 1985;8(4):106-107.

[16] King CH. Some Further Properties of the Fibonacci Numbers. Master's thesis. San Jose, CA: San Jose State; 1960.

[17] David GG. Permanent of a square matrix. European Journal of Combinatorics. 2010;31;1881-1891.